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SOME NEW VIEWPOINTS IN DIFFERENTIAL GEOMETRY IN THE LARGE[1]

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1. **Introduction.** Differential geometry in the large is concerned with relations which exist between the local properties of a geometric being[2] given in a manifold and the properties of the manifold as a whole. The manifold is differentiable in the sense that it is covered by a set of coordinate neighborhoods, each having the same number of coordinates, and that two different systems of coordinates in a common region are related by a differentiable transformation of class not less than 1. The latter assumption allows the use of differentiation in studying the local geometry and thus leads to a number of geometric properties the study of which was initiated by Euler, Gauss, and Monge.

To give an example of a problem of this nature we consider a closed surface S, differentiably imbedded (of class not less than 2) in a Euclidean space of three dimensions. Let K be the Gaussian curvature and dA the surface element of S. Then the classical Gauss-Bonnet formula asserts that

$$\frac{1}{2\pi}\int\int_S KdA = 2(1-p), \tag{1}$$

where p is the genus of S. This formula expresses the genus p of S, a topological invariant, in terms of a differential invariant. In other words, p is completely determined by the local properties of S.

As another example consider a closed curve C imbedded in the Euclidean plane. If C is rectifiable with length l and bounds a region of area A, then

$$l^2 - 4\pi A \geqq 0. \tag{2}$$

The equality sign holds only when C is a circle. This so-called isoperimetric inequality has recently been derived in an unexpected

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[1] Dedicated to Professor Elie Cartan.

[2] The term *geometric object* was first suggested by Professor Oswald Veblen, who now prefers to use *geometric being*, a translation of the French term "être géométrique" due to Cartan.

way in integral geometry, which deals with the integrals of differential invariants.

In problems of this nature there are two aspects, a local one and a global one. During several decades of intensive work the local aspect has been extensively studied and developed, culminating in the doctrine of tensor analysis. While tensor analysis gives an adequate tool for handling most local problems, it is natural that the study of global problems will necessitate the introduction of new concepts and the modifications of the classical treatment which has so far been followed. It is the main aim of the present article to emphasize the importance, for the global study of manifolds with a geometric being, of drawing into consideration new topological spaces associated with the manifold. In fact, this idea is important for both local and global aspects of differential geometry. For local problems it is certainly very familiar to Elie Cartan, who introduced the notion of tangent space ("espace tangent") for his general theory of geometric beings (affine connections, projective connections, conformal connections, and so on). The tangent space in the sense of Cartan is not always the space of tangent vectors and therefore constitutes one of the sources of difficulty for the understanding of his work. On the other hand, recent works on fibre bundles in topology (Stiefel, Whitney, Feldbau, Ehresmann, Pontrjagin, Steenrod, and so on) seem to lay the foundation for a global theory of the ideas of Cartan. It is a conviction of the author that a mingling of these two streams of thought will give the ideas and tools better adaptable to the study of differential geometry in the large than hitherto achieved. The present article will be devoted to a discussion of different aspects of the problem arising from this viewpoint.

Before going into details, we shall give a brief summary of the main points of our discussion. Our problem will be the study, on a differentiable manifold, of a geometric being given in terms of each local coordinate system by components which obey a definite transformation law under change of the local coordinates. We emphasize that for each such problem it is in general possible to define in a certain sense a natural fibre bundle associated with the manifold. The geometric being then defines in a unique way a set of linear differential forms in the fibre bundle, which gives all the local properties of the geometric being. For Riemannian manifolds the natural fibre bundle is the space of all frames on the manifold and the corresponding linear differential forms give what is essentially known as the parallelism of Levi-Civita. The nature of the fibre bundle to be associated is best decided by the solution of the so-called problem of equivalence, which is the form problem in the case of Riemannian geometry. With the set of linear differential forms in the fibre bundle, differential forms of higher degree can be obtained by operations of Grassmann analysis. Such differential forms define cochains and also cocycles, when they are exact. A study of the interrelations between the cochains and cocycles in the given manifold and the associated fibre bundles, which arise from the projection of the latter into the former, will give a deeper understanding of the global theory of the geometric being. An attempt is made in the following to illustrate in the simplest cases the results that can be derived from this idea. Although this will be our main concern in the present paper, it does not imply that this is the only service that the theory of fibre bundles could render to differential geometry. In fact, there are indications that other possible applications could be fruitfully made. The whole field seems therefore to deserve a more thorough exploitation.

2. **Grassmann algebra.** At the fore of the Cartan scheme is the so-called Grassmann algebra. In a formal algebraic way this can be defined as follows: Let K be a field of characteristic zero and let $V(n, K)$ be a vector space of n dimensions over K. The Grassmann algebra H over $V(n, K)$ is a hypercomplex system over K satisfying the following conditions:

(1) H contains a unit element 1 and all elements of $V(n, K)$ and is generated by these elements (by operations of the hypercomplex system).

(2) If x, y belong to $V(n, K)$, their multiplication satisfies the alternating rule:

$$xy = - yx. \tag{3}$$

(3) The elements of H satisfy no other relations than those derived from (1) and (2).

Let $e_1, \cdots, e_n$ be a set of basis elements of $V(n, K)$. From the above conditions the following conclusion can be drawn: Every element of H can be expressed in one and only one way as a linear combination, with coefficients in K, of the elements $1, e_{i_1} \cdots e_{i_m}$, $i_1 < \cdots < i_m$, $i_1, \cdots, i_m = 1, \cdots, n$. In symbols we write, if x belongs to H,

$$x = \sum_{m=0}^{n} x^{(m)}, \tag{4}$$

where

$$x^{(m)} = \sum_{i_1, \cdots, i_m = 1; i_1 < \cdots < i_m}^{n} a_{i_1 \cdots i_m} e_{i_1} \cdots e_{i_m}. \tag{5}$$

The element x is said to be of degree m, if

$$x^{(m)} \neq 0, \qquad x^{(m+1)} = \cdots = x^{(n)} = 0.$$

It is called an exterior form, or simply a form, if $x^{(i)}=0$, $i\neq m$. The degree of an element is independent of the choice of the set of basis elements, and the same is true of the property that an element be a form.

Let W be a vector subspace of dimension m of $V(n, K)$. We take a set of basis vectors $f_1, \cdots, f_m$ in W and form the product $w=f_1 \cdots f_m$. It is easy to prove that w is determined by W up to a nonzero factor of K. The form w is called the associated form of W. If w_1 and w_2 are the associated forms of two vector subspaces W_1 and W_2, of degrees m and p respectively, then W_1 and W_2 have a nonzero intersection when and only when $w_1w_2=0$. Moreover, if $w_1w_2\neq 0$, it is the associated form of the ambient vector subspace of dimension $m+p$ of W_1, W_2. Such are simple instances of ways in which the Grassmann algebra can be applied to the study of the geometry in a vector space.

3. **The differential calculus of Elie Cartan.** The Grassmann algebra was applied with success to the Pfaffian problem by Frobenius and Darboux. The so-called bilinear covariant is an exterior differential form of degree two. But it was Elie Cartan who started the systematic use of a calculus closely related to the Grassmann algebra and pertaining to exterior differential forms of higher degree.

Let M^n be a differentiable manifold of dimension n and class not less than 2. At a point P of M^n the contravariant vectors and the covariant vectors constitute two vector spaces (over the real field) which are dual to each other in the sense that one is the space of the linear forms in the other. The notion of a linear differential form at P is identical with that of a covariant vector. For let $x^1, \cdots, x^n$ be local coordinates at P. With respect to the coordinates x^i a covariant vector will have the components X_i. The differential form

$$\omega = \sum_{i=1}^{n} X_i dx^i \tag{6}$$

is then intrinsic in the sense that it is independent of the choice of the local coordinate system. Conversely, an intrinsic form ω has a set of components in each local coordinate system, which constitute a covariant vector.

Over the vector space of linear differential forms at a point P of M^n we construct the Grassmann algebra. A form of this Grassmann algebra is called an exterior differential form or simply a differential form. In a local coordinate system x^i an exterior differential form can be written

$$\omega = \sum_{i_1, \cdots, i_m=1}^{n} a_{i_1 \cdots i_m}(x^1, \cdots, x^n) dx^{i_1} \cdots dx^{i_m}, \tag{7}$$

where we can assume the coefficients $a_{i_1 \cdots i_m}$ to be skew-symmetric in their indices. In the terminology of tensor analysis the components $a_{i_1 \cdots i_m}$ are the components of an alternating covariant tensor of order m.

In a neighborhood of P in which the local coordinates x^i are valid let $a_{i_1 \cdots i_m}$ be of class not less than 2. We define a differential form of degree $m+1$, called the exterior derivative of ω, by means of the relation

$$d\omega = \sum_{i_1, \cdots, i_m=1}^{n} da_{i_1 \cdots i_m} dx^{i_1} \cdots dx^{i_m}, \tag{8}$$

where $da_{i_1 \cdots i_m}$ is an ordinary differential. This process of exterior differentiation has the following properties, which can be easily verified:

(1) If ω, θ are two differential forms of degrees m and p respectively, then

$$d(\omega\theta) = d\omega \cdot \theta + (-1)^m \omega d\theta. \tag{9}$$

(2) For any ω:

$$d(d\omega) = 0. \tag{10}$$

(3) The process of exterior differentiation is invariant under coordinate substitutions, by which we mean the following: Let the local coordinates x^i in a neighborhood of P represent a set of points S in an open set D in a Euclidean space of n dimensions. Let an open set E in a Euclidean space of r dimensions with the coordinates $y^1, \cdots, y^r$ be mapped into D by the mapping

$$x^i = f^i(y^1, \cdots, y^r), \qquad i = 1, \cdots, n, \tag{11}$$

such that S is the image of a set $T \subset E$, the functions f^i being supposed to be of class not less than 2. By the definition

$$F(x^1, \cdots, x^n) \to F(x^1(y), \cdots, x^n(y)), \qquad dx^i \to \sum_{k=1}^{r} \frac{\partial x^i}{\partial y^k} dy^k;$$

this mapping induces in a natural way (that is, with the multiplication of the Grassmann algebra preserved) a mapping of a differential form ω into a form of the space $(y^1, \cdots, y^r)$, which we denote by $\tilde{\omega}$. Then we have

$$d\tilde{\omega} = \widetilde{d\omega}. \tag{12}$$

In particular, when $r=n$, this signifies that the process of exterior differentiation is intrinsic, that is, independent of the local coordinate system under which it is performed.

The exterior differential forms are the forms under the integral sign in the theory of multiple integrals. In fact, let ω be a form of degree m defined on the manifold M^n. Let K^m be a chain of dimension m (in the sense of combinatorial topology) on M^n, which is a sum, with integral, rational, or real coefficients, of images of simplexes in a Euclidean space by mappings which are differentiable of class not less than 2. Then it is possible to define the integral of ω over K^m. If ∂K^{m+1} denotes the boundary of a chain K^{m+1} of dimension $m+1$, the theorem of Stokes can be written

$$\int_{\partial K^{m+1}} \omega = \int_{K^{m+1}} d\omega. \tag{13}$$

To see the connection between this formulation of the Stokes' Theorem and the usual one, notice that

$$d(Pdx + Qdy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy, \tag{14a}$$

$$\begin{aligned} d(Pdx + Qdy + Rdz) \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx \\ &\quad + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \end{aligned} \tag{14b}$$

4. Fibre bundles.

A simple example of a fibre bundle is the manifold formed by all the nonzero vectors tangent to a sphere of three-dimensional Euclidean space. It is a topological manifold, but a very special one. However, it turns out that manifolds with similar properties play an important rôle in the application of topology to differential geometry. In the example of vectors tangent to a sphere the following three facts deserve attention:

(1) The spaces formed by the vectors with the same origin are homeomorphic to each other, and therefore to a fixed space which we call F_0.

(2) The tangent vectors drawn at all points of a neighborhood U

are homeomorphic to a topological product of U and F_0. In particular, if $P \in U$, there is a homeomorphism T which carries the tangent vectors with the origin P to F_0, T depending on P and U.

(3) In the notation of (2) let us write $T(P, U)$ for T. If P belongs to a second neighborhood V, the mapping $T(P, V)T^{-1}(P, U)$ is a homeomorphism of F_0 onto itself. It is possible to show that T can be chosen so that $T(P, V)T^{-1}(P, U) \in G$, where G is either the rotation group or the affine group in F_0.

The last statement can be demonstrated as follows: Let the neighborhoods be coordinate neighborhoods, such that U is the set of all points whose local coordinates u^1, u^2 satisfy the inequalities $|u^i| < \epsilon$, $i=1, 2$. With respect to the local coordinate system u^1, u^2 a vector $\mathfrak{x}$ has the components X^1, X^2. The vectors with origins in U are thus decomposed into a topological product in an obvious way. If a point belongs to a neighborhood V with the local coordinates v^1, v^2 and the vector $\mathfrak{x}$ has the components Y^1, Y^2 in the coordinate system v^1, v^2, then it is well known that

$$Y^1 = \frac{\partial v^1}{\partial u^1} X^1 + \frac{\partial v^1}{\partial u^2} X^2, \qquad Y^2 = \frac{\partial v^2}{\partial u^1} X^1 + \frac{\partial v^2}{\partial u^2} X^2,$$

which is an affine transformation.

Guided by this particular example, we shall give the definition of a general fibre bundle. For the terms and notation which will be used consistently later the following table of reference is given:

General case	Special example	Notation
Fibre bundle	Manifold of tangent vectors of a sphere	$\mathfrak{F}$
Point of fibre bundle	Tangent vector	P
Base space	Sphere	M
Fibre	Vectors with same origin	F
Projection	Mapping of a tangent vector into its origin	$\pi(P) \in M$
Transformation group in fibre	Affine or rotation group in space of vectors with same origin	G

We define a fibre bundle $\mathfrak{F}$ as a topological space which has the following properties:

(1) There exists a (continuous) mapping π of F onto another topological space M: $\pi(F) = M$. The space M is called the base space and the mapping π is called the projection. The complete inverse image

$\pi^{-1}(p)$ of a point $p \in M$ is called the fibre at p. It will follow from assumptions made later that all the fibres are homeomorphic to each other and hence to a definite topological space F_0.

(2) There exists a family of neighborhoods which cover M. If U is a neighborhood of the family, the inverse image $\pi^{-1}(U)$ is a topological product, which means that there exists a homeomorphism ψ_U, depending on U, such that

$$\psi_U\{\pi^{-1}(U)\} = U \times F_0,$$

and that for every $p \in U$

$$\psi_U\{\pi^{-1}(p)\} = p \times F_0.$$

(3) Let U, V be two such neighborhoods of M, and let $p \in U \cap V$. The mapping $\psi_V\{\psi_U^{-1}(p \times F_0)\}$ is a homeomorphism of $p \times F_0$, and hence of F_0, into itself. This homeomorphism belongs to a group G given in advance in F_0.

For the cases in which we are interested both M and $\mathfrak{F}$ will be supposed to be differentiable manifolds of class not less than 1, and the group G will be a Lie group.

In order to show the scope of the notion of a fibre bundle, the following further examples are given—all pertaining to a differentiable manifold M of dimension n and class not less than 1:

(1) The tangent vectors of M constitute a fibre bundle with M as the base space.

(2) Let $\mathfrak{e}_1, \cdots, \mathfrak{e}_p$, $1 \leqq p \leqq n$, be tangent vectors of M with the common origin P. The elements $(P; \mathfrak{e}_1, \cdots, \mathfrak{e}_p)$ constitute a fibre bundle over M. Another fibre bundle is constituted by the elements satisfying the condition that $\mathfrak{e}_1, \cdots, \mathfrak{e}_p$ are linearly independent. These fibre bundles have been studied by Stiefel and Whitney [26, 33].[3]

(3) We call a scalar density of weight k a geometrical being, which has a component in each local coordinate system and whose components f and f^* in the local coordinate systems x^i and x^{*i} are connected by the relation

$$f^* = f \cdot J^k, \tag{15}$$

where

$$J = \partial(x^1, \cdots, x^n)/\partial(x^{*1}, \cdots, x^{*n}). \tag{15a}$$

The scalar densities or the nonzero scalar densities in M constitute a fibre bundle over M.

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[3] Numbers in brackets refer to the Bibliography at the end of the paper.

(4) A general procedure of defining an important class of fibre bundles is as follows: Let M be imbedded in a Euclidean space E^{n+N} of dimension $n+N$. At a fixed point O of E^{n+N} let $H(n, N)$ be the Grassmann manifold formed by all the oriented linear subspaces of dimension n through O. A mapping $T(M)\subset H(n, N)$ defines a fibre bundle over M such that the fibre at each point P of M is the n-dimensional linear subspace through P parallel to $T(P)$.

One of the important results in the theory of fibre bundles is the introduction of the so-called characteristic cocycle. To give its definition we assume M to be a polyhedron, and let K be a simplicial decomposition of M. Let K^r be the r-dimensional skeleton of K, that is, the subcomplex of K consisting of all simplexes of dimension not greater than r. We suppose the fibre bundle to be orientable in the sense explained by Steenrod [24]. Let F_0 be connected and have an abelian fundamental group. Under these assumptions the following theorems can be established:

THEOREM 4.1. *Let $H^i(F_0)$ be the ith homology group (with integral efficients) of F_0. If*

$$H^1(F_0) = \cdots = H^{r-1}(F_0) = 0,$$

it is possible to define a continuous mapping ϕ of K^r into $\mathfrak{F}$ such that $\pi\phi(p) = p$, $p \in K^r$.

THEOREM 4.2. *With the same notation as in Theorem 4.1, if*

$$H^1(F_0) = \cdots = H^{r-1}(F_0) = 0, \qquad H^r(F_0) \neq 0,$$

there exists a cohomology class (with $H^r(F_0)$ as the coefficient group) γ^{r+1} of dimension $r+1$, whose vanishing is a necessary and sufficient condition that a continuous mapping ϕ of K^{r+1} into $\mathfrak{F}$ can be defined such that $\pi\phi(p) = p$, $p \in K^{r+1}$. This cohomology class is a topological invariant of $\mathfrak{F}$.

The cohomology class γ^{r+1} is called the characteristic cohomology class and any one of its cocycles a characteristic cocycle. If M is an orientable manifold, the dual of γ^{r+1} is a homology class of dimension $n-r-1$ and is called the characteristic homology class. In particular, if M is a differentiable manifold and $\mathfrak{F}$ the fibre bundle of nonzero tangent vectors of M, the characteristic cocycle γ is of dimension n and its value for the fundamental cycle of M is the Euler-Poincaré characteristic of M. In this way we see that the theory of characteristic class generalizes the classical theory of vector fields on a differentiable manifold.

5. Riemannian geometry. In the preceding sections are sketched the preliminaries for the theory of a geometric being. As an illustration we shall apply the tools so established to Riemannian geometry, the "simplest" and most important among the geometric beings. It will be seen that the results are essentially those given by Cartan and their relations with the now classical treatment by tensor analysis will also be indicated; but we have put ourselves in a more general viewpoint, which will lead to further developments.

Let M be a differentiable manifold of dimension n and class not less than 4. In M suppose a Riemannian metric be given, which, in terms of a local coordinate system x^i, is defined by a positive definite quadratic differential form[4]

$$ds^2 = \sum_{i,j} g_{ij}(x^k)(dx^i dx^j), \qquad g_{ij} = g_{ji}. \tag{16}$$

The manifold M is then called a Riemannian manifold.

Consider a point P of M and the contravariant tangent vectors of M having the origin P. With the differential form (16) (or the tensor g_{ij}) the scalar product of two vectors $\mathfrak{e}$, $\mathfrak{f}$ can be defined, which we denote by $\mathfrak{e}\cdot\mathfrak{f}$. A vector $\mathfrak{e}$ is called a unit vector if $\mathfrak{e}^2=1$. An ordered set of n vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ is said to constitute a frame (ennuple), if

$$\mathfrak{e}_i\mathfrak{e}_j = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i, j = 1, \cdots, n. \tag{17}$$

A differentiable curve through P is defined by the equations

$$x^i = x^i(s), \tag{18}$$

where s is the arc length of the curve and the functions are of class not less than 1. We denote by dP/ds the unit tangent vector of the curve at P. It is easy to show, by referring to a local coordinate system, that there exists a vector (which we shall denote by dP) whose components with respect to a frame are linear differential forms and which is equal to the product of ds and the unit tangent vector along a curve. This vector is intrinsic, that is, independent of the choice of coordinates. Referred to a frame $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$, we can write

$$dP = \omega_1\mathfrak{e}_1 + \cdots + \omega_n\mathfrak{e}_n, \tag{19}$$

where $\omega_1, \cdots, \omega_n$ are linear differential forms. Then we have, along a curve, $(dP/ds)^2=1$ or

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[4] The differential form in question is an ordinary and not an exterior differential form. As a means of distinction, we insert a parenthesis about the differentials when the former is the case.

$$ds^2 = (\omega_1)^2 + \cdots + (\omega_n)^2. \tag{20}$$

The forms ω_i, $i=1, \cdots, n$, are therefore the forms the sum of whose squares is equal to the given (ordinary) quadratic differential form.

As the origin P runs over M, the frames $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ constitute a fibre bundle with M as the base space. Each fibre is topologically the space of all frames having the same origin. The fibre bundle is of dimension $n(n+1)/2$ and each fibre is of dimension $n(n-1)/2$. The group which operates in each fibre is the group of orthogonal transformations.

There is yet no intrinsic meaning attached to the vectors $d\mathfrak{e}_i$. We want to see whether it is possible to define intrinsically certain differential forms ω_{ij} such that

$$d\mathfrak{e}_i = \sum_j \omega_{ij}\mathfrak{e}_j. \tag{21}$$

From (17) it follows that ω_{ij} has to be skew-symmetric in its indices:

$$\omega_{ij} + \omega_{ji} = 0. \tag{22}$$

The answer to our question is given by the following theorem:

THEOREM 5.1. *There exists one, and only one, set of linear differential forms* ω_{ij} *in the fibre bundle such that the equation* (21) *and the equation*

$$d(dP) = 0 \tag{23}$$

hold.

In fact, the condition (23) gives, when expanded,

$$d\omega_i - \sum_j \omega_j\omega_{ji} = 0. \tag{24}$$

To calculate $d\omega_i$ we make use of a local coordinate system x^i and take a definite decomposition of ds^2 as a sum of squares:

$$ds^2 = (\theta_1)^2 + \cdots + (\theta_n)^2, \tag{25}$$

where

$$\theta_i = \sum_j a_{ij}(x)dx^j. \tag{25a}$$

Since (20) and (25) give two decompositions of the same ordinary quadratic differential form as sums of squares, we have

$$\omega_i = \sum u_{ij}\theta_j, \tag{26}$$

where u_{ij} are the elements of an orthogonal matrix:

(26a) $$U = (u_{ij}), \qquad UU' = I = \text{identity matrix}.$$

Forming the exterior derivative of the equation (26), we get

$$d\omega_i = \sum du_{ij}\theta_j + \sum u_{ij}d\theta_j,$$

which shows that $d\omega_i$ is of the form

(27) $$d\omega_i = \sum \omega_j\phi_{ji},$$

where ϕ_{ji} are linear differential forms in the variables x^i, u_{ij} and can be explicitly calculated. Substitution into (24) gives

$$\sum \omega_j(\omega_{ji} - \phi_{ji}) = 0,$$

which gives in turn

(28) $$\omega_{ji} - \phi_{ji} = \sum \lambda_{jik}\omega_k, \qquad \lambda_{jik} = \lambda_{kij}.$$

In this equation the forms ϕ_{ji} are known and the quantities λ_{jik} will be determined in order that ω_{ji} be determined. Since ω_{ij} is skew-symmetric in its indices, we have

$$-(\phi_{ij} + \phi_{ji}) = \sum (\lambda_{ijk} + \lambda_{jik})\omega_k.$$

This equation shows that $\phi_{ij}+\phi_{ji}$ is of the form

(29) $$\phi_{ij} + \phi_{ji} = -\sum A_{ijk}\omega_k, \qquad A_{ijk} = A_{jik},$$

and that then

(30) $$\lambda_{ijk} + \lambda_{jik} = A_{ijk}.$$

The quantities λ_{ijk}, being symmetric in the first and third indices, are then uniquely determined and are given by

(31) $$\lambda_{ijk} = (-A_{ikj} + A_{jki} + A_{ijk})/2.$$

This proves our theorem.

The equation (21) may be interpreted as giving a means of transporting a vector at an infinitesimally near point of P to a vector at P. It is essentially the notion known as the parallelism of Levi-Civita.

By forming the exterior derivative of (24) and making use of these equations themselves, we get

$$\sum_j \omega_j\left(\sum_k \omega_{ik}\omega_{kj} - d\omega_{ij}\right) = 0.$$

It follows easily from this that we can write

(32) $$d\omega_{ij} = \sum_k \omega_{ik}\omega_{kj} + \Omega_{ij},$$

where Ω_{ij} are of the form

$$\Omega_{ij} = \sum R_{ij,kl}\omega_k\omega_l, \qquad R_{ij,kl} + R_{ij,lk} = 0, \qquad \Omega_{ij} + \Omega_{ji} = 0. \tag{33}$$

We shall call the forms Ω_{ij} the curvature forms.

In a slightly different version our results may be summarized in the theorem:

THEOREM 5.2. *Let M be a Riemannian manifold of dimension n and let $\mathfrak{F}$ be the fibre bundle of all frames of M. There is a unique way to define in $\mathfrak{F}$ (of dimension $n(n+1)/2$) a set of $n(n+1)/2$ linearly independent linear differential forms ω_i, ω_{ij}, which satisfy the equations* (19), (21), (24).

Let us indicate briefly the relations of these considerations to the ordinary treatment by means of tensor analysis. In tensor analysis emphasis is laid particularly on the local coordinate system. The procedure is as follows: With respect to the local coordinates x^i let $\mathfrak{f}_k$ be the velocity vector of the coordinate curve $x^i = \text{const.}$, $i \neq k$, with x^k as the time. Then we have

$$\mathfrak{f}_i \cdot \mathfrak{f}_k = g_{ik}, \tag{34}$$

and

$$dP = dx^1 \cdot \mathfrak{f}_1 + \cdots + dx^n \cdot \mathfrak{f}_n. \tag{35}$$

The components X^i of a vector $\mathfrak{x}$ with respect to the vectors $\mathfrak{f}_k$ are defined by the equation

$$\mathfrak{x} = \sum X^i \mathfrak{f}_i. \tag{36}$$

By an argument analogous to the proof of Theorem 5.1 it can be shown that there exists a set of linear differential forms such that

$$d\mathfrak{f}_i = \sum \theta_i^j \mathfrak{f}_j, \tag{37}$$

and

$$d(dP) = 0.$$

The vector $\mathfrak{x} + d\mathfrak{x}$ is called parallel to $\mathfrak{x}$ if $d\mathfrak{x} = 0$, which can be written, by using equations (36) and (37),

$$dX^i + \sum \theta_j^i X^j = 0. \tag{38}$$

These equations are easily recognized to be the well known equations which define the parallelism of Levi-Civita.

The important fact in Riemannian geometry is the necessity of taking into consideration not only the Riemannian manifold itself but the fibre bundle over it. In our approach we consider directly the fibre bundle. The usual tensor approach avoids it by laying emphasis on the local coordinate system. Quantities of geometrical interest are those which obey a simple transformation law under changes of the local coordinate systems. For Riemannian geometry the two methods are essentially the same, so far as local problems are concerned. It is, however, to be remarked that even for local problems their generalizations to other geometric beings will lead to different formulations. We shall give in the next section a systematic way of treating the local theory of a geometric being.

6. **Method of equivalence.** So far the introduction given in the above section of the fibre bundle over a Riemannian manifold M seems to be accidental. The underlying reason will be most clear, if we start with the following problem:

Let two Riemannian metrics be given by the positive definite quadratic differential forms

$$ds^2 = \sum_{i,j} g_{ij}(x)(dx^i dx^j), \qquad g_{ij} = g_{ji}, \tag{39}$$

$$ds^{*2} = \sum_{i,j} g_{ij}^*(x^*)(dx^{*i} dx^{*j}), \qquad g_{ij}^* = g_{ji}^*, \tag{40}$$

in two coordinate neighborhoods U, U^* respectively. We determine the conditions that a neighborhood $V \subset U$ can be mapped by a differentiable mapping (of sufficiently large class) with nonvanishing functional determinant into U^* such that

$$ds^{*2} = ds^2.$$

This so-called form problem was solved by Christoffel and Lipschitz. It is clear that if the problem has a solution, the local theories of the two Riemannian metrics are essentially the same.

To give a different formulation of our problem we determine two sets of linear differential forms $\theta_i(x, dx)$ and $\theta_i^*(x^*, dx^*)$ such that

$$ds^2 = (\theta_1)^2 + \cdots + (\theta_n)^2, \qquad ds^{*2} = (\theta_1^*)^2 + \cdots + (\theta_n^*)^2.$$

Our problem will have a solution if and only if the differentiable mapping in question and the functions $u_{ij}(x)$ can be determined in a neighborhood $V \subset U$ such that

$$\theta_i^*(x^*, dx^*) = \sum_j u_{ij}(x)\theta_j(x, dx), \tag{41}$$

where the matrix

$$U = (u_{ij}) \tag{42}$$

is an orthogonal matrix.

In this way our problem, originally formulated in terms of quadratic differential forms, is reduced to one in linear differential forms. In this formulation the problem is a particular case of a more general problem, which was first stated and solved by Elie Cartan. Cartan's problem is as follows:

Let $\theta_i(x, dx)$ and $\theta_i^*(x^*, dx^*)$ be two sets of n linearly independent linear differential forms in the coordinate neighborhoods U, U^* of the coordinates x^i, x^{*i}, $i=1, \cdots, n$. Let Γ be a linear group in the vector space of n dimensions. We wish to determine the conditions that there exists a differentiable mapping with a nonvanishing functional determinant of a neighborhood $V \subset U$ into U^* such that

$$\theta_i^* = \sum_j u_{ij}(x)\theta_j, \tag{43}$$

where the linear transformation belongs to the given linear group Γ.

In our case Γ is the orthogonal group.

The discussion of this local problem requires Cartan's theory of Pfaffian systems in involution and is not simple. For details we refer to [**3**].

We only remark that if Γ consists of the identity only, the solution of this problem is relatively simple. The general procedure is to reduce the problem to this particular case by the introduction of new variables. We shall illustrate this method by considering our form problem. Let u_{ij} and u_{ij}^* be two sets of new variables which are elements of the orthogonal matrices

$$U = (u_{ij}), \qquad U^* = (u_{ij}^*),$$

and let us put

$$\omega_i = \sum u_{ij}\theta_j, \qquad \omega_i^* = \sum u_{ij}^*\theta_j^*. \tag{44}$$

Our form problem will have a solution if and only if there exists a differentiable mapping of a neighborhood $V \subset U$ into U^* and an orthogonal matrix X, whose elements are functions in V, such that under this mapping and the mapping between (u_{ij}) and (u_{ij}^*) defined by

$$U^*X = U, \tag{45}$$

we have

$$\omega_i^* = \omega_i. \tag{46}$$

From the last equation we derive

$$d\omega_i^* = d\omega_i. \tag{47}$$

By an argument exactly identical to one used in the last section it can be shown that there exists one and only one set of linear differential forms ω_{ij} in x^i, u_{ij}, which are skew-symmetric in their indices and which satisfy the equations

$$d\omega_i = \sum \omega_j \omega_{ji}.$$

Similarly, we find a set ω_{ij}^* $(= -\omega_{ji}^*)$ of linear differential forms such that

$$d\omega_i^* = \sum \omega_j^* \omega_{ji}^*.$$

When (46) and (47) hold, it is easy to derive

$$\omega_{ij}^* = \omega_{ij}. \tag{48}$$

Let V be a sufficiently small neighborhood of U and let $O(n)$ be the group manifold of the orthogonal group. We consider the topological product $V \times O(n)$. Similarly, we have the topological product $V^* \times O^*(n)$. In order that our form problem possess a solution it is necessary and sufficient that there exist a differentiable mapping between $V \times O(n)$ and $V^* \times O^*(n)$ such that under this mapping the equations (46) and (48) hold. It is easy to see that the forms ω_i, ω_{ij}, whose number $n(n+1)/2$ is equal to the dimension of $V \times O(n)$, are linearly independent. Thus our problem is reduced to Cartan's general problem with Γ consisting of the identity only.

It can therefore be seen that the discussions of the last section are nothing else than a geometrical treatment of the above analytic considerations. We summarize these results as follows:

Given a Riemannian manifold M of dimension n with a positive definite quadratic differential form ds^2. To M is associated the fibre bundle, with M as its base space, of all sets of linear differential forms (that is, covariant vectors) the sum of whose squares is equal to ds^2. The fibre bundle is of dimension $n(n+1)/2$ and there exists in it a set of $n(n+1)/2$ linearly independent linear differential forms ω_i, ω_{ij} which are invariant. There exists locally a differentiable mapping which carries one ds^2 into another when and only when such a mapping exists in the corresponding fibre bundles under which the forms ω_i, ω_{ij} are respectively equal.

We may say that the deep reason that this very fibre bundle (namely, the one whose fibres are topologically the manifolds $O(n)$) is attached to a Riemannian manifold comes as a result of the solution of the form problem. From the case of the Riemannian manifold it is clear that, for the theory of other kinds of geometric beings, the solution of a corresponding local "equivalence problem" is a natural preliminary step for the establishment of a proper theory in the large. In fact, from the solution of the equivalence problem the nature of the fibre bundle to be attached to the manifold can be determined.

There is a great variety of geometric beings which are of interest in differential geometry. In order to illustrate our method we shall take as a further example the so-called geometry of paths. Let M be a differentiable manifold of dimension n. In a neighborhood with the local coordinates x^i let there be given a system of differential equations of the second order:

$$\frac{d^2x^i}{dt^2} + \sum \Gamma^i_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = F(x, t) \frac{dx^i}{dt}, \tag{49}$$

where the functions Γ^i_{jk}, F are supposed to be differentiable of class not less than 3. The integral curves of the differential system (49) are called the paths. They have the property that through every point of M and tangent to every vector through that point there passes one and only one path. The independent variable t is a parameter on the path. It is irrelevant in the sense that a change of parameter $t=\phi(\tau)$, where $\phi(\tau)$ is differentiable, gives the same path. In short, our manifold M carries a system of paths which is defined in each local coordinate system by a differential system of the form (49) and our problem is to study the geometrical properties in M arising from the paths. An important particular case of this problem is the case that M is the projective space of n dimensions and the paths are the straight lines in M, which, in a suitable local coordinate system, are defined by the system (49), with $\Gamma^i_{jk}=0$.

Following the spirit of our solution of the form problem we start by considering the following problem of equivalence: In another coordinate neighborhood with the local coordinates x^{*i} let a system of paths be given by a differential system analogous to (49). Determine the conditions under which a differentiable mapping between the coordinates x^i and x^{*i}, which carries one system of paths into another, exists.

To apply the method of equivalence we shall write the system (49) in a system of total differential equations. For simplicity let us suppose that, for the integral curves of (49) under consideration, x^n is not a constant. We can then take x^n to be the parameter along the curves, and the system (49) can be written

$$\frac{dx^\alpha}{dx^n} = y^\alpha, \qquad \frac{dy^\alpha}{dx^n} + \sum \Gamma^\alpha_{jk} y^j y^k = F y^\alpha, \qquad \sum \Gamma^n_{jk} y^j y^k = F,$$

where the index α runs from 1 to $n-1$ and $y^n = 1$. This system can be written as a system of total differential equations as follows:

$$\begin{aligned} \theta^\alpha &= dx^\alpha - y^\alpha dx^n = 0, \\ \pi^\alpha &= dy^\alpha + \left\{ \sum \Gamma^\alpha_{jk} y^j y^k - y^\alpha \sum \Gamma^n_{jk} y^j y^k \right\} dx^n = 0. \end{aligned} \tag{50}$$

We introduce also the form dx^n, so that θ^α, dx^n constitute a set of linearly independent linear differential forms in the coordinates x^i. On the other manifold with the local coordinates x^{*i} we shall introduce in a similar way the new variables $y^{*\alpha}$ and the forms $\theta^{*\alpha}$, dx^{*n}, $\pi^{*\alpha}$. Let us write

$$\Theta = \begin{pmatrix} dx^n \\ \theta^\alpha \\ \pi^\alpha \end{pmatrix}, \qquad \Theta^* = \begin{pmatrix} dx^{*n} \\ \theta^{*\alpha} \\ \pi^{*\alpha} \end{pmatrix}, \tag{51}$$

and let us introduce the matrix

$$U = \begin{pmatrix} A(1,1) & B(1, n-1) & 0 \\ 0 & C(n-1, n-1) & 0 \\ 0 & D(n-1, n-1) & E(n-1, n-1) \end{pmatrix}, \tag{52}$$

whose nonzero elements are independent variables, and where the numbers in the parentheses denote the numbers of rows and columns of the sub-matrices in question. It is easy to show that our problem of equivalence possesses a solution when and only when there is a differentiable mapping between two neighborhoods (x^i, y^α) and $(x^{*i}, y^{*\alpha})$ and the elements of U can be determined as functions of x^i, y^α, such that under this mapping the matrix equation

$$\Theta^* = U\Theta \tag{53}$$

holds. Now in the linear vector space of $2n-1$ dimensions all the matrices U form a group. Hence in this formulation our problem of equivalence is a particular case of the problem of equivalence of Cartan stated above.

The problem can be solved by applying Cartan's general procedure. We shall not give the details here. The solution of a problem which contains this problem as a particular case was given in [12].

Our final result can be stated in the theorem:

THEOREM 6.1. *In a differentiable manifold M of dimension n and* *class not less than 3 let a system of paths be given, defined locally by the differential equations* (49). *It is possible to introduce in a neighborhood of M with the local coordinates* x^i, $n(n+1)$ *other variables* u_A, $A=1, \cdots, n(n+1)$, *and to define* $n(n+2)$ *linearly independent linear differential forms* ω_B, $B=1, \cdots, n(n+2)$, *in the variables* x^i, u_A, *having the following property: A necessary and sufficient condition that a local differentiable mappings exists, which carries one system of paths into another with the local coordinates* x^{*i} *and the new variables* u_A^*, *and so on, is that there exists a differentiable mapping between two neighborhoods of the spaces* (x^i, u_A) *and* (x^{*i}, u_A^*) *under which the forms* ω_B *and* ω_B^* *are mapped into each other.*

With respect to a local coordinate system x^i the variables x^i, u_A, $A=1, \cdots, n(n+1)$, determine a topological product. We are thus led to consider the fibre bundle over M, which is of dimension $n(n+2)$ and is locally given by the coordinates x^i, u_A. In the fibre bundle there are defined exactly $n(n+2)$ intrinsic linearly independent linear differential forms. The significance of this fibre bundle to the differential geometry of paths is evident. It is important to point out that in the geometry of paths it is, in view of our Theorem 6.1, this fibre bundle, and not the fibre bundle of tangent vectors of M, which plays the fundamental rôle.

From the above discussions of Riemannian geometry and the geometry of paths it can be seen that the method of equivalence offers the best weapon for a frontal attack on the problem of a geometric being. Only from the outcome of the solution of the problem of equivalence can it be decided the most important kind of fibre bundle to be defined over the manifold. It is of course possible to define, in a more or less intuitive way, other kinds of fibre bundles over the manifold, but they will play a minor rôle. We wish also to remark that the solution of this local problem of equivalence is not always simple. So far as the writer is aware, the problem of equivalence for a skew-symmetric covariant tensor of order two, for instance, has not been explicitly solved (although the solution is theoretically always possible).

7. Relations between the fibre bundle and its base manifold. We have attempted to show in the above, in the cases of Riemannian geometry and the geometry of paths, how the local theory of a geometric being leads naturally to the fibre bundle to be associated to the manifold on which the geometric being is defined. The fact remains true of any geometric being, but its proof requires the fundamental theorem on partial differential equations to the effect that every system of partial differential equations can be "prolonged" to a system in involution, and we shall not enter into its discussion. It is to be remarked that for problems of this sort, as for any problem concerned with both pure and applied mathematics, the general result and the particular cases offer entirely different problems, one being an existence theorem and the other being an explicit solution. In our examples we solve the problems explicitly, thereby insuring the existence of the solution.

After defining over the given manifold M a fibre bundle $\mathfrak{F}$, the next step is to study their relationship. Before proceeding farther, let us remark that the fibre bundle $\mathfrak{F}$ will give rise to new fibre bundles over M by identification. Consider, for instance, the case that M is a Riemannian manifold. Let $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ be a frame in M. For any integer p, $1 \leqq p \leqq n$, we shall define a class of frames to be all frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ such that the point P and the vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_p$ are identical. These classes of frames constitute, with a natural topology, a fibre bundle over M, which we denote by $\mathfrak{F}^{(p)}$. $\mathfrak{F}^{(p)}$ is the fibre bundle of all ordered sets of p mutually perpendicular unit vectors of M. In particular, $\mathfrak{F}^{(1)}$ is the fibre bundle of unit vectors of M and $\mathfrak{F}^{(n)}$ is the fibre bundle $\mathfrak{F}$ itself. For distinction we shall call $\mathfrak{F}$ the principal fibre bundle of M and all others obtained by identification the associated fibre bundles. We shall consistently assume that the identification is made only in each fibre, so that the projection of the associated bundle onto the base manifold is the one induced by that of the principal bundle. It is seen that this process is very general.

Let $\mathfrak{F}$ be the principal fibre bundle over M and $\mathfrak{S}$ an associated fibre bundle over M. There is a projection $\pi(\mathfrak{F}) \subset M$ and a projection $\pi_1(\mathfrak{S}) \subset M$, and also a projection $\pi_2(\mathfrak{F}) \subset \mathfrak{S}$, defined by assigning to a point of $\mathfrak{F}$ the class to which it belongs during the identification. From the remark made at the end of the last paragraph it is clear that $\pi = \pi_1\pi_2$. We also remember that a set of intrinsic linearly independent linear differential forms is defined in $\mathfrak{F}$ by the geometric being, whose number is equal to the dimension of $\mathfrak{F}$. Our main aim will be to see how these differential forms behave under the projections π, π_1, π_2.

In order to understand this relationship more clearly we shall make a digression to recall some elementary facts in combinatorial topology. Let K be a finite complex of dimension n, whose simplexes we denote by σ_i^r, $0 \leqq r \leqq n$, $1 \leqq i \leqq \alpha_r$. Let R be a commutative ring. A chain of dimension r or an r-chain is a sum

$$\overset{r}{C} = \sum_{i=1}^{\alpha_r} \lambda_i \overset{r}{\sigma}_i, \qquad \lambda_i \in R. \tag{54}$$

To the chains C^r we introduce a boundary operator ∂, which is linear:

$$\partial(C_1^r + C_2^r) = \partial C_1^r + \partial C_2^r, \qquad \partial(\lambda C^r) = \lambda \partial C^r, \qquad \lambda \in R, \tag{55}$$

and is defined for the simplexes σ_i^r by the so-called incidence relations:

$$\partial \sigma_i^r = \sum_{j=1}^{\alpha_{r-1}} \eta_{ij}^{(r)} \sigma_j^{r-1}. \tag{56}$$

The relations (55) and (56) completely determine the boundary operator ∂ and the chain ∂C^r of dimension $r-1$ is called the boundary of C^r. It is easy to verify that

$$\partial\partial C^r = 0. \tag{57}$$

A chain whose boundary is zero is called a cycle. Equation (57) shows that a chain which is the boundary of another chain is a cycle, called a bounding cycle. The cycles form an abelian group and the bounding cycles a subgroup of it. Their quotient group is called the r-dimensional homology group of K (with the additive group of R as its coefficient group).

A cochain γ^r of dimension r is a linear function of the r-chains. We shall denote the function by the notation $\gamma^r \cdot C^r$, which has therefore the properties:

$$\gamma^r \cdot (C_1^r + C_2^r) = \gamma^r \cdot C_1^r + \gamma^r \cdot C_2^r, \quad \gamma^r \cdot (\lambda C^r) = \lambda \gamma^r \cdot C^r, \qquad \lambda \in R. \tag{58}$$

From a cochain γ^r of dimension r we define a cochain $\delta\gamma^r$ of dimension $r+1$, called its coboundary, by means of the relation

$$\delta\gamma^r \cdot C^{r+1} = \gamma^r \cdot (\partial C^{r+1}). \tag{59}$$

It then follows from (57) and (59) that

$$\delta\delta\gamma^r = 0. \tag{60}$$

A cochain whose coboundary is zero is called a cocycle. All the cocycles of dimension r form an abelian group, which contains as subgroup the group of all cocycles which are coboundaries. Their quotient group is called the r-dimensional cohomology group of K.

Now, let M be a differentiable manifold of dimension n and class not less than 2. Let ω be a differential form of degree r in M, whose coefficients are of class not less than 2. As we have remarked in §3, it is possible to define in a rigorous manner the integral of ω over an r-simplex and also the integral of ω over an r-chain. Moreover, the functional $\int_{C^r} \omega$ is linear in the r-chains C^r. Hence it defines an r-cochain with the ring of real numbers as the coefficient ring. When there is no confusion, we shall speak of ω as an r-cochain.

From (13) the Theorem of Stokes can be written in the form

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$$\int_{C^{r+1}} d\omega = \int_{\partial C^{r+1}} \omega. \tag{61}$$

Comparison with (59) shows that if ω defines an r-cochain, then $d\omega$ defines its coboundary. It follows that the r-cochain ω is a cocycle if and only if $d\omega = 0$ (in which case ω is called exact), and that ω is a coboundary if there exists a form θ whose exterior derivative $d\theta$ is equal to ω (in which case ω is called derived).

It was de Rham [22] who proved the important theorem that to every r-cochain γ (with real or rational coefficient group) there exists a differential form ω of degree r which defines γ according to the above process. We see from §4 that from the fibre bundles over M there are the characteristic cocycles in M which are invariants of the fibre bundles. A theorem in the differential geometry in the large will be obtained, whenever it is possible to define a characteristic cocycle by means of a differential form constructed locally from the geometric being.

Suppose now that a simplicial mapping f exists, which maps a complex K into a complex K^*. The mapping f induces a mapping ϕ of the chains of K into the chains of K^*. From φ a mapping ψ of the cochains of K^* into the cochains of K can be defined. In fact, if γ^* is a cochain of dimension r of K^*, we define $\psi\gamma^* \cdot \sigma^r = \gamma^* \cdot \phi\sigma^r$ for every simplex σ^r of K. This so-called inverse mapping ψ maps cocycles of K^* into cocycles of K and cocycles of the same cohomology class into cocycles of the same cohomology class. Therefore it induces a mapping of each cohomology group of K^* into the cohomology group of the same dimension of K and it can be proved that it is a homomorphism. It can also be proved that the homomorphism remains unchanged, if f is replaced by a mapping homotopic to f. The result is that we can define from a continuous mapping of a polyhedron P into a polyhedron P^* a homomorphism of a cohomology group of P^* into one of the same dimension of P.

In the case of a fibre bundle $\mathfrak{F}$ over M in which we are interested, there is a continuous mapping of $\mathfrak{F}$ into M, namely the projection. We assume that both $\mathfrak{F}$ and M are differentiable manifolds. According to the above, the projection induces a mapping of a cohomology group of M into one of the same dimension of $\mathfrak{F}$. In particular, an exact differential form of M will be mapped into an exact differential form of $\mathfrak{F}$.

Consider the case that M is a differentiable manifold in which a geometric being is given. Let $\mathfrak{F}$ be the principal fibre bundle and $\mathfrak{S}$ an associated fibre bundle over M. It is known that there is a projection $\pi_2(\mathfrak{F}) \subset \mathfrak{S}$. There are also linear differential forms defined in $\mathfrak{F}$

by the geometric being in M. From the linear differential forms it is possible to construct, by operations of the Grassmann algebra and the Cartan calculus, differential forms of higher degree. For any such differential form it is important to decide whether it is the image of a differential form of $\mathfrak{G}$ under the inverse mapping induced by π_2. There are several criterions for this problem. We shall give a criterion for the case that M is a Riemannian manifold, but the process can be readily generalized to other geometric beings.

Let $\mathfrak{G}=\mathfrak{F}^{(p)}$ and let the forms ω_i, ω_{ij}, Ω_{ij} be defined in the principal fibre bundle $\mathfrak{F}$. Let Π be constructed from ω_i, ω_{ij}, Ω_{ij} by operations of the Grassmann algebra. We wish to decide whether Π is the image of a form in $\mathfrak{G}$.

For this purpose let us use in this paragraph the following ranges of indices:

$$1 \leqq \alpha, \beta \leqq p, \qquad p+1 \leqq r, s \leqq n, \qquad 1 \leqq i, k \leqq n.$$

Let δ be an operation (not coboundary) under which the figure $P\mathfrak{e}_1 \cdots \mathfrak{e}_p$ remains unchanged, so that

$$\omega_i(\delta) = 0, \qquad \omega_{\alpha i}(\delta) = 0. \tag{62}$$

We also put

$$\omega_{rs}(\delta) = e_{rs}. \tag{62a}$$

From (24) and (32) we have

$$\begin{aligned} &\delta\omega_\alpha = 0, \quad \delta\omega_r = \sum e_{rs}\omega_s, \quad \delta\omega_{\alpha\beta} = 0, \quad \delta\omega_{\alpha r} = -\sum e_{sr}\omega_{\alpha s}, \\ &\delta\Omega_{\alpha\beta} = 0, \quad \delta\Omega_{\alpha r} = \sum e_{rs}\Omega_{\alpha s}, \quad \delta\Omega_{rs} = \sum (e_{rt}\Omega_{ts} - \Omega_{rt}e_{ts}). \end{aligned} \tag{63}$$

Then we have the following theorem:

THEOREM 7.1. *A form* Π *is the image of a differential form in* $\mathfrak{F}^{(p)}$ *if and only if* $\delta\Pi = 0$.

This theorem is also true for $p=0$, if we make the convention $\mathfrak{F}^{(0)}=M$.

If a form in $\mathfrak{F}$ is the image of a form in $\mathfrak{F}^{(p)}$, we shall say simply that it is a form of $\mathfrak{F}^{(p)}$.

It can easily be verified, by means of our criterion, that the following forms are forms of M:

$$\begin{aligned} &\Delta_m = \sum \Omega_{i_1 i_2}\Omega_{i_2 i_3} \cdots \Omega_{i_{2m} i_1}, \quad 1 \leqq m \leqq n/4, \\ &\Delta_0 = \sum \epsilon_{i_1 \cdots i_n}\Omega_{i_1 i_2} \cdots \Omega_{i_{n-1} i_n}, \quad \text{if } n \text{ is even}, \end{aligned} \tag{64}$$

where $\epsilon_{i_1 \cdots i_n}$ is the Kronecker index, which is $+1$ or -1, according as $i_1, \cdots, i_n$ forms an even or odd permutation of $1, \cdots, n$, and is otherwise zero.

Important is the case of an exact form of M which is not derived in M, but has an image in a fibre bundle which is derived. The form Δ_0 in (64) has, for instance, an image in $\mathfrak{F}^{(1)}$, which is derived. It is on this fact that the generalized Gauss-Bonnet formula was proved in a simple way [**10**]. A further example to illustrate this idea will be given for Riemannian manifolds of four dimensions. (Compare [**14**].)

8. Theorem of Pontrjagin.

From a topological viewpoint Pontrjagin has in two recent notes [**20, 21**] studied a problem related to ours. We shall summarize here his results in a slightly more general version which will allow its generalization to complex analytic Hermitian manifolds.

Let E^{n+N} be a real oriented Euclidean space of dimension $n+N$. In E^{n+N} we consider the Grassmann manifold $H(n, N)$, formed by all the oriented n-dimensional linear spaces through the origin. To each submanifold M' of $H(n, N)$ there is defined in a natural way a sphere bundle with M' as the base space, that is, a fibre bundle whose fibres consist of spheres cut on the unit hypersphere about the origin of E^{n+N} by the n-dimensional linear subspaces of M'. Given any manifold M, and a mapping $f(M) \subset H(n, N)$, a sphere bundle can be defined over M as base space by taking as the fibre attached to a point P of M the unit hypersphere in $f(P)$. With a proper definition of equivalence of sphere bundles, Whitney and Steenrod have proved the following theorem [**25, 33**]:

THEOREM 8.1. *To a given bundle of spheres of dimension* $n-1$ *over a compact manifold* M *there exists a mapping* $f(M) \subset H(n, N)$, *which defines a sphere bundle over* M *equivalent to the given one, provided that* $\dim M \leqq N$. *Two sphere bundles over* M *defined by the mappings* $f_i(M) \subset H(n, N)$, $i=1, 2$, *are equivalent, when and only when the mappings* f_1 *and* f_2 *are homotopic.*

From the mapping $f(M) \subset H(n, N)$ the cocycles of dimension not greater than $\dim M$ of $H(n, N)$ are mapped by the inverse mapping into cocycles of M. The latter part of Theorem 8.1 asserts that the image cocycles of M are cohomologous, if the sphere bundles are equivalent. The image of a cohomology class of $H(n, N)$ under the inverse mapping is therefore independent of the choice of the mapping f, provided that the sphere bundle induced remains equivalent. Such a cohomology class in M is called a generalized characteristic class.

If M is a differentiable manifold and $\mathfrak{F}$ the tangent sphere bundle over M, we can find the mapping f in question by imbedding M in a Euclidean space E^{n+N} and defining $f(P)$ to be the n-plane of $H(n, N)$ parallel to the tangent n-plane of M at P. M has then a Riemannian metric induced by the Euclidean metric of E^{n+N}.

In E^{n+N} consider the frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_{n+N}$. We use the ranges of indices

$$1 \leqq i, k \leqq n, \qquad n+1 \leqq r, s \leqq n+N, \qquad 1 \leqq A, B, C \leqq n+N,$$

and put

$$\omega_{AB} = d\mathfrak{e}_A \cdot \mathfrak{e}_B.$$

Then we have

$$d\omega_{AB} = \sum_C \omega_{AC}\omega_{CB}.$$

To each n-plane of $H(n, N)$ we attach the frames $O\mathfrak{e}_1 \cdots \mathfrak{e}_n\mathfrak{e}_{n+1} \cdots \mathfrak{e}_{n+N}$ such that $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ belong to the n-plane. Then the nN linear differential forms $\omega_{\alpha r}$ constitute a set of linearly independent forms of $H(n, N)$. To represent an integral cocycle of $H(n, N)$ in the sense of §7 we define

$$\Omega_{ik} = -\sum_r \omega_{ir}\omega_{kr}. \tag{65}$$

It is then a consequence of the first main theorem in the theory of vector invariants for orthogonal groups [31] that every integral cocycle of dimension not greater than n of $H(n, N)$ can be represented by a differential form which is a linear combination, with constant coefficients, of the products of the forms Δ_m, Δ_0 defined by (64), where the Ω_{ik} are here given by (65).

The forms Ω_{ik} defined in (65) are forms of $H(n, N)$. But their images in M induced by the mapping f are the curvature forms of M derived from the Riemannian metric of M induced by E^{n+N}. In fact, let us attach to each point P of M all frames $P\mathfrak{e}_1 \cdots \mathfrak{e}_{n+N}$ such that $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ are tangent vectors. On M we have then

$$\omega_r = 0,$$

and

$$\begin{aligned} d\omega_i &= \sum_k \omega_k\omega_{ki}, \\ d\omega_{ik} &= \sum_j \omega_{ij}\omega_{jk} - \sum_r \omega_{ir}\omega_{kr} = \sum_j \omega_{ij}\omega_{jk} + \Omega_{ik}. \end{aligned} \tag{66}$$

From (66) it is seen that the images of the Ω_{ik} in (65) are the curvature forms of the Riemannian metric induced on M.

In this way we can prove the following theorem of Pontrjagin:

THEOREM 8.2. *Let M be a compact orientable differentiable manifold imbedded in a Euclidean space and let Ω_{ik} be the curvature forms derived from the induced Riemannian metric. An integral characterisic cocycle of M can be represented by a differential form which is a linear combination, with constant coefficients, of the products of the forms Δ_m, $0 \leqq m \leqq n/4$.*

It is highly probable that the relations asserted in Theorem 8.2 remain true, if the Ω_{ik} are the curvature forms of a Riemannian metric given intrinsically on M. But so far this has only been established for the case of the form Δ_0 and for a few other cases. The result will follow if it can be proved that a Riemannian manifold can be isometrically imbedded in a Euclidean space. Unfortunately this imbedding question is not settled, even for $n=2$.

9. **Hermitian geometry.** Our results are more satisfactory in the case of Hermitian geometry. By an Hermitian manifold M we shall mean a complex analytic manifold in which there is given an Hermitian metric:

$$ds^2 = \sum_{i,k=1}^{n} g_{ik}(z, \bar{z})(dz_i d\bar{z}_k), \qquad \bar{g}_{ik} = g_{ki}, \tag{67}$$

where z_i are the local complex coordinates and where the bar denotes the operation of taking the conjugate complex.

As in the case of Riemannian manifolds, the principal fibre bundle for an Hermitian manifold is one whose fibres consist of a set of complex vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$, such that

$$\mathfrak{e}_i \cdot \bar{\mathfrak{e}}_j = \delta_{ij}, \tag{68}$$

where the scalar product is understood in the sense of Hermitian geometry. Like the Riemannian case, the fundamental formulas of Hermitian geometry can be established to be

$$ds^2 = \sum (\omega_i \bar{\omega}_i), \quad d\omega_i = \sum \omega_j \omega_{ji}, \quad d\omega_{ik} = \sum \omega_{ij}\omega_{jk} + \Omega_{ik}, \tag{69}$$

where

$$\omega_{ij} + \bar{\omega}_{ji} = 0, \quad \Omega_{ij} + \bar{\Omega}_{ji} = 0, \quad \Omega_{ij} = \sum_{k,l} R_{ij,kl}\omega_k \bar{\omega}_l. \tag{70}$$

From Ω_{ij} we construct the differential forms

$$(71) \qquad \Delta_m = \sum_{i_1,\cdots,i_m} \Omega_{i_1 i_2}\Omega_{i_2 i_3}\cdots\Omega_{i_m i_1}, \qquad 1 \leqq m \leqq n,$$

and also the differential forms

$$(72) \qquad \Psi_m = \sum \delta(i_1, \cdots, i_m; j_1, \cdots, j_m)\Omega_{i_1 j_1}\cdots\Omega_{i_m j_m}, \qquad 1 \leqq m \leqq n,$$

where $\delta(i_1, \cdots, i_m; j_1, \cdots, j_m)$ is the Kronecker symbol which is equal to $+1$ or -1 according as $j_1, \cdots, j_m$ constitutes, an even or odd permutation of $i_1, \cdots, i_m$, and is otherwise zero, and where the summation is extended to all $i_1, \cdots, i_m$ from 1 to n. It is easy to see that every Ψ_m can be expressed as a polynomial of $\Delta_1, \cdots, \Delta_m$ with constant coefficients, and conversely.

Over the compact Hermitian manifold M we can construct the fibre bundle $\mathfrak{F}^{(p)}$ such that the fibre over a point P of M consists of ordered sets of vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_p$, satisfying the relations

$$(73) \qquad \mathfrak{e}_i - \bar{\mathfrak{e}}_j = \delta_{ij}, \qquad 1 \leqq i, j \leqq p.$$

Then we can prove the following theorem:

THEOREM 9.1. *The characteristic cocycle of $\mathfrak{F}^{(p)}$ in M in the sense of* §4 *is of dimension $2n-2p+2$ and can be defined, up to a constant factor, by the form Ψ_{n-p+1}.*

These characteristic cocycles, altogether n of them, corresponding to the values $p=1, \cdots, n$, we shall call the basic characteristic cocycles. On the other hand, the imbedding process of §8 can be carried over to the present case, resulting in a correspondence between the equivalent fibre bundles over M (in the sense of Steenrod) and the homotopy classes of mappings of M into $H(n, N, C)$, where $H(n, N, C)$ is the Grassmann manifold of all n-dimensional complex linear spaces through the origin in a complex Euclidean space of $n+N$ (complex) dimensions. This allows us to define a generalized characteristic cohomology class of M as the inverse image of a cohomology class of dimension not greater than $2n$ of $H(n, N, C)$. But in the complex case the first main theorem on vector invariants for the unitary group is of simpler form than the corresponding theorem for the orthogonal group, and we can prove the following theorem:

THEOREM 9.2. *Every cohomology class of $H(n, N, C)$ of dimension not greater than $2n$ can be obtained by operations of the cohomology ring from n basic classes, namely, the classes whose inverse images are the basic characteristic classes of M.*

It is in this sense that we can say that for complex analytic Hermitian manifolds the characteristic classes arising from the consideration of the principal fibre bundle are completely determined by the Hermitian metric. We also remark that complex analytic Hermitian manifolds play an important rôle in algebraic geometry and in the theory of analytic functions of several complex variables.

The proofs of the results announced in this section will be published elsewhere.

10. The kinematic principal formula in integral geometry. In the above sections we have discussed the interrelations between local differential-geometric properties and topology. There are, however, other geometric properties in the large which are equally of interest and which are closely related to the ideas set forth above. Moreover, they also illustrate the important rôle played by the fibre bundle in the theory of a geometric being.

Let M be a Riemannian manifold of n dimensions, and let $\mathfrak{F}$ be its principal fibre bundle. We have defined in the space $\mathfrak{F}$ of dimension $n(n+1)/2$ the forms ω_i, ω_{ik}. The differential form of degree $n(n+1)/2$:

$$\mathfrak{P} = \prod \omega_i \cdot \prod_{i<k} \omega_{ik} \tag{74}$$

can serve as a kind of volume element in $\mathfrak{F}$. If M is the Euclidean space of n dimensions, $\mathfrak{P}$ is called the kinematic density, which was first introduced by Poincaré for the case $n=2$.

To illustrate how the kinematic density can be utilized in geometrical problems suppose M to be the Euclidean space of n dimensions. Then $\mathfrak{F}$ is the space of frames in M and is topologically a product. In M we consider two closed hypersurfaces S_1, S_2 of class not less than 2. For each S_i let $r_1^{(i)}, \cdots, r_{n-1}^{(i)}$ be the principal curvatures and let

$$\frac{1}{C_{n-1,k}} \int_{S_i} \sum r_1^{(i)} \cdots r_k^{(i)} dO_i = H_k^{(i)}, \quad i = 1, 2; \; k = 0, \cdots, n-1, \tag{75}$$

where the integrand is the kth elementary symmetric function of the principal curvatures and dO_i is the surface element of S_i. Denote by $V^{(i)}$ the volume bounded by S_i in M. Let S_1 be fixed in the space and let S_2 take all possible positions. For each position of S_2 denote by $\chi(S_1 S_2)$ the Euler-Poincaré characteristic of the intersection of S_1, S_2. Then we have the following so-called kinematic principal formula:

$$\begin{aligned} \int_{S_2} \chi(S_1 \cdot S_2) \mathfrak{P} = J_n \Big\{ & H_{n-1}^{(1)} V^{(2)} + V^{(1)} H_{n-1}^{(2)} \\ & + \frac{1}{n} \sum_{k=0}^{n-2} C_{n,k+1} H_k^{(1)} H_{n-2-k}^{(2)} \Big\}, \end{aligned} \tag{76}$$

where J_n is the kinematic measure about a point in M and is given by

$$J_n = O_{n-1}O_{n-2} \cdots O_1, \tag{76a}$$

O_{p-1} being the area of the unit sphere in a Euclidean space of p dimensions.

The formula (76) was proved by Blaschke for $n=2, 3$ [2], and was established for general n by Yien and the present writer [13]. It has numerous interesting consequences. In particular, when $n=2$ and when both S_1 and S_2 are convex curves, it becomes

$$\int_{S_1 \cdot S_2 \neq 0} \mathfrak{P} = 2\pi(A_1 + A_2) + L_1L_2, \tag{77}$$

where A_i is the area bounded by S_i and L_i is the length of S_i, $i=1, 2$. Formula (77) was first established by Santalò and can be used to derive a sharpening of the isoperimetric inequality (2) [23]. It is thus through all the round-about discussions on differential geometry and topology that we arrived at the relation between the theory of fibre bundles and the isoperimetric inequality given at the beginning of this article.

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INSTITUTE FOR ADVANCED STUDY AND
TSING HUA UNIVERSITY

MEMORANDUM CONCERNING THE DEFERMENT OF GRADUATE STUDENTS, TEACHERS, AND RESEARCH WORKERS IN PHYSICAL SCIENCE

On November 26 two important changes in Selective Service regulations were announced. These fundamentally affect the situation with respect to the deferment of graduate students and teachers of physical science and engineering.

Local Board Memorandum No. 115 is amended so as to read: "A registrant age 18 through 25 may be retained or placed in Class II-A if the local board finds that he is *necessary to and regularly engaged in* an activity in the support of the national health, safety or interest." This is a liberalization of the previous Memorandum 115 for the requirement that the registrant be "indispensable and irreplaceable" is dropped. Form 42A (Special-Revised) will continue to be used for the making of requests for the occupational deferment of registrants ages 18 through 25, except those found to be disqualified for any military service or found to be qualified for limited military service only. For this latter group, Form 42 should be used.

Memorandum 115-M, which is reproduced in full and attached to this communication, concerns special consideration for certain students, teachers and research workers in the physical sciences. This memorandum represents a recognition by Selective Service of: (1) the shortage of scientific personnel, due to the curtailment of advanced study in science during the war; (2) the necessity for the resumption of study in these fields.

Selective Service officials believe that these two memoranda, 115 and 115-M, adequately provide for the deferment of scientific and technical personnel. Cases now active where registrants seem to have been improperly classified should be reopened at once in the light of the new releases. In view of the change in policy, it is advisable that the department chairman request re-consideration of such cases.

T. H. HILDEBRANDT,
President
J. R. KLINE,
Secretary

December 10, 1945.

LOCAL BOARD MEMORANDUM NO. 115-M. ISSUED: 11/26/45

SUBJECT: SPECIAL CONSIDERATION FOR CERTAIN STUDENTS, TEACHERS, AND RESEARCH WORKERS IN THE PHYSICAL SCIENCES

PART I—STATEMENT OF GENERAL POLICIES AND PROCEDURES

1. *General.*—The demands of the armed forces and industry during the emergency have resulted in a curtailment in advanced studies for men having high technical and scientific qualifications. Since the cessation of active fighting, the demands of long range national interest require a resumption of these studies in order to fully develop the technical and scientific skills which have been acquired and to provide adequate teaching facilities for returning veterans who desire to resume their studies in these fields.

2. *Reconversion working committee on deferment and selective release.*—(a) Pursuant to a request from the Director of War Mobilization and Reconversion, the Reconversion Working Committee on Deferment and Selective Release has been established to assist in the accomplishment of this purpose. The Committee is composed of representatives of the (1) Office of Scientific Research and Development; (2) Civilian Production Administration; (3) Office of Rubber Reserve; (4) Petroleum Administration for War; (5) War Department; (6) Navy Department; (7) United States Employment Service; (8) Office of Education; (9) National Roster of Scientific and Specialized Personnel; and the (10) Selective Service System. The Chairman of the Committee will be the Director of War Mobilization and Reconversion, or a staff member designated by him. The functions of the Committee are to:

(1) Indicate to the Director of War Mobilization and Reconversion the specific occupations in which shortages of personnel threaten to interfere with the national health, safety, or interest.

(2) Formulate the specific standards indicating that a man is qualified to engage in a selected occupation.

(3) Certify to the Director of the Selective Service System those individuals meeting the standards established by the Committee.

(4) Indicate to the War and Navy Departments the categories of occupations in which shortages detrimental to the national interest could be relieved by release of men from the armed forces.

PART II—CERTIFICATION PLAN

1. *Operation of plan.*—The Director of the Office of War Mobilization and Reconversion will examine all proposals by the Committee relating to deferment and release, and shall transmit to the Selective Service System and the War and Navy Departments all approved proposals. The Director will transmit the recommendations for deferment to local boards through the appropriate State Directors.

2. *Classification policies.*—Pursuant to the provisions of this memorandum and under the general authority contained in paragraph 5 of Part II of Local Board Memorandum No. 115, as amended, local boards will give serious consideration to the occupational deferment of registrants engaged in the technical and scientific fields set forth in Part III of this memorandum.

PART III—STANDARDS AND PROCEDURES

1. *Advanced studies in the physical sciences or engineering.*—(a) Any registrant who is accepted by an accredited college or university as a candidate for a Master's or Doctor's degree in the physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity.

(b) The fact that a candidate for a Master's or a Doctor's degree may engage in part-time employment or other activities will not affect his certification under this paragraph so long as his academic standing is satisfactory.

2. *University teaching in the physical sciences or engineering.*—Any registrant who is to be employed by an accredited college or university as a teacher of physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity.

3. *University research in the physical sciences or engineering.*—(a) Any registrant (1) who is to be employed by or attached to the staff of an accredited college or university for research in the physical sciences or engineering and (2) who signifies his intention to engage in such an activity may be certified to the Director of Selective Service as essential to the national interest in a civilian capacity.

(b) Such a registrant will be certified only if (1) the research to be undertaken by the registrant contributes significantly to the national interest, and (2) inability of the individual registrant to undertake the research will result in its delay.

4. *Submission of information in certain cases.*—Any registrant who wishes to be certified under the provisions of paragraphs 1, 2, and 3 above, must present to the Office of War Mobilization and Reconversion, Washington, D. C., the following documents in triplicate:

(a) A notarized statement of his intention:

(1) To undertake graduate studies leading to a Master's or Doctor's degree in the physical sciences or engineering; or

(2) To engage in the teaching of physical sciences or engineering; or

(3) To undertake advanced research in the physical sciences or en-

gineering on the staff of or attached to a college or university.

(b) A statement from an accredited college or university signed by a responsible official of the college or university, indicating that the registrant:

(1) Has been accepted as a candidate for a Master's or Doctor's degree in the physical sciences or engineering; or

(2) Has been accepted as a teacher of physical sciences or engineering; or

(3) Is to be employed by or attached to the staff of the college or university for research in the physical sciences or engineering, together with a statement indicating in detail the nature of the research to be performed by the registrant, the scope of the registrant's responsibilities for the research, and the necessity for securing the individual registrant for the work.

(c) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

5. *Study leading to a B.S. or B.A. degree in physical sciences or engineering.*—(a) Any registrant who has satisfactorily completed at least three years of work leading to a Bachelor's degree in the physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity, provided such registrant has served for a period of not less than two years in a project directly connected with the war effort.

(b) Any registrant who wishes to be certified under the provisions of this paragraph must present to the Office of War Mobilization and Reconversion, Washington, D. C., the following documents in triplicate:

(1) A notarized statement of his intention to continue undergraduate studies leading to a B.S. or B.A. degree in the physical sciences or engineering.

(2) A statement from an accredited college or university, signed by a responsible official of the college or university, indicating that the registrant has been accepted for the fourth year of study in a course leading to a B.S. or B.A. degree in the physical sciences or engineering.

(3) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

(4) A statement from the registrant's employer that he has been engaged for at least two years in scientific war work.

LEWIS B. HERSHEY, *Director*

THE ANNUAL MEETING OF THE SOCIETY

The fifty-second Annual Meeting and twenty-seventh Colloquium of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, Friday and Saturday, November 23 and 24, 1945, in conjunction with meetings of the Mathematical Association of America. Over three hundred persons registered, including the following two hundred forty members of the Society:

A. A. Albert, G. E. Albert, N. R. Amundson, T. W. Anderson, K. J. Arnold, Max Astrachan, M. C. Ayer, W. L. Ayres, R. W. Babcock, Reinhold Baer, R. M. Ballard, R. H. Bardell, R. W. Barnard, J. L. Barnes, Walter Bartky, G. E. Bates, J. C. Bell, J. L. Bell, R. F. K. Benton, Lipman Bers, S. F. Bibb, R. H. Bing, K. E. Bisshopp, G. A. Bliss, L. M. Blumenthal, H. F. Bohnenblust, W. M. Borgman, D. G. Bourgin, G. F. Bradfield, C. P. Brady, H. R. Brahana, C. C. Bramble, A. J. Brandt, Richard Brauer, R. H. Bruck, E. L. Buell, R. S. Burington, H. E. Burns, I. W. Burr, L. E. Bush, S. S. Cairns, W. D. Cairns, R. H. Cameron, H. H. Campaigne, W. B. Carver, Subrahmanyan Chandrasekhar, W. F. Cheney, E. W. Chittenden, A. M. Christiansen, R. V. Churchill, Paul Civin, H. M. Clark, E. H. Clarke, C. J. Coe, K. S. Cole, Nancy Cole, B. H. Colvin, J. J. Corliss, Richard Courant, N. A. Court, H. S. M. Coxeter, M. D. Donsker, J. L. Doob, T. L. Downs, W. L. Duren, William H. Durfee, John Dyer-Bennet, Paul Erdös, D. H. Erkiletian, Martinus Esser, H. J. Ettlinger, H. P. Evans, H. S. Everett, I. A. Flodin, L. R. Ford, Tomlinson Fort, J. S. Frame, Evelyn Frank, M. R. Freundlich, A. J. Froseth, B. H. Gere, H. H. Germond, J. W. Givens, A. M. Gleason, L. M. Graves, L. W. Griffiths, V. G. Hachmeister, D. W. Hall, P. R. Halmos, O. H. Hamilton, M. L. Hartung, A. E. Heins, E. D. Hellinger, R. G. Helsel, Fritz Herzog, M. R. Hestenes, E. H. C. Hildebrandt, T. H. Hildebrandt, J. D. Hill, D. L. Holl, M. C. Holland, Carl Holtom, G. B. Huff, R. C. Huffer, S. P. Hughart, H. K. Hughes, Ralph Hull, H. D. Huskey, M. H. Ingraham, R. L. Jeffery, A. W. Jones, B. W. Jones, F. B. Jones, Irving Kaplansky, William Karush, H. P. Kean, D. E. Kearney, A. J. Kempner, R. B. Kershner, Fred Kiokemeister, J. R. Kline, W. C. Krathwohl, Joseph Landin, E. P. Lane, R. E. Langer, G. A. Larew, H. D. Larsen, J. W. Lasley, B. A. Lengyel, Charles Loewner, Z. L. Loflin, A. T. Lonseth, W. C. McDaniel, E. J. McShane, C. C. MacDuffee, Murray Mannos, Morris Marden, J. W. Marshall, Ella Marth, W. T. Martin, J. R. Mayor, Karl Menger, E. J. Mickle, A. N. Milgram, D. D. Miller, H. J. Miser, W. L. Miser, E. J. Moulton, M. E. Munroe, J. R. Musselman, S. B. Myers, J. L. Nagle, C. V. Newsom, K. L. Nielsen, Ivan Niven, E. P. Northrop, F. S. Nowlan, J. M. H. Olmsted, Isaac Opatowski, Philip Peak, P. M. Pepper, L. C. Plant, J. E. Powell, G. B. Price, A. L. Putnam, Tibor Radó, J. F. Randolph, R. B. Rasmusen, C. H. Rawlins, Mina Rees, W. T. Reid, Haim Reingold, E. K. Ritter, G. de B. Robinson, W. H. Roever, R. E. Root, G. F. Rose, A. E. Ross, J. B. Rosser, Hans Samelson, R. G. Sanger, J. B. Scarborough, A. T. Schafer, R. D. Schafer, O. F. G. Schilling, J. E. Schubert, G. E. Schweigert, M. E. Shanks, Seymour Sherman, D. M. Smiley, M. F. Smiley, C. V. L. Smith, G. W. Smith, H. L. Smith, R. D. Specht, Abraham Spitzbart, C. E. Springer, M. P. Steele, N. E. Steenrod, R. C. Stephens, B. M. Stewart, C. N. Stokes, M. H. Stone, R. G. Stoneham, E. B. Stouffer, W. T. Stratton,

J. S. Stubbe, R. L. Swain, J. L. Synge, H. P. Thielman, J. M. Thomas, W. R. Transue, E. F. Trombley, H. F. Tuan, A. W. Tucker, H. L. Turrittin, S. M. Ulam, E. H. Umberger, E. P. Vance, G. B. Van Schaack, J. I. Vass, M. F. Vaudreuil, H. E. Vaughan, S. E. Warschawski, C. P. Wells, E. T. Welmers, M. E. Wescott, P. M. Whitman, W. M. Whyburn, L. R. Wilcox, R. L. Wilder, F. B. Wiley, R. B. Wiley, S. S. Wilks, R. S. Wolfe, M. A. Wurster, G. S. Young, J. W. T. Youngs, Daniel Zelinsky, J. W. Zimmer.

The twenty-seventh Colloquium consisted of four lectures on *Length and area* by Professor Tibor Radó of Ohio State University. The lectures took place at 9:30 A.M. and 2:00 P.M. on Friday and on Saturday, and were presided over by Professors J. M. Thomas, C. C. MacDuffee, President T. H. Hildebrandt and Secretary J. R. Kline.

Sessions for the reading of short papers were held Friday at 10:45 A.M. and 3:15 P.M. and Saturday at 3:15 P.M. Presiding officers for these sessions were Professors A. A. Albert, L. M. Graves and W. L. Ayres.

The nineteenth Josiah Willard Gibbs Lecture was delivered by Professor J. C. Slater of the Massachusetts Institute of Technology on Friday evening at 7:30 P.M., with President T. H. Hildebrandt presiding. The title of Professor Slater's address was *Physics and the wave equation.* About three hundred people attended.

The annual business meeting and election of officers was held on Saturday morning, President T. H. Hildebrandt presiding. Dean Tomlinson Fort of Lehigh University presented resolutions expressing the thanks and appreciation of the members of the Society and the Association to the officials of the Museum of Science and Industry and to the local committee for the excellent arrangements. Further proceedings are included later in the report.

At 11:00 A.M. Saturday, following Professor Radó's third lecture, Professor S. M. Ulam of the University of Southern California presented an address entitled *On the stability of functional equations.* Professor R. E. Langer presided.

Sessions of the Mathematical Association of America were held on Saturday evening and Sunday morning.

At the meeting of the Board of Trustees at 6:00 P.M., November 23, 1945, in the Hotel Windermere East, there was no quorum present and the Board adjourned to January 4, 1946. The Council met at 9:00 P.M. on November 23, 1945, in the Hotel Windermere East.

The Secretary announced the election of the following seventeen persons to ordinary membership in the Society:

Mr. Arthur Bernstein, Carnegie Institution of Washington;
Miss Nancy Brixey, Davis and Gilbert, New York, N. Y.;
Mr. Willie Russell Callahan, Department of Electrical Engineering, Yale University;
Professor Clarence Selmer Carlson, St. Olaf College;
Mr. Kenneth Collwell Cartwright, P.P.G. Co., Barberton, Ohio;
Mr. Peter Jennings Gill, Magnetic Observatory, Christchurch, New Zealand;
Mr. Frank Harary, New York University;
Mr. J. Halcombe Laning, Jr., Watertown Arsenal, Watertown, Mass.;
Professor Harvey M. Lashier, Pacific Union College, Angwin, Calif.;
Mr. John H. Lewis, N.D.R.C., New York, N. Y.;
Mr. Jesse Eugene Sherwood, N.D.R.C., Cumberland, Md.;
Professor Yu-Why Tschen, National University of Peking, Peking, China;
Mr. Edmund Forrest Tyler, Los Angeles, Calif.;
Mr. Stephen Eugene Walkley, U.S.N.R.;
Dr. Everett C. Westerfield, Costa Mesa, Calif.;
Professor L. C. Young, University of South Africa, Capetown;
Mr. John Anthony Zulon, U.S.A.

The following appointments by President T. H. Hildebrandt were reported: as representative of the Society at the Seventy-fifth Anniversary of Wilson College on October 13, 1945, Professor F. W. Owens; as representative of the Society at the inauguration of William Allison Shimer as President of Marietta College on October 20, 1945, Professor J. L. Synge; as representative of the Society at the inauguration of George Henry Armacost as President of the University of Redlands on November 4, 1945, Professor Morgan Ward; as representative of the Society at the inauguration of Howard S. McDonald as President of Brigham Young University on November 14, 1945, Professor M. T. Bird; as representative of the Society at the inauguration of Franklin Stewart Harris as President of Utah State Agricultural College on November 16, 1945, Professor I. O. Horsfall; as a Committee on Arrangements for the 1945 Annual Meeting in Chicago, Illinois, Dean R. G. Sanger (Chairman), Professors R. H. Bruck, B. W. Jones, W. T. Reid, and L. R. Wilcox; as tellers for the election at the 1945 Annual Meeting, Professor W. C. Krathwohl, Drs. Mary E. Ladue and W. C. Strodt; as a committee to study the implementation of the Birkhoff Memorial Committee report, Professors M. H. Stone (Chairman), R. M. Foster, and R. E. Langer; as a Committee on the Award of the Cole Prize in Theory of Numbers, to be awarded at the 1946 Annual Meeting for papers published during the period 1941–1945, Professors E. T. Bell (Chairman), Emil Artin, and H. W. Brinkmann; as Chairman of the Committee on Visiting Lectureships, term to expire December 31, 1947, Professor Hassler Whitney; as a member of the Committee on Visiting Lectureships, Professor Norbert Wiener. (Professor

Wiener was reappointed for a period of three years, 1946–48; committee now consists of Professor Hassler Whitney, Chairman, Dean R. G. D. Richardson, and Professor Wiener.)

At the annual election which closed on November 24, and at which 544 votes were cast, the following officers were elected:

Vice Presidents, Professors L. R. Ford and Saunders MacLane.

Associate Secretaries, Professors W. L. Ayres, R. H. Bruck and A. C. Schaeffer.

Member of the Editorial Committee of the Bulletin, Professor Deane Montgomery.

Member of the Editorial Committee of the Transactions, Professor A. A. Albert.

Member of the Editorial Committee of the Colloquium Publications, Professor J. F. Ritt.

Member of the Editorial Committee of Mathematical Reviews, Professor O. E. Neugebauer.

Member of the Editorial Committee of Mathematical Surveys, Professor A. W. Tucker.

Member of the Editorial Committee of the American Journal of Mathematics, Professor L. M. Graves.

Members at large of the Council, Dr. R. P. Boas, Professors R. H. Cameron, R. V. Churchill, Churchill Eisenhart, and A. P. Morse.

Professor M. H. Stone was elected by the Council as a member of the Editorial Committee of Mathematical Reviews to fill the unexpired term of Professor J. D. Tamarkin.

It was reported that Professor Marston Morse had been selected to act as Chairman of the Emergency Committee of the International Congress of Mathematicians.

It was reported that Professor Subrahmanyan Chandrasekhar had accepted the invitation to deliver the twentieth Josiah Willard Gibbs Lecture at the 1946 Annual Meeting and that Professor Oscar Zariski had accepted the invitation to deliver a series of Colloquium Lectures at the 1947 Summer Meeting on the subject *Abstract algebraic geometry.*

The Secretary reported that he had submitted to the Rockefeller Foundation a report on the activities of the War Policy Committee for the period August 1, 1944–September 30, 1945.

Certain invitations to give addresses were announced: Professor A. D. Wallace for the February, 1946, meeting in New York City; Professor O. F. G. Schilling for the April, 1946, meeting in Chicago; Dr. P. F. Reichelderfer for the November, 1946, meeting in the midwest; Professor N. E. Steenrod for the April, 1947, meeting in the midwest.

The Secretary reported the completion of the biennial membership campaign held during the latter part of October when approximately 3,200 college teachers of mathematics, including 950 members of the Mathematical Association of America, were invited to membership in the Society. The Secretary is pleased to report at this time that the ordinary membership in the Society is now 2828, including 210 nominees of institutional members and 63 life members. There are also 87 institutional members. The total attendance at all meetings in 1945 was 628; the number of papers read was 196; there were 5 hour addresses, 5 symposium addresses, 1 Gibbs Lecture, and four Colloquium Lectures; the number of members attending at least one meeting was 570.

The Council adopted the following resolution on the death of Professor J. D. Tamarkin:

> On November 18, 1945, this Society lost one of its most active and beloved members through the untimely death of Jacob David Tamarkin in his fifty-eighth year of life. His health had been failing for some time, but we had hoped that he would recover his strength and usefulness which was more than life itself to him.
>
> J. D. was the friend of us all and of mathematicians all over the world for whom he kept open house. He was intensely human, generous, gregarious, and warm-hearted; he loved music, a good table and a friendly chat. He took an active interest in his many friends, their personal problems, and their scientific work and gave his time and counsel without stint. His hearty laughter and booming voice brought cheer wherever he went.
>
> He came as a stranger to our shores two decades ago but soon found an active place in American mathematical life to the development of which he was to contribute so much. He read widely and with passion and many of us have profited from his constructive criticism and advice. His critical ability was soon discovered and utilized: he served on the Editorial Committees of the Transactions, the Colloquium, the Mathematical Reviews, and the Mathematical Surveys. He was a member of the Council since 1931 and a Vice President in 1942–1943.
>
> His intellectual honesty, high standards, and civil courage made themselves felt everywhere. He helped materially in placing Brown University in the mathematical front rank and his editorial activities during a critical time were a driving force in the growth and development of mathematical research in this country. He saw the problems of the Society clearly and in their proper perspective and fought fearlessly for his convictions. His vigorous personality never became overbearing, he fought well but also cleanly. To his many collaborators and students he was a source of ideas, information, and inspiration, a friendly guide and critic, and his influence on their mathematical development was profound. His own contributions to classical as well as modern functional analysis were many and of lasting value.
>
> The Council of the American Mathematical Society voices its deep sense of loss in the decease of J. D. Tamarkin. He lives in his work and in our fond memories.

This resolution was read at the annual business meeting and the members of the Society stood for a minute in respect to the memory of their distinguished colleague.

On recommendation of the Committee on Places of Meetings, the Council voted to accept the invitation of Cornell University to hold the 1946 Summer Meeting at that institution at a date to be decided.

Since the 1945 Annual Meeting was held before the close of the Society's fiscal year, it was impossible to present a full report of the Treasurer. (Excerpts of the final report of the Treasurer, however, appear on pages 44–47 of this Bulletin.) It was reported that the Society's finances were in excellent shape, that the net excess of receipts over disbursements for the year would be somewhat more than $10,000. The Society's investment portfolio now includes government bonds 27 per cent, other bonds 18 per cent, preferred stocks 13 per cent, common stock 35 per cent, cash in banks 7 per cent, and total assets amount to approximately $270,000. During the year the Trustees directed the transfer to the Endowment Fund Principal from surplus of a sufficient amount to bring the principal to $71,000, slightly more than its value before the charges for depreciation of securities were made in 1940. While there is a considerable surplus, this is desirable because we shall soon be faced with a large volume of publication at considerably increased costs.

The Librarian reported the following additions to the Library: 29 books, 89 bound volumes of periodicals, 162 pamphlets (most of them are printed dissertations), and 229 dissertations on microfilm. He also reported that the use of the library had been twice as great during 1944–1945 as in 1943–1944. A great deal of work has been done during the past year by the staff of the New York office and by a special assistant on the revision of the Catalogue of the Library. A new edition of this Catalogue will be published and distributed to members of the Society early in 1946. The Librarian is establishing archives of the Society for the preservation of letters and documents of importance. Members are invited to send to the Librarian documents in their possession which may be placed in the archives.

The Bulletin Editorial Committee reported that it would publish slightly under 1,000 pages in 1945. It was reported that Professors E. G. Begle and Otto Szász had been appointed as Assistant Editors, to replace Professors J. H. Curtiss and D. H. Lehmer.

The Transactions Editorial Committee reported that it would publish 965 pages during 1945. During the past year Professor Oscar Zariski has been on leave of absence from his university for scientific work in Brazil. His duties in connection with the Transactions have been carried on by Professor R. L. Wilder. Professor Oystein Ore will retire as an Associate Editor and will be replaced by Professor O. F. G. Schilling. The backlog of papers is about the same as last year but there is some expectation that a large number of new manuscripts will be submitted in the near future.

The Mathematical Reviews Editorial Committee reported that during the latter part of 1945 the size of the issue was increased from 28 to the original 32 pages and it is expected that this increase in size will be the minimum increase in 1946. The gaps in the foreign literature are being filled although there are still quite a few remaining, particularly in the Italian literature. In 1945 Mathematical Reviews obtained the sponsorship of the Institute of Mathematical Statistics. The subscription list contains 1,382 names. This is a remarkable record when one considers the fact that most foreign mathematicians have been unable to subscribe during this period.

The Council appointed Professors K. O. Friedrichs, Norman Levinson, and R. L. Wilder as representatives of the Society on the Editorial Board of the Annals of Mathematics for a period of three years beginning 1946

The Council voted to continue for another year the plan of reduced dues to members of the Society who are serving as enlisted men in the armed forces of the United States and Canada. Application for this privilege must be made in accordance with instructions which are mailed to members with annual bills on January 1.

The Council approved a plan, submitted by President Hildebrandt and Secretary Kline, for establishing a Policy Committee for Mathematics, and adopted procedures for the election of the representatives of the American Mathematical Society on this committee.

Titles and cross references to papers read at the meeting follow below. Papers 1 to 5 and paper 34 were read Friday morning. Papers 6 to 12 were read Friday afternoon. Papers 13, 14, 16, 17, 19, 43 and 44 were read Saturday afternoon. Except as noted above, papers 15 through 45 were read by title; their abstract numbers are followed by the letter *t*. Paper 7 was read by Professor Cameron, 8 by Dr. Kaplansky, 11 by Professor Bers, 17 by Professor Samelson, 43 by Professor Heins and 44 by Professor Herzog. Professor Schiffer was introduced by Dr. Stefan Bergman.

1. B. W. Jones: *Equivalence of quadratic forms over the ring of 2-adic integers.* (Abstract 52-1-3.)

2. H. S. M. Coxeter: *The Petrie polygon of a regular solid.* (Abstract 51-11-235.)

3. Richard Brauer: *On the arithmetic in a group ring.* II. Preliminary report. (Abstract 52-1-1.)

4. Ivan Niven: *Sums of squares of integral quaternions.* (Abstract 51-1-5.)

5. G. B. Huff: *Inequalities connecting solutions of Cremona's equations.* (Abstract 52-1-34.)

6. S. E. Warschawski: *On the modulus of continuity of the mapping function at the boundary in conformal mapping.* (Abstract 52-1-25.)

7. R. H. Cameron and W. T. Martin: *The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals.* (Abstract 52-1-15.)

8. Paul Erdös and Irving Kaplansky: *The asymptotic number of Latin rectangles.* (Abstract 52-1-2.)

9. Paul Civin: *Fourier coefficients of dominant functions.* (Abstract 51-11-212.)

10. R. H. Bing: *Solution of a problem of J. R. Kline.* (Abstract 52-1-46.)

11. Lipman Bers and Abe Gelbart: *A topological property of solutions of partial differential equations.* (Abstract 52-1-12.)

12. F. B. Jones: *Concerning the separability of certain locally connected metric spaces.* (Abstract 52-1-48.)

13. Isaac Opatowski: *Markoff chains and Tchebychev polynomials.* (Abstract 52-1-44.)

14. P. R. Halmos: *The theory of unbiased estimation.* (Abstract 52-1-43.)

15. Lipman Bers and Abe Gelbart: *A remark on the Lebesgue-Sperner covering theorem.* (Abstract 52-1-45-*t*.)

16. S. B. Myers: *Equicontinuous sets of mappings.* (Abstract 52-1-50.)

17. Deane Montgomery and Hans Samelson: *Fiberings with singularities.* (Abstract 51-11-251.)

18. J. L. Kelley and Everett Pitcher: *Applications of natural homomorphism sequences.* (Abstract 52-1-49-*t*.)

19. Charles Loewner: *On pairs of quadratic forms in Hilbert space.* (Abstract 52-1-19.)

20. Richard Brauer: *On the representation of a group of order g in the field of the gth roots of unity.* (Abstract 51-11-201-*t*.)

21. S. A. Kiss: *Transformations on lattices and structures of logic.* (Abstract 52-1-4-*t*.)

22. A. R. Schweitzer: *On the genesis of number systems.* I. (Abstract 52-1-39-*t*.)

23. A. R. Schweitzer: *On the genesis of number systems.* II. (Abstract 52-1-40-*t*.)

24. D. G. Bourgin: *Quadratic forms.* (Abstract 51-11-246-*t*.)

25. R. P. Agnew: *A simple sufficient condition that a method of summability be stronger than convergence.* (Abstract 51-11-208-*t*.)

26. J. W. T. Youngs: *Various definitions of surface and area.* Preliminary report. (Abstract 52-1-26-*t*.)

27. Mark Kac: *Distribution of eigenvalues of certain integral equations with an application to roots of Bessel functions.* Preliminary report. (Abstract 52-1-18-*t*.)

28. C. F. Stephens: *Solutions of systems of nonlinear difference equations in the neighborhood of a singular point.* (Abstract 52-1-24-*t*.)

29. D. G. Bourgin: *Complete sets of functions.* (Abstract 52-1-14-*t*.)

30. N. R. Amundson: *On the boundary value problem of third kind for the quasi-linear parabolic differential equation.* (Abstract 52-1-9-*t*.)

31. D. G. Bourgin: *A class of generating functions.* (Abstract 52-1-13-*t*.)

32. H. S. M. Coxeter: *The order of the symmetry group of the general regular hyper-solid.* (Abstract 51-11-234-*t*.)

33. L. M. Blumenthal: *Metric characterization of elliptic space.* (Abstract 52-1-31-*t*.)

34. C. E. Springer: *Rectilinear congruences whose developables intersect a surface in its lines of curvature.* (Abstract 52-1-36.)

35. Edward Kasner and John DeCicco: *Heat surfaces.* (Abstract 52-1-27-*t*.)

36. John DeCicco: *Differential geometry in the Kasner plane.* (Abstract 52-1-33-*t*.)

37. J. L. Doob: *Markoff chains—denumerable case.* (Abstract 51-11-242-*t*.)

38. Isaac Opatowski: *Calculation of Markoff chains by incomplete gamma and beta functions and by Charlier polynomials.* (Abstract 51-11-244-*t*.)

39. Will Feller: *On the normal approximation to the binomial.* (Abstract 52-1-42-*t*.)

40. Will Feller: *Note on the law of large numbers and "fair" games.* (Abstract 52-1-41-*t*.)

41. Menahem Schiffer: *Hadamard's formula and variation of domain-functions.* (Abstract 52-1-23-*t*.)

42. S. C. Chang: *A new foundation of the projective differential theory of curves in five-dimensional space.* (Abstract 52-1-32-*t*.)

43. A. E. Heins and Norbert Wiener: *A generalization of the Wiener-Hopf integral equation.* (Abstract 52-1-17.)

44. Fritz Herzog and J. D. Hill: *The Bernstein polynomials for discontinuous functions.* (Abstract 51-7-115.)

45. Gordon Pall: *On generalized quaternions.* (Abstract 52-1-6-*t*.)

R. H. BRUCK, *Associate Secretary*

APPENDIX

Excerpts from Report of Treasurer

December 18, 1945

To the Board of Trustees of the
American Mathematical Society

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1945. The following comments may be of interest.

Investment Portfolio

On November 30, 1945, the market value of securities held for Invested Funds exceeded book value by $26,930, and the market value of securities held for Current Funds exceeded book value by $1,863, thus reflecting the recent rise in security prices. In view of these figures and of the conservative character of many of the securities, the reserves held in accounts "Reserve for Investment Losses" and "Profit on Sales of Securities" are adequate protection against contingent depreciation in market value.

The size of the investment portfolio suggests that it may be helpful to include here a summary of the changes in security holdings made on authorization of the Board of Trustees during the year. I need hardly remind you in this connection of the value to the Society of the services of the City Bank Farmers Trust Company as Custodian.

Acquired by Purchase:

50 shares	Crane Co. Cum. Pfd. $3\frac{3}{4}$
25 shares	Union Carbide and Carbon Cap.
200 shares	National Dairy Products Com.
$5,000	New York, Chicago and St. Louis RR. Ref. $3\frac{3}{4}$ 1975
$5,000	U. S. Treasury $2\frac{1}{2}$ 1972–67
$5,200	U. S. Savings Ser. G Ref. $2\frac{1}{2}$ 1957

Acquired as Gift:

$25	U. S. Savings Ser. F. 1957

Acquired through Exchange:

50 shares	Tidewater Associated Oil Co. Cum. Pfd. $4\frac{1}{2}$ for 50 shares Tidewater Associated Oil Co. Cum. Pfd. $3\frac{3}{4}$
200 shares	Sears Roebuck and Co. Com. for 50 shares, in stock split.

Called:

50 shares	Crane Co. Cum. Pfd. 5
$5,000	Montana Power Co. 1st and Ref. 3¾ 1966.

The investment portfolio, valued at market November 30, 1945, now includes Government bonds 28 per cent, other bonds 16 per cent, preferred stock 13 per cent, common stock 37 per cent, cash in savings banks 6 per cent.

Income from Investments

Income received during the year from investment of Current Funds amounted to $1,245, exclusive of $98 earmarked for International Congress. This represents a return of something less than 2.1 per cent computed on average book value of investments. Income on Invested Funds amounted to $7,367, representing a return of approximately 3.8 per cent. Total investment income from all sources was thus $8,710, representing a return somewhat under 3.37 per cent. This rate of return is very slightly over the corresponding rate for the fiscal year 1944, increased common stock dividend payments having been more than sufficient to offset lower yields from fixed income securities.

Income from the Henderson Estate was $4,750, which is $239 more than was received in 1944 from this source.

Increase in Surplus and Assets

Surplus account shows an increase of $4,637 after transferring $5,427 from Surplus to Endowment Fund Principal, and after setting up a reserve of $2,500 for payment for the Library Catalogue. The transfer to the Endowment Fund brought the principal to $71,000, slightly more than its value before the charges for depreciation of securities were made in 1940. The increase in Surplus is about one-half of the corresponding figure for the preceding year. It is to be expected that costs will increase in the next fiscal year when, in addition to the Library Catalogue and the biennial List of Members, at least two Colloquium volumes will be printed and one reprinted.

The net increase in assets for the year amounted to nearly $14,500.

Respectfully submitted,
BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

Assets

	November 30, 1945	November 30, 1944
CURRENT FUNDS:		
Cash	$ 24,868.41	$ 27,973.79
Investments	54,433.13	41,839.37
	$ 79,301.54	$ 69,813.16
INVESTED FUNDS:		
Cash	$ 5,333.06	$ 336.09
Investments	137,543.57	188,059.77
	$193,376.63	$188,395.86
TOTAL ASSETS	$272,578.17	$258,209.02

Liabilities

CURRENT FUNDS:		
Mathematical Reviews	$ 13,637.34	$ 11,936.09
Colloquium	9,310.80	7,368.15
Mathematical Surveys	1,250.95	1,236.10
Library Catalogue	1,868.68	
Reprinting Funds	3,310.24	3,765.26
Prize Funds and Other Special Funds Accumulated Income	4,518.11	3,332.75
International Congress	6,331.71	6,233.39
War Policy Committee	143.25	1,719.41
Sinking Fund	820.91	768.96
Profit on Sales of Securities	413.36	413.36
Miscellaneous	91.60	72.56
Surplus	37,604.59	32,967.13
	$ 79,301.54	$ 69,813.16
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 65,573.18
Prize Funds and Other Special Funds	31,033.22	31,033.22
Life Membership and Subscription Reserve	3,866.88	4,032.68
Colloquium	5,000.00	5,000.00
Mathematical Reviews	65,000.00	65,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	13,090.64	13,370.89
	$193,376.63	$188,395.86
TOTAL LIABILITIES	$272,678.17	$258,209.02

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1944–1945

	1945		1944	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$18,346.37		$18,060.64	
Dues—Contributing Memberships	978.64		948.06	
Dues—Institutional Memberships	7,013.32		7,510.02	
Initiation Fees	445.00		793.95	
Investment Income	9,694.54		8,836.16	
Miscellaneous	261.74		353.84	
Grant from Rockefeller Foundation for War Policy Committee	2,500.00		3,750.00	
GENERAL DISBURSEMENTS:				
Secretaries		$ 6,277.52		$ 5,642.71
Treasurer		1,153.96		1,070.77
Officers' Traveling		386.69		600.00
Library		1,015.67		984.46
War Policy Committee Expenses		4,076.16		2,030.59
Emergency		15.82		158.38
Total	$39,239.61	$12,925.82	$40,252.67	$10,486.91
Excess of General Receipts	$26,313.79		$29,765.76	
PUBLICATION:				
Bulletin	$ 2,195.05	$13,194.69	$ 2,462.10	$11,189.68
Bulletin Reprinting	936.16	1,629.45	856.40	
Transactions	6,592.76	7,918.76	7,097.63	8,644.20
Transactions Reprinting	491.75	253.48	669.76	
Colloquium	3,266.38	1,323.73	3,210.08	1,603.62
Mathematical Reviews	15,684.45	13,983.20	15,453.49	12,770.02
Mathematical Surveys	220.49	205.64	713.82	317.75
Semicentennial Publications	53.09	1.14	180.65	4.03
American Journal		2,500.00		2,500.00
Library Catalogue	2,500.00	631.32		
Total	$31,940.13	$41,641.41	$30,643.93	$37,029.30
Excess Cost of Publication		$ 9,701.28		$ 6,385.37
OTHER:				
Special Funds (including Congress)	$ 1,283.68		$ 1,223.90	$ 100.00
Profit/Loss on Sales of Securities		$ 280.25	141.03	
Miscellaneous	19.04			8.43
Surplus:				
Mathematical Reviews		1,000.00		1,000.00
Library Catalogue		2,500.00		
War Policy Committee	334.17			
Total	$ 1,636.89	$ 3,780.25	$ 1,364.93	$ 1,108.43
Difference		$ 2,143.36	$ 256.50	
Net Change in Assets	$14,469.15		$23,636.89	
ASSETS BEGINNING OF YEAR	$258,209.02		$234,572.13	
ASSETS END OF YEAR	$272,678.17		$258,209.02	

THE NOVEMBER MEETING IN PASADENA

The four hundred thirteenth meeting of the American Mathematical Society was held at the California Institute of Technology on Saturday, November 24, 1945. The attendance included the following forty members of the Society:

O. W. Albert, H. M. Bacon, Harry Bateman, E. F. Beckenbach, Clifford Bell, E. T. Bell, F. L. Campbell, F. M. Clarke, D. R. Curtiss, P. H. Daus, H. C. Dieckmann, R. P. Dilworth, W. H. Glenn, J. W. Green, W. S. Gustin, O. G. Harrold, P. G. Hoel, D. G. Humm, D. H. Hyers, Glenn James, P. B. Johnson, M. B. Lehman, Hans Lewy, R. T. Luginbuhl, A. D. Michal, Sam Perlis, W. T. Puckett, W. C. Randels, A. C. Schaeffer, G. E. F. Sherwood, Ernst Snapper, R. H. Sorgenfrey, D. V. Steed, J. D. Swift, A. E. Taylor, F. A. Valentine, L. F. Walton, Morgan Ward, E. R. Worthington, Max Zorn.

In the first part of the meeting three papers were presented with Professor Max Zorn presiding. After a brief intermission Professor Hans Lewy delivered the invited hour address entitled *Water waves on sloping beaches* with Professor E. T. Bell presiding. Following this the remaining papers were presented.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title.

1. E. F. Beckenbach: *On a characteristic property of linear functions.* (Abstract 51-9-154.)

2. William Gustin: *Countable connected spaces.* Abstract 51-11-249.)

3. R. H. Sorgenfrey: *Concerning continua irreducible about n points.* (Abstract 52-1-51.)

4. Ernst Snapper: *Polynomial matrices in several variables.* (Abstract 52-1-8.)

5. F. A. Valentine: *Set properties determined by conditions on linear sections.* (Abstract 52-1-52.)

6. O. G. Harrold: *The* ULC *property of certain open sets.* I. *Euclidean domains.* (Abstract 52-1-47.)

7. H. S. M. Coxeter: *Quaternions and reflections.* (Abstract 51-11-233-*t*.)

8. Ernst Snapper: *Polynomial matrices in one variable, differential equations and module theory.* (Abstract 52-1-7-*t*.)

9. T. Y. Thomas: *Topological theory of dynamical systems.* (Abstract 52-1-38-*t*.)

10. T. Y. Thomas: *Absolute scalar invariants and the isometric correspondence of Riemann spaces.* (Abstract 52-1-37-*t*.)

11. A. C. Sugar: *On the numerical treatment of forced oscillations.* (Abstract 52-1-30-*t*.)

12. E. M. Beesley and A. P. Morse: ϕ-*Cantorian functions and their convex moduli.* (Abstract 52-1-11-*t*.)

A. C. SCHAEFFER,
Associate Secretary

BOOK REVIEWS

Collected works. Vol. 1. *Theory of numbers*. By P. L. Chebyshev (Tchebychef). Moscow-Leningrad, Academy of Sciences of the U.S.S.R., 1944. 342 pp.+1 plate (Russian).

The first edition of the collected works of P. L. Chebyshev in two large volumes which appeared in 1899 and 1907 contained all the memoirs and notes published by the illustrious Russian mathematician during his lifetime with the exception of three dissertations, two of which had been published previously in book form, while the third existed only in manuscript. The Academy of Sciences of the U.S.S.R. has now undertaken the publication of the second edition of the works of Chebyshev.

In a short preface to the first volume S. N. Bernstein states the motives which guided the Academy in undertaking this new edition: "The investigations of P. L. Chebyshev which had played such an important part in the development of contemporary mathematics preserve their vital significance even in our time and many ideas of our great countryman, whose work afforded a shining example of the unity of theory and practice in mathematics, are not as yet completely explored."

According to the plan the new edition will include not only the papers from the first edition, but also the three above-mentioned dissertations together with some hitherto unpublished material. For the convenience of readers who might be interested only in certain phases of the work of Chebyshev the editors of the new edition do not follow the chronological order of publication but distribute the papers in five volumes according to their content. This first volume contains all the papers of Chebyshev pertaining to the theory of numbers, including his well known book, *Teorija Sravenij* (*Theory of congruences*). The influence which some of these papers exercised on the development of mathematical science is well known and it is hardly necessary to stress it in this review. Besides the papers published by Chebyshev himself there is a reproduction of a note by A. Markoff in which he reconstructs the proof of a theorem on prime divisors of numbers of the form $4x^2+1$ communicated orally by Chebyshev to Hermite, and two interesting commentaries: one by A. Gelfond on the famous Chebyshev memoir on prime numbers, and another by B. Delaunay on a series of investigations which grew out of Chebyshev's memoir *Ob odnom arifmetičescom voprose* (*On a certain arithmetical question*).

J. V. USPENSKY

Lectures on the theory of functions. By J. E. Littlewood. Oxford University Press, 1944. 8+244 pp. 17s 6d.

This book is neither a systematic treatise on the theory of functions nor a monograph on some definite branch of this theory, but rather it considers such parts of the theory of functions as are near to the center of the sphere of interest of the author. The book has a strong and unmistakable personal character and can give not only to those who know the author personally but also to those who read between the lines an impression of his power and penetration. Mathematical arguments are selected and arranged with exceptional regard for methodological points, an approach which is extremely instructive and valuable.

The book is divided into three parts: an extensive "Introduction" which occupies more than one-third of the book, Chapter I and Chapter II. According to the author's original plan, further chapters were to be incorporated in a second volume which has not yet been published. The various matters collected for reference in the Introduction provide a mathematical background for the understanding of the book, but much of the preparation is actually intended for the unpublished second volume.

The Introduction contains a systematic treatment of the inequalities of Hölder and Minkowski. A section on the theory of functions of a real variable provides the background needed in that field of the theory of functions of a complex variable which concerns "boundary values." Another section is devoted to the general theory of harmonic functions, and a section at the end sets out the behavior of certain special functions whose role is to provide counter examples.

Chapter I is composed of selected topics from classical theory of functions of a complex variable, and ends with a discussion of the theory of conformal mapping.

Chapter II is centered around the following general problem. Suppose that some restriction is placed on the set of values taken by a function $f(z)$ which is regular in the unit circle $|z| <1$. What is the influence of this restriction on the behavior of the function? In particular, how does it restrict the modulus of the function and its coefficients? For example, the point $w=f(z)$ may be restricted to move, for varying z, on some given Riemann surface (subordination). The chapter begins with a section on subharmonic functions (a topic not prima facie related to the main problem), and proceeds to a discussion of the principle of subordination (Lindelöf principle). Applications of this principle are made to functions subordinate to the elliptic modular functions, and also to functions subordinate to schlicht functions.

Most of the chapter is based on the author's own research, but some of the material has been supplied by W. W. Rogosinski, who helped the author in the final writing of this chapter.

Some readers will miss an index, and many more will miss bibliographical references.

D. C. SPENCER

General equilibrium theory in international trade. By Jacob L. Mosak. (Cowles Commission Monograph No. 7.) Bloomington, Ind., Principia Press, 1944. 187 pp. $2.50.

This work can be recommended to mathematicians who are interested in learning what modern economic theory is about or what mathematical methods have been found useful in this field. It is an interesting historical fact that a statistically significant fraction of the great literary or non-mathematical economists began their training as mathematicians. On the other hand, a no less significant proportion of those who have made important contributions to mathematical economics started out with poor early trainings in mathematics. And with a few notable exceptions, the excursions into economics of well-trained creative mathematicians of reputations have not resulted in the most important advances in this field.

Like Willard Gibb's work in classical thermodynamics, but unlike his statistical mechanics, the tools employed in this book are the mathematically elementary ones capable of being taught in advanced calculus and undergraduate algebra classes: maxima of functions of many variables, Jacobians, Hessians, determinants, and so on. However, some of the important aspects of these topics are rarely taught in such courses although there is no good reason why they should not be. Among these is the statement of sufficient secondary conditions for an extremum of a function of many variables subject to a number of constraints; or what is the same thing, the conditions for definiteness of a quadratic form of variables subject to linear constraints, with the implied inequalities of certain bordered Hessians. The Weierstrassian treatment of this problem is not widely available. (Compare, however, Harris Hancock's *Maxima and minima*, now out of print, and Carathéodory's *Partiale Differentialgleichungen erster Ordnung*.) Because Pareto was unfamiliar with this theory, progress was held up for fifteen years until the Russian mathematician Slutsky put the cart back on the tracks.

In connection with asymmetrical determinants which arise in the study of the problem of stability of multiple markets, some less familiar mathematical problems and theorems occur. Typical of these is the theorem proved by the author that any matrix which has positive diagonal elements and negative off-diagonal elements and whose principal minors are all positive has an inverse whose elements are *all* positive. Another economist (Metzler) has proved the further theorem of Routhian type that the real parts of all the characteristic roots of such a matrix are all positive.

PAUL A. SAMUELSON

NOTES

Linear transformations in Hilbert space and their applications to analysis, by M. H. Stone, volume 15 of the American Mathematical Society Colloquium Publications, is again available. A limited number of copies have been prepared photographically from the original edition. The price is unchanged, $8.00, with the usual 25 per cent discount to members of the Society.

The editors of the Bulletin wish to make grateful public acknowledgement of the services rendered by the following persons who have refereed papers: C. R. Adams, R. P. Agnew, A. A. Albert, Melvin Avramy, Reinhold Baer, M. A. Basoco, R. A. Beaumont, E. F. Beckenbach, P. O. Bell, Richard Brauer, A. E. Brown, Herbert Busemann, Claude Chevalley, Alonzo Church, R. V. Churchill, M. M. Day, John DeCicco, R. P. Dilworth, F. G. Dressel, Nelson Dunford, Samuel Eilenberg, Paul Erdös, Will Feller, A. D. Fialkow, H. H. Goldstine, L. M. Graves, L. J. Green, V. G. Grove, O. G. Harrold, M. H. Heins, E. D. Hellinger, T. H. Hildebrandt, J. D. Hill, Einar Hille, Harold Hotelling, Witold Hurewicz, Dunham Jackson, Nathan Jacobson, B. W. Jones, F. B. Jones, E. P. Lane, Solomon Lefschetz, Joseph Lehner, Walter Leighton, Howard Levi, Norman Levinson, N. H. McCoy, E. J. McShane, J. K. L. MacDonald, C. C. MacDuffee, G. W. Mackey, H. B. Mann, Morris Marden, W. T. Martin, C. N. Moore, A. P. Morse, F. J. Murray, C. J. Nesbitt, John von Neumann, C. V. Newsom, Ivan Niven, Gordon Pall, Harry Pollard, Hans Rademacher, Tibor Radó, G. Y. Rainich, E. D. Rainville, W. C. Randels, W. T. Reid, R. M. Robinson, L. J. Savage, A. C. Schaeffer, O. F. G. Schilling, W. Seidel, I. M. Sheffer Seymour Sherman, N. E. Steenrod, J. J. Stoker, Otto Szász, Gabor Szegö, S. M. Ulam, J. V. Uspensky, R. J. Walker, H. S. Wall, A. D. Wallace, Hermann Weyl, A. L. Whiteman, Hassler Whitney, H. S. Zuckerman, Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 57 and 58: Emil Artin, P. O. Bell, Salomon Bochner, A. T. Brauer, Richard Brauer, R. H. Bruck, Claude Chevalley, W. Z. Chien, A. H. Copeland, H. S. M. Coxeter, Herbert Federer, Will Feller, K. O. Friedrichs, Nathan Jacobson, M. S. Knebelman, C. C. MacDuffee, G. W. Mackey, W. T. Martin, A. M. Mood, Oystein Ore, Gordon Pall, Harry Pollard, C. E. Rickart, O. F. G. Schilling,

E. R. Stabler, Otto Szász, Gabor Szegö, G. T. Whyburn, S. S. Wilks, John Williamson.

The following American mathematicians attended the annual meeting of the Mexican Mathematical Society during the week of May 28th, 1945, at Guadalajara: Nelson Dunford, Solomon Lefschetz, F. D. Murnaghan, Rufus Oldenburger, and Norbert Wiener. The first evening of the congress was devoted to a memorial to the late G. D. Birkhoff, with addresses presented by Messrs. Graef, Lefschetz, and Wiener.

Professor Garrett Birkhoff of Harvard University has been elected an honorary fellow of the Mexican Mathematical Society.

One of the Theodore Roosevelt Distinguished Service Medals of Honor for 1945 was awarded to Dr. Vannevar Bush of the Carnegie Institution of Washington in recognition of distinguished service in the field of science.

Professor Hermann Weyl of The Institute for Advanced Study has been awarded the honorary degree of Doctor of Mathematics by the Eidgenössische Technische Hochschule in Zürich.

The Duodecimal Society of America has conferred its Annual Award for 1944 on F. E. Andrews, Tenafly, New Jersey, and for 1945 on G. S. Terry, Hingham, Massachusetts.

Mr. L. Mirsky has been appointed assistant lecturer at the University of Sheffield.

Dr. Beulah M. Armstrong of the University of Illinois has been promoted to an assistant professorship.

Professor Alfredo Baños of the University of Mexico has been appointed professor of physics at the University of California at Los Angeles.

Dr. J. J. Barron of Brown University has been appointed to an assistant professorship at Marquette University.

Dr. R. C. F. Bartels of the University of Michigan has been promoted to an assistant professorship.

Associate Dean Walter Bartky of the University of Chicago has been made dean.

Mr. P. T. Bateman has been appointed lecturer in statistics at Bryn Mawr College.

Dr. Lipman Bers has been appointed to an assistant professorship at Syracuse University.

Associate Professor D. G. Bourgin of the University of Illinois has been promoted to a professorship.

Dr. F. A. Butter of Lawrence College has been appointed to an acting assistant professorship at Stanford University.

Mr. F. L. Celauro has been appointed to an assistant professorship at Lehigh University.

Assistant Professor Max Coral of Wayne University has been promoted to an associate professorship.

Professor Emeritus L. L. Dines of the Carnegie Institute of Technology has been appointed to a visiting professorship at the University of Saskatchewan.

Associate Professor J. L. Doob of the University of Illinois has been promoted to a professorship.

Associate Professor Samuel Eilenberg of the University of Michigan is at present a visiting lecturer at Princeton University.

Assistant Professor Churchill Eisenhart of the University of Wisconsin has been promoted to an associate professorship.

Miss Mary L. Elveback has been appointed to an assistant professorship at Rockford College.

Assistant Professor F. C. Gentry of the University of New Mexico has been appointed to an associate professorship at Louisiana Polytechnic Institute.

Mr. R. A. Good of the University of Wisconsin has been appointed to an assistant professorship at the University of Maryland.

Dr. P. E. Guenther of the Case School of Applied Science has been promoted to an assistant professorship.

Associate Professor E. H. Hadlock of Hastings College, Nebraska, has been promoted to a professorship.

Assistant Professor Israel Halperin of Queen's University, Kingston, Ontario, Canada, has been promoted to an associate professorship.

Mr. J. R. Hanna of the University of Wichita has been promoted to a professorship.

Dr. E. D. Hellinger of Northwestern University has been appointed to a professorship.

Assistant Professor J. D. Hill of Michigan State College has been promoted to an associate professorship.

Professor W. L. Johnson of Mississippi Southern College has been appointed dean of men.

Dr. H. A. Jordan of Harvard University has been appointed to an associate professorship at Mount Holyoke College.

Assistant Professor K. D. Kelly of Fenn College, Cleveland, Ohio, has been promoted to an associate professorship.

Associate Professor S. C. Kleene of Amherst College has been appointed to an associate professorship at the University of Wisconsin.

Professor Lincoln LaPaz of Ohio State University has been appointed to a professorship at the University of New Mexico.

Dr. C. C. Lin of the California Institute of Technology has been appointed to an assistant professorship at Brown University.

Associate Professor M. S. Macphail of Acadia University has been promoted to a professorship.

Dr. H. B. Mann of Brown University has been appointed to an associate professorship at Ohio State University.

Dr. Margaret E. Mauch of Michigan State College has been appointed to an assistant professorship at the University of Akron.

Associate Professor R. R. Middlemiss of Washington University has been promoted to a professorship.

Dr. Josephine M. Mitchell of Connecticut College has been appointed to an associate professorship at Winthrop College.

Dr. H. E. Nelson of Gustavus Adolphus College, St. Peter, Minnesota, has been promoted to an associate professorship.

Dr. R. S. Pate has been appointed to an associate professorship at the University of South Carolina.

Associate Professor R. M. Pinkerton of the Agricultural and Mechanical College of Texas has been promoted to a professorship.

Dr. A. R. Poole has been appointed to an assistant professorship at Montana State College.

Assistant Professor R. F. Rinehart of the Case School of Applied Science has been promoted to an associate professorship.

Mr. Fred Robertson of Iowa State College has been promoted to an assistant professorship.

Dr. L. D. Rodabaugh of Oberlin College has been appointed statistician in the Bureau of the Census.

Assistant Professor O. F. G. Schilling of the University of Chicago has been promoted to an associate professorship.

Associate Professor Samuel Selby of the University of Akron has been promoted to a professorship.

Dr. Seymour Sherman has been appointed to an assistant professorship at the University of Chicago.

Dr. C. L. Siegel has been appointed to a professorship at the Institute for Advanced Study.

Dr. Ernst Snapper of Princeton University has been appointed to an assistant professorship at the University of Southern California.

Assistant Professor N. E. Steenrod of the University of Michigan has been promoted to an associate professorship.

Dr. W. C. Taylor of the University of Cincinnati has been promoted to an assistant professorship.

Assistant Professor C. J. Thorne of Louisiana State University has been appointed to an assistant professorship at the University of Utah.

Dr. W. R. Transue has been appointed to an associate professorship at Kenyon College.

Dr. B. R. Ullsvik of the State Teachers College in Eau Claire, Wisconsin, has been appointed to an associate professorship at Illinois State Normal University, Normal, Illinois.

Dr. Mary C. Varnhorn of Trinity College, Washington, D. C., has been promoted to an assistant professorship.

Dr. G. C. Vedova has been appointed to a professorship at Pennsylvania Military College, Chester, Pennsylvania.

Dr. Bernard Vinograde of Tulane University has been appointed to an assistant professorship at Iowa State College.

Assistant Professor G. L. Walker of the University of Delaware has been appointed to an assistant professorship at Temple University.

Associate Professor S. E. Warschawski of Washington University has been appointed to a professorship at the University of Minnesota.

Dr. E. L. Welker of the University of Illinois has been promoted to an assistant professorship.

Dr. Jacob Wolfowitz has been appointed to an associate professorship at North Carolina State College.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1946–1947. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, N. J., and are returnable by March 1, 1946.

The following appointments to instructorships are announced: Brown University: Dr. W. M. Chen, Dr. Herbert Federer, Mr. M. H. Protter; University of Chicago: Dr. Irving Kaplansky; Drake University, Des Moines, Iowa: Dr. A. A. Grau; Everett Junior College, Everett, Washington: Mr. G. H. Van Arkel; University of Georgia: Mr. J. L. Howell; Johns Hopkins University: Dr. Norman Davids; University of Kansas: Mr. Edison Greer; University of Michigan: Dr. R. D. Schafer; University of Missouri: Mr. J. C. Koken; North Carolina State College: Mr. G. C. Watson; Ohio State University: Dr. Joshua Barlaz; Oregon State College: Dr. Edmund Pinney; Princeton University: Dr. G. W. Whitehead; University of Saskatchewan: Mr. J. S. Vigder; Southern College of Optometry, Memphis, Tennessee: Mr. R. L. Coker; Texas Technological College: Miss Virginia E. Bowman; Vanderbilt University: Mr. J. A. Tierney; University of Wisconsin: Dr. B. H. Colvin, Dr. Abraham Spitzbart.

Professor Emeritus M. J. Babb of the University of Pennsylvania died on October 27, 1945, at the age of seventy-five years. He had been a member of the Society since 1906.

Professor Conrad Bilgery of Regis College, Denver, Colorado, died August 13, 1945, at the age of sixty-five years.

Professor Emeritus H. F. Blichfeldt of Stanford University died November 16, 1945, at the age of seventy-two years. He had been a member of the Society since 1900.

Assistant Professor Annie W. Fleming of Iowa State College died on September 19, 1945.

Professor J. D. Tamarkin of Brown University died November 18, 1945, at the age of fifty-seven years. He had been a member of the Society since 1925.

Mr. L. E. Widmark of the Star Electric Motor Company, Bloomfield, New Jersey, died September 27, 1945, at the age of sixty-one years. He had been a member of the Society since 1923.

Attention is called to the fact that income from the Permanent Science Fund, by Agreement and Declaration of Trust, is disbursed by the American Academy of Arts and Sciences in support of scientific research in the fields of Mathematics, Physics, Chemistry, Astronomy, Geology, Geography, Zoology, Botany, Anthropology, Psychology, Sociology and Economics, History and Philology, Engineering, Medicine, Surgery, Agriculture, Manufacture and Commerce, Education, or any other science of any nature or description. Applications for Grants-in-Aid are receivable on multiple forms which will be supplied upon request to the chairman of the Committee, and are considered by the Permanent Science Fund Committee of the Academy on March 1 and October 1. It is a condition of grants made by the Academy from the Permanent Science Fund that they are not for financial support of work the results of which comprise partial fulfillment of requirements for an academic degree. Disbursements will be made upon authorization of the Academy by its treasurer directly to recipients, as named in the awards, and not to institutions with which recipients may be affiliated. Annual accounting for expenditure of these funds is required as a condition of the grant, and the Committee appreciates being informed of the state of progress of projects supported and the manner in which grants have been utilized. Communications should be addressed to: John W. M. Bunker, Chairman, Permanent Science Fund Committee, Massachusetts Institute of Technology, Cambridge, Massachusetts.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. Richard Brauer: *On the arithmetic in a group ring.* II. Preliminary report.

The paper is a continuation of an earlier paper (Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) pp. 109–114). The arithmetic in the group ring of a group of finite order over suitable algebraic number fields is studied. In particular, the prime ideals are considered. In the first paper, a connection has been established between these prime ideals and the prime ideals of the group rings of certain subgroups. New relations of this kind are given. On the basis of these results, a number of statements concerning the characters of the group can be made, provided that the corresponding statements concerning the characters of the subgroups are known. If p^a is the highest power of a rational prime p dividing the order of the group, it is shown that the number of characters belonging to the same p-block lies below a bound depending only on p^a. This implies a corresponding result for the prime ideal divisors of p belonging to the same block. A refinement of the orthogonality relations for group characters is given. (Received October 16, 1945.)

2. Paul Erdös and Irving Kaplansky: *The asymptotic number of Latin rectangles.*

It has been conjectured that the number of n by k Latin rectangles is asymptotic to $(n!)^k e^{-k(k-1)/2}$. In this paper the conjecture is proved not only for k constant (as $n \to \infty$) but for $k < (\log n)^{2-\epsilon}$. Certain closer approximations are also found. (Received October 18, 1945.)

3. B. W. Jones: *Equivalence of quadratic forms over the ring of 2-adic integers.*

Any two quadratic forms f and g with coefficients in $R(p)$, the ring of p-adic integers, are equivalent in $R(p)$ to $2^{t_1}f_1+2^{t_2}f_2+\cdots+2^{t_k}f_k$ and $2^{s_1}g_1+2^{s_2}g_2+\cdots+2^{s_j}g_j$ where $t_1<t_2<\cdots<t_k$, $s_1<s_2<\cdots<s_j$ and each f_i and g_i is a form in $R(p)$ of unit determinant. If h is that portion of f containing $f_i, f_{i+1}, \cdots, f_{i+r}$ it is called an 8-block if it satisfies the following two conditions for $\lambda=8$: (1). $i=1$ or $2^{t_i}f_i \equiv 0 \pmod{4\cdot 2^{t_{i-1}}}$, (2). $i+r=k$ or $2^{t_{i+r+1}}f_{i+r+1} \equiv 0 \pmod{2^{t_{i+r}}\cdot\lambda}$. If h is not an 8-block but satisfies conditions 1 and 2 for $\lambda=4$, it is called a 4-block. These two concepts are included in the term λ-block for $\lambda=8$ or 4. The following result is proved: if f and g are two forms written as above they are equivalent in $R(2)$ if and only if $k=j$, for every i, $s_i=t_i$, and f_i and g_i

have the same number of variables and are both properly primitive or improperly primitive, and for every pair F and G of corresponding λ-blocks of f and g the following conditions hold: (a) if $\lambda=8$, then F is equivalent to G in the *field* of 2-adic numbers; (b) if $\lambda=4$, then the Hasse invariants $c_2(F)$ and $c_2(G)$ are equal and $|F|/|G|$ is a square unit or 5 times the square of a unit in $R(2)$. This criterion is simpler than previously known ones. (Received October 17, 1945.)

4. S. A. Kiss: *Transformations on lattices and structures of logic.*

Any binary operational system $(\cdot)$ is transformed into an isomorphic system $(+)$ by a one-one transformation t on its elements: $xt+yt=(x\cdot y)t$ or $(xt+yt)t^{-1}=x\cdot y$ and $x+y=(xt^{-1}\cdot yt^{-1})t$. The lattice of a Boolean algebra B^n has $n!2^n$ "link-preserving" transformations with which 2^n distinct transformed systems can be constructed from $(\cap)$. These are mutually distributive over each other and constitute, together with the "principal" transformations, a logical classification structure with 2^n truth classes in which normal forms, quantifiers and identically true formulae exist for each truth class. Consequently, there are as many different classes of logic as finite Boolean algebras B^n $(n=1, 2, 3, 4, \cdots)$ although hitherto only the 2-class logic corresponding to B^1 has been known and used. A link is an immediate connection between two elements of a discrete lattice. For example, the diagram of B^2 balanced on a vertex is a square; its 8 link-preserving (4 principal) transformations are the symmetries of the square (rectangle). These transformations define 4 distinct systems two of which are the known $(\cap)$ and $(\cup)$ and two are new. Link-preserving transformations of certain infinite distributive lattices are also studied and their connection with the number systems (integers, complex and hypercomplex numbers) is established. (Received October 12, 1945.)

5. Ivan Niven: *Sums of squares of integral quaternions.*

Every integral quaternion $a+2bi+2cj+2dij$ is expressible as a sum of three squares of integral quaternions, but not every one is a sum of two squares. (Received October 20, 1945.)

6. Gordon Pall: *On generalized quaternions.*

Let $(a_{\alpha\beta})$ denote a symmetric ternary matrix $(\alpha, \beta=1, 2, 3)$, and $(A_{\alpha\beta})$ its adjoint. Assume that the $a_{\alpha\alpha}$ and $2a_{\alpha\beta}$ are integers, and choose $\epsilon_\alpha=0$ or 1 to secure $4A_{\alpha\beta}\equiv\epsilon_\alpha\epsilon_\beta$ mod 2. Then the form $F=(t_0+2^{-1}\sum\epsilon_\alpha t_\alpha)^2+\sum A_{\alpha\beta}t_\alpha t_\beta$ is an integral quaternary quadratic form, and is the norm-form of the system of generalized quaternions associated with $(a_{\alpha\beta})$. The arithmetic of these quaternions is maximal if and only if F is fundamental, that is, F cannot be derived by an integral linear transformation from another norm-form of smaller determinant. There is a close connection between the arithmetics of these quaternions and the integral automorphs of F, and of $(a_{\alpha\beta})$, the correspondence being one-to-one in the maximal case. The factors of any given norm of a primitive quaternion are always unique, up to a unit factor. However, unless the genus of F contains only one class, such factorization is in general not possible, even when all the obviously necessary conditions are satisfied. There exist exactly 39 positive-definite classes of norm-forms in genera of one class, hence exactly 39 such systems in which factorization is in general possible as well as unique. However, only five of these are maximal. If the genus of a positive-definite norm-form F contains two different classes of norm-forms, then it also contains a class of quaternary forms which are not norm-forms. (Received November 23, 1945.)

7. Ernst Snapper: *Polynomial matrices in one variable, differential equations and module theory.*

This paper establishes the foundation for the theory of matrices $A=(\alpha_{ij})$, where $(\alpha_{ij})\in P[x_1, \cdots, x_n]$. Part I treats the case $n=1$. Contrary to the classical procedure which uses sub-determinants of A, the theory is developed intrinsically in terms of the column space C and row space R of A. The meanings of the irreducible factors and multiplicities of the norm and elementary divisor of A for C and R thus become clear. Systems of linear differential equations and algebraic equations are fully discussed. Part II reviews and extends the ideal theoretic module theory, developed by P. M. Grundy in *A generalization of additive ideal theory*, Proc. Cambridge Philos. Soc. vol. 38 (1942), and by the author in *Structure of linear sets*, Trans. Amer. Math. Soc. vol. 52 (1942). This theory is the foundation for the case $n>1$. A general theory of systems of linear equations over any ring $\mathfrak{r}$ is developed. All known criteria for the solvability of such systems for special rings are corollaries of the *criterion of lengths* of this general theory. If $\mathfrak{r}=P[x]$, the theory becomes the theory of Part I. (Received October 7, 1945.)

8. Ernst Snapper: *Polynomial matrices in several variables.*

This paper discusses the theory of matrices $A=(\alpha_{ij})$, where $\alpha_{ij}\in P[x_1, \cdots, x_n]$. The module theory, discussed in Part II of the author's paper *Polynomial matrices in one variable, differential equations and module theory*, associates several invariants to the column space C and the row space R of A, for example the associated primes $\mathfrak{p}_j$, the $\mathfrak{p}_j$-lengths, the $\mathfrak{p}_j$-elementary divisors, and so on. Since R and C are *polynomial* modules, the theory of the Hilbert characteristic function can be developed for them which gives rise to one further invariant, called the $\mathfrak{p}_j$-degree. In terms of these invariants, the theory of the system of linear partial differential equations and algebraic equations, represented by A, is investigated. Furthermore, the irreducible factors and multiplicities of the norm and elementary divisor of A, as defined by the author in *The resultant of a linear set*, Amer. J. Math. vol. 66 (1944), are explained in terms of the above invariants. (Received October 7, 1945.)

ANALYSIS

9. N. R. Amundson: *On the boundary value problem of third kind for the quasi-linear parabolic differential equation.*

The author considers the quasi-linear parabolic equation with boundary conditions of the *third* kind for the open rectangle, that is, $u_{xx}=f(x, y, u, p, q)$; $-a_1u_x+b_1u=c_1(y)$, when $x=0$; $a_2u_x+b_2u=c_2(y)$, when $x=l$; $u=\phi(x)$, when $y=0$, where $c_i(y)$ and $\phi^{(iv)}(x)$ are continuous and b_i/a_i are non-negative constants. By use of the Green's function for the problem the above system is shown to be equivalent to a nonlinear integro-differential equation. Assuming that $f(x, y, u, p, q)$ is continuous in all five variables, and that its partial derivatives with respect to y, u, p, q satisfy a Lipschitz condition in u, p, q and are bounded, the *existence* of a solution $u(x, y)$ of the integro-differential equation is proved by an iteration method. Under the further assumption the u_x and u_y satisfy a Hölder condition with respect to y, the *uniqueness* of the solution $u(x, y)$ is established. M. Gevrey (*Thèse*, Journal de mathématique (6) vol. 9 (1913) and vol. 10 (1914)) considers the same differential equation for boundary conditions of the first kind. (Received October 19, 1945.)

10. E. W. Barankin: *On the eigen-values of infinite matrices and of linear integral equations.* Preliminary report.

In a recent paper (Bull. Amer. Math. Soc. vol. 51 (1945)) the author established several symmetric upper bounds for the characteristic roots of a finite matrix. The methods there employed are carried over to the discretely infinite and continuous cases, to give, under weak restrictions to insure convergence, the corresponding results. For an infinite matrix the bounds are of the same form (with some small exceptions) as in the finite case. For the general homogeneous linear integral equation $\int_0^\infty K(s, t)f(t)dt = \lambda f(s)$ the bounds obtained are $|\lambda|^2 \leqq \text{l.u.b.}_{(s)} \{\int_0^\infty |K(s, t)| dt \cdot \int_0^\infty |K(t,s)| dt\}$, $|\lambda| \leqq \text{l.u.b.}_{(s)} \int_0^\infty |K(s, t)| dt$, $|\lambda| \leqq \text{l.u.b.}_{(s)} \int_0^\infty |K(t, s)| dt$, and certain generalizations of the first of these. (Received November 17, 1945.)

11. E. M. Beesley and A. P. Morse: *ϕ-Cantorian functions and their convex moduli.*

A set function ϕ^* may be associated with a function ϕ which is nondecreasing on $[0, \infty)$ with $\phi(0+) = \phi(0) = 0$. A set $A \subset I$ is said to be ϕ-Cantorian if it is the intersection of a denumerable sequence of sets each of which is associated with a family of intervals satisfying certain conditions involving ϕ and I. If $f(x) = \phi^*((-\infty, x] \cdot A)$ where A is a ϕ-Cantorian set, then f is called a ϕ-Cantorian function. The convex modulus Φ of a function f has the property that $|f(t_2) - f(t_1)| \leqq \Phi(t_2 - t_1)$ and satisfies certain other conditions. Methods for construction of ϕ-Cantorian sets and for determination of the convex modulus are considered. These methods are employed to construct in any interval a set of Lebesgue measure zero which cannot be covered by any sequence of intervals whose lengths are equal to a prescribed sequence of numbers whose sum is less than the length of the original interval. A symmetric product measure is constructed and a perfect plane set is exhibited which has measure one but which cannot be expressed as a sum of a denumerable sequence of sets of finite measure after being subjected to a properly chosen shear or rotation. (Received November 19, 1945.)

12. Lipman Bers and Abe Gelbart: *A topological property of solutions of partial differential equations.*

Given a function $u(x, y)$ defined in a domain D and satisfying the elliptic differential equation $(au_x + bu_y)_x + (bu_x + cu_y)_y = 0$, where a, b, c are analytic functions of x and y; then there exists a homeomorphism of D into a domain Δ of the ξ, η-plane which takes u into a harmonic function of ξ and η. The proof is based on Stoilow's topological characterization of analytic functions of a complex variable. (Received October 19, 1945.)

13. D. G. Bourgin: *A class of generating functions.*

This note continues previous work on orthonormal sequences of the type $\{f(nx)\}$. Among other things it is shown that if the associated $\phi(z)$ belongs to K' and has a finite base, then $\phi(z)$ is a quasi elementary solution. (Received October 19, 1945.)

14. D. G. Bourgin: *Complete sets of functions.*

(A) Let $\{g_n(x)\}$ be O.N. and complete in $L_2(E)$. Let $F_n(x) = f_n(x) - g_n(x)$. Then, under certain minor convergence conditions, a sufficient condition for completeness of $\{f_n(x)\}$ is that the Grammian of $\{F_n(x)\}$ have a bound inferior to 1. In particular, the restriction $\sup_{1\leqq i<\infty}\sum_j |[F_i, F_j]| < 1$ is sufficient. (B) If $g_n(x) = \sin nx$ and $f_n(x) = f(nx)$ with $f(x) \sim \sin x - \sum b_i \sin ix$, then a sufficient condition for completeness of $\{f_n(x)\}$ in the space of odd functions in $L_2(-\pi, \pi)$ is essentially that $\sum [F_i, F_j] \cos (\log i - \log j)x \leqq \theta < 1$ where the sum is taken over all relatively prime integers i and j. If only a finite number of b_i's are nonvanishing, θ can be 1. Some theorems of Szász type are also given. (Received October 18, 1945.)

15. R. H. Cameron and W. T. Martin: *The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals.*

The authors show that certain products of Hermite functions of orthogonal linear functionals form a closed orthonormal set in the space L_2^* of nonlinear (that is, not necessarily linear) functionals of the Wiener integrable square. In terms of this set, each functional F of L_2^* can be developed in an orthogonal series which converges to F in the mean of L_2^*. (Received October 19, 1945.)

16. Herman Chernoff: *Complex solutions of partial differential equations.* I.

The author considers certain classes $\mathcal{C}$ of complex solutions of equations $\Delta u + Au_x + Bu_y + Cu = 0$. Bergman demonstrated certain properties of these classes (see Bergman, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155 and vol. 57 (1945) pp. 299–331). The author investigates the distribution of b-points of functions $u \in \mathcal{C}$. He introduces in the usual manner the notion of the index of a b-point (that is, generalized concept of multiplicity) and denotes by $n[r, (u-b)^{-1}]$ the sum of the indices of points where $u = b$ in $|z| < r$ $(z = x + iy)$. Let $2\pi m[r, u] = \int_0^{2\pi} \log^+ |u| \, d\theta$ where $z = re^{i\theta}$. Using Nevanlinna's Second Fundamental Theorem, the author gives an upper bound for $m[r, u]$ in terms of $\sum_{\nu=1}^{q} n[r, (u-\alpha_\nu)^{-1}]$. This upper bound holds for a certain restricted set of positive r. However in certain cases it is shown that r belongs to this set if r is large enough and $u \neq \alpha_\nu$ on $|z| = r$. For certain subsets of functions $u \in \mathcal{C}$, it is proved (in analogy of the Picard Theorem) that u attains all finite values. (Received November 14, 1945.)

17. A. E. Heins and Norbert Wiener: *A generalization of the Wiener-Hopf integral equation.*

A method is given which enables one to find the solutions of the integral equations of the type (*) $f(x) = \lambda \int_0^\infty K(x+y) f(y) dy$ where $K(x) = O(e^{-x})$, $x \to \infty$, and $O(1/x)$, $x \to 0$. This is accomplished by decomposing (*) into an infinite sequence of bilateral faltung integral equations, each of which depends on the solution of the previous one. The final result appears as an infinite series of integral operators, applied to a known function. An example for which the final answer appears in closed form is given. (Received November 13, 1945.)

18. Mark Kac: *Distribution of eigenvalues of certain integral equations with an application to roots of Bessel functions.* Preliminary report.

Let $\rho(u) \in L(-\infty, \infty)$, $\rho(u) = \rho(-u)$ and let $F(\xi) = \int_{-\infty}^{\infty} \rho(u) \cos u\xi du$ be also in $L(-\infty, \infty)$. Consider the eigenvalues $\lambda_1(a), \lambda_2(a), \cdots$, of the integral equation (*) $\int_{-a}^{a} \rho(s-t) f(t) dt = \lambda f(s)$. It is shown that for $n \geqq 2$ the limit of $(2a)^{-1} \sum \lambda_j^n$, as $a \to \infty$, is $\pi^{-1} \int_0^\infty F^n(\xi) d\xi$. Denoting by $N(\alpha, \beta; a)$ the number of eigenvalues of (*) which fall within (α, β) (not containing 0) it is easy to calculate explicitly $\lim_{a\to\infty}(2a)^{-1}N(\alpha, \beta; a)$ in terms of $F(\xi)$. If $\gamma>0$ and $\lambda_1(\gamma), \lambda_2(\gamma), \cdots$, are the eigenvalues of the integral equation (**) $\int_{-\infty}^{\infty} \exp(-\gamma|s|)\rho(s-t)\exp(-\gamma|t|)f(t)dt=\lambda f(s)$, then the limit of $\gamma\sum\lambda_j^n$, as $\gamma\to 0$, is $(\pi n)^{-1}\int_0^\infty F^n(\xi)d\xi$. Denoting by $N(\alpha, \beta; \gamma)$ the number of eigenvalues of (**) which fall within (α, β) it is again easy to calculate $\lim_{\gamma\to 0}\gamma N(\alpha, \beta; \gamma)$ in terms of $F(\xi)$. If in (**) one puts $\rho(u)'=2^{-1}\exp(-|u|)$ the eigenvalues are expressible in terms of roots of $J_{1/\gamma}(x)$ and $J_1'{}_{/\gamma}(x)$. As a consequence one obtains the following result: If $N(\alpha; \nu)$ denotes the number of positive roots of $J_\nu(x)$ which are less than $\alpha\nu$ $(\alpha>1)$ then $\nu^{-1}N(\alpha; \nu)\to\pi^{-1}\{(\alpha^2-1)^{1/2}-\text{arc tan }(\alpha^2-1)^{1/2}\}$. This in turn implies, among others, that if $r_\nu(\nu)$ is the νth positive root of $J_\nu(x)$, then $\nu^{-1}r_\nu(\nu)\to x_1^2+1$, where x_1 is the first positive root of $\tan x=x$. (Received October 18, 1945.)

19. Charles Loewner: *On pairs of quadratic forms in Hilbert space.*

One may ask which of the infinitely many linear transformations in n-space transforming two given positive definite forms $A(x, x)=\sum_1^n a_{\rho\sigma}x^\rho x^\sigma$, $B(x, x)=\sum_1^n b_{\rho\sigma}x^\rho x^\sigma$ into each other is the most "economical." The answer will depend on the method by which the economy is measured. It is natural to introduce a metric in the space of linear transformations and to call that transformation the most economical which is closest to the identity. Again, it is natural to use a metric that is intrinsically connected with the two given forms. A positive definite form represents a metric in the vector space. In a well known manner one can derive from it a metric in an arbitrary tensor space, especially in the space of linear transformations. It is remarkable that the two minimum problems derived in this way from the given forms have the same unique solution. Its matrix T_0 can be expressed by the matrices A and B of the given forms as that square root $(A^{-1}B)^{1/2}$ whose characteristic values are all positive. These elementary considerations can be generalized to Hilbert space by a suitable modification of the extremum problem. It can be shown that a suitable $(A^{-1}B)^{1/2}$ has similar extremum properties as in the finite-dimensional case. (Received October 22, 1945.)

20. L. B. Robinson: *Solution of an integral equation by alternating successive approximations.* I.

In the integral equation $u(x)=\lambda\sum_{i=1}^3(P_{in}(x)/Q_{in}(x))\int_0^x(1+s^2)^{-1}(\overline{P}_{in}(s)/\overline{Q}_{in}(s))\cdot u(s^{-p})ds+P_{4n}(x)/Q_{4n}(x)+\sum_{i=1}^3 u_{0i}(P_{in}(x)/Q_{in}(x))$, P and Q are polynomials of order n and p is an integer. The adjoint is $u(x^{-1})=\lambda\sum_{i=1}^3(P_{in}(x^{-1})/Q_{in}(x^{-1}))\int_0^x-(1+s^2)^{-1}\overline{P}_{in}(s^{-1})/Q_{in}(s^{-1})u(s^{+p})ds+P_{4n}(x^{-1})/Q_{4n}(x^{-1})+\sum_{=1}V_{0i}(P_{in}(x^{-1})/Q_{in}(x^{-1}))$. By alternating successive approximations obtain $u(x)$ and $u(x^{-1})$ and then weld them together by calculating the v_0 as linear functions of the u_0. The u_0 may be all zero. The author made his calculations using the number 3 but 3 can be replaced by any integer. (Received November 2, 1945.)

21. L. B. Robinson: *Solution of an integral equation by alternating successive approximations.* II.

Write $u(x)=\lambda\int_0^x(1+s^2)^{-1}G(x, \pi s/(1+s))u(s^{-2})ds+F(x)$, $\pi s/(1+s)=\sigma$, $\pi/(1+s)=\tau$. G, F are finite in the real domain with period π, $-\pi$. Then $u(x)=\lambda\int_0^x(1+s^2)^{-1}\{a_0(x)+\sum_{m=1}^\infty[a_m(x)\cos m\sigma+b_m(x)\sin m\sigma]\}u(s^{-2})ds+F(x)$, $u(x^{-1})=\lambda\int_0^x-(1+s^2)^{-1}\{a_0(x^{-1})+\sum_{m=1}^\infty[a_m(x^{-1})\cos m\tau+b_m(x^{-1})\sin m\tau]\}u(s^{+2})ds+F(x^{-1})+\sum_{m=1}^\infty[a_m(x^{-1})u_m+b_m(x^{-1})v_m]$. Solve by alternating successive approximations. $u(x)$ and $u(x^{-1})$ converge within the unit circle. The two can be welded together if the constants u_m, v_m

are selected properly. The results can be extended to the complex domain. (Received November 2, 1945.)

22. Robert Schatten (National Research Fellow) and John von Neumann: *The cross-space of linear transformations*. II.

The present notation is that of Bull. Amer. Math. Soc. Abstract 51-9-162. For any crossnorm $\alpha \geqq \lambda$, $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ may be considered as a Banach space of linear transformations from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ (from $\mathfrak{B}_2$ into $\mathfrak{B}_1^*$), while $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^* \subset (\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ may be considered as the Banach space of all those linear transformations of $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ which may be approximated in norm by linear transformations with finite-dimensional range. In particular for a Hilbert space $\mathfrak{H}$, $\mathfrak{H} \otimes_\gamma \mathfrak{H} = (\mathfrak{H} \otimes_\lambda \mathfrak{H})^*$ represents the "trace-class," that is, all linear transformations T on $\mathfrak{H}$ with finite trace; the norm $\gamma(T) =$ trace $(T'T)^{1/2}$, where T' represents the adjoint of T. It is also shown that λ does not necessarily represent the least crossnorm, that is, there exist crossnorms whose associates are not crossnorms. (Received November 13, 1945.)

23. Menahem Schiffer: *Hadamard's formula and variation of domain-functions.*

Hadamard's formula for the variation of the Green's function $g(x, y)$ for a varying domain D holds for smooth boundaries only. Considering special variations of D and using Green's identity, one may express this variation in terms of g and its derivatives at interior points of D only, resulting in a formula valid for general domains D. This was previously derived by the author in another way (Amer. J. Math. vol. 65 (1943) pp. 341–360). The same variation formula holds for many other domain-functions, for example, log $|f(x)|$, where $f(x)$ maps D on the exterior of a circle slit along concentric arcs and radial "stretches." In the case of smooth boundaries, the formula may be transformed into one of Hadamard's type, but is different for each of the domain-functions mentioned. The general formula is of wide applicability in extremum problems of conformal representation. Utilizing the periods of the analytic function with real part $g(x, y)$, one obtains variation formulae for the harmonic measures and the capacity constants of the different boundary continua. The method also permits applications in the theory of Riemann surfaces. The variation of the elementary integrals and their periods satisfy very similar formulae for a general class of deformations. (Received October 22, 1945.)

24. C. F. Stephens: *Solutions of systems of nonlinear difference equations in the neighborhood of a singular point.*

Consider the system of nonlinear difference equations (1) $y_i(x+1) = x^k \sum_{j=1}^{n} b_{ij}(x) y_j(x) + x^k f_i(y_1(x), \cdots, y_n(x); x)$ where the f_i begin with terms of the second degree in $y_i(x)$, are analytic functions of $y(x)$, and continuous functions of x in the neighborhood of $(x = \infty,\ y_1(x) = 0, \cdots, y_n(x) = 0)$. k is taken to be a positive integer and $f_i(0, \cdots, 0; x) \equiv 0$. By making use of the transformation $y_i(x) = [\Gamma(x)] z_i(x)$, where $\Gamma(x)$ is the well known gamma function, and the earlier results of the author (Bull. Amer. Math. Soc. Abstract 50-5-148), one shows that there exist many solutions of equations (1). These solution functions are continuous functions of x in a certain domain extending to infinity on the left and approach zero as a limit as x approaches infinity on rays parallel to the negative axis of reals. (Received October 18, 1945.)

25. S. E. Warschawski: *On the modulus of continuity of the mapping function at the boundary in conformal mapping.*

The author proves the following theorem: Let D be a simply connected region such that (i) D contains the unit circle and is contained in $|w|<R$; (ii) if D is divided into two parts by a crosscut of diameter $\delta<1$, then the diameter d of the subregion of D which does not contain the origin satisfies the inequality $d\leqq\mu\delta+\eta$ (μ and η are constants, $\mu\geqq1$, $\eta\geqq0$). Suppose that $w=f(z)$ maps $|z|<1$ conformally onto D ($f(0)=0$, $f'(0)>0$). Let k be an arbitrary constant, $k\geqq4R^2$, $k>\mu^2$. If z_0 is any point on $|z|=1$ and if z_1 and z_2 are in $|z|<1$ with $|z_i-z_0|\leqq r\leqq\exp[-k\pi^2/4]$, then $|f(z_1)-f(z_2)|\leqq(kr^\alpha/\mu)+\eta k^{1/2}/(k^{1/2}-\mu)$ where $\alpha=(2/(\pi^2k))(\log k-2\log\mu)$. This result is applied to the following problem. Let C_1 and C_2 be closed Jordan curves containing $w=0$ in their interiors D_i, such that C_2 is in the ϵ-neighborhood of C_1 (that is, every point of C_2 is within a circle of radius ϵ about some point of C_1) and C_2 in the ϵ-neighborhood of C_1. Let $w=f_i(z)$ map $|z|<1$ onto D_i ($f_i(0)=0$, $f_i'(0)>0$). Then a function $\Phi(\epsilon)$ is determined, which, aside from ϵ, depends only on certain parameters characterizing the C_i, such that $|f_1(z)-f_2(z)|\leqq\Phi(\epsilon)$ for $|z|\leqq1$. (Received October 19, 1945.)

26. J. W. T. Youngs: *Various definitions of surface and area.* Preliminary report.

This paper lists several definitions of the word "surface" and uses recent results in the field to show how the term "area" can be applied to each. The principal result is that in each case the area is a lower semi-continuous function of the surface. (Received October 17, 1945.)

APPLIED MATHEMATICS

27. Edward Kasner and John DeCicco: *Heat surfaces.*

If a region of space is heated by conduction, the temperature v at a time t at a point (x, y, z) is $v=\phi(x, y, z, t)$, where ϕ satisfies the Fourier heat equation. Kasner has introduced the term heat surfaces for those along which $v=$const. and $t=$const. In general, there are ∞^2 heat surfaces. In the present work, the authors extend to space certain theorems of Kasner concerning heat families in the plane, published in 1932–1933, Proc. Nat. Acad. Sci. U.S.A. There are no systems of ∞^2 planes or ∞^2 spheres which form a heat family except in the imaginary domain. The only sets of ∞^1 planes which form a heat family are the pencils. A system of ∞^1 spheres is a heat family if and only if it is a concentric set. The only isothermal systems of planes are pencils and the only isothermal sets of spheres are concentric families. The cases where there are only ∞^1 heat surfaces are connected with the equations of Laplace, Poisson, and Helmholtz-Pockels. Finally, these results are extended to n dimensions. (Received October 11, 1945.)

28. H. E. Salzer: *Coefficients for repeated integration with central differences.*

The present paper is aimed toward facilitating double or k-fold repeated quadrature of a function which is tabulated at a uniform interval, with its central differences of even order (see Abstract 51-9-172). When the Everett interpolation formula is integrated k times over an interval of tabulation, one obtains a formula for stepwise multiple quadrature in the form (1) $\int_{x_0}^{x_1}\cdots\int_{x_0}^{x}\int_{x_0}^{x}f(x)(dx)^k=h^k[A_0^{(k)}f_0+B_0^{(k)}f_1$

$+\sum_{s=1}^{m}(A_{2s}^{(k)}\delta_0^{2s}+B_{2s}^{(k)}\delta_1^{2s})]+R_{2m}$. This article tabulates: I. $A_{2s}^{(2)}$, $B_{2s}^{(2)}$, exact values for $2s=0, 2, \cdots, 20$ and sixteen decimal places for $2s=22, \cdots, 48$. II. $A_{2s}^{(k)}$, $B_{2s}^{(k)}$, $k=3(1)6$, $2s=2, 4, \cdots, 20$, eight significant figures (but exactly for $s=0$). Simple recursion formulas are obtained for $A_{2s}^{(2)}$ and $B_{2s}^{(2)}$ in terms of $M_{2s}\equiv 2A_{2s}^{(1)}$, an application of which is the expression of (1) in terms of δ_0^{2s} and $\delta_{1/2}^{2s+1}$, analogous to the forward version of the Newton-Gauss interpolation formula. Expressions are derived for $A_{2s}^{(2)}$ and $B_{2s}^{(2)}$ in terms of $B_\nu^{(n)}(x)$, Bernoulli polynomials of degree ν and order n. Cumulative recursion formulas are derived for $A_{2s}^{(k)}$ in terms of $B_{2s}^{(i)}$, and for $B_{2s}^{(k)}$ in terms of $A_{2s}^{(i)}$, $i=1, 2, \cdots, k$. (Received November 22, 1945.)

29. H. E. Salzer: *Table of coefficients for obtaining the first derivative without differences.*

When a function $f(x)$ is known for n equally spaced arguments at interval h, an approximation to the derivative at a point $x=x_0+ph$ may be obtained, by the differentiation of the well known Lagrangian interpolation formula, in the form $f'(x_0+ph)\sim(1/hC(n))\sum_{i=-[(n-1)/2]}^{i=[n/2]}C_i^{(n)}(p)f(x_0+ih)$, where $[m]$ denotes the largest integer in m, $C_i^{(n)}(p)$ are polynomials in p of the $(n-2)$th degree, and $C(n)$ denotes the least positive integer which enables $C_i^{(n)}(p)$ to have integral coefficients. The present table gives the exact values of these polynomials $C_i^{(n)}(p)$, for p ranging from $-[(n-1)/2]$ to $[n/2]$. For $n=4$, 5 and 6, the polynomials $C_i^{(n)}(p)$ are tabulated at intervals of 0.01; for $n=7$, they are tabulated at intervals of 0.1. (Received November 6, 1945.)

30. A. C. Sugar: *On the numerical treatment of forced oscillations.*

The solution of the equation $\ddot{x}+\omega^2x=a(t)$, $x(0)=0=\dot{x}(0)$, is given by $x=(1/\omega)\cdot\int_0^t a(\tau)\sin\omega(t-\tau)d\tau$. In this paper a simple approximation of x and hence of $\ddot{x}$ is found. Easy vector methods of obtaining max $|x|$ and max $|\ddot{x}|$ are discussed. (Received November 17, 1945.)

Geometry

31. L. M. Blumenthal: *Metric characterization of elliptic space.*

In this paper the first characterizations of finite and infinite dimensional elliptic spaces to be expressed wholly and explicitly in terms of distance relations are obtained. The characterizations are secured by direct, elementary geometric arguments. Only the simplest properties of elliptic space are used and no reference whatever is made to topological theorems. (Received October 3, 1945.)

32. S. C. Chang: *A new foundation of the projective differential theory of curves in five-dimensional space.*

As a preliminary a covariant triangle of reference and unit point for a plane curve is determined in an elementary and geometric manner using neighborhoods of order six. For a point P on a curve Γ in five dimensions a covariant triangle PP_1P_2 and unit point is first determined for the curve of intersection C of osculating plane and developable hypersurface of Γ. PP_1P_2 are three vertices of a quadrilateral Q on a covariant quadric generated by certain Bompiani osculants. The fourth vertex of Q is chosen as P_3. Similarly P_4, P_5 can be defined leading to a covariant pyramid for Γ. The Frenet-Serret formulas for the cases of P an ordinary and a k-ic ($k=6, 7, 8$) point follow from the corresponding canonical expansions. The method has the advantage of facilitating the geometric interpretation of the projective invariants of Γ. (Received November 1, 1945.)

33. John DeCicco: *Differential geometry in the Kasner plane.*

A horn angle is defined as the configuration formed by two curves which pass through a point in a common direction. The qualitative and quantitative aspects of a horn angle have been under consideration since ancient times. In discussions from 1909–1912, Kasner showed that each category has an invariant under the conformal group, essentially unique for each category. To study a horn angle of category n, it has been found convenient to introduce an auxiliary $(n+1)$-dimensional space, termed the Kasner space K_{n+1}, where the metric corresponds to the measure of a horn angle. Consider the simplest case $n=1$. The metric in the Kasner plane is $ds=dx^2/dy$. In the present paper, the author begins the study of the differential geometry of the Kasner plane. The concept of curvature and the theory of evolutes and involutes are developed. Also the arc curvature is considered. This is the limit of the ratio of arc to chord. (Received October 11, 1945.)

34. G. B. Huff: *Inequalities connecting solutions of Cremona's equations.*

An integer solution $x=(x_0; x_1, \cdots, x_\rho)$ of Cremona's equations $x_0^2-x_1^2-\cdots-x^\rho=d+p-1$, $3x_0-x_1-\cdots-x_\rho=d-p+1$ is said to be *proper* if x_0 is the order and $x_1, \cdots, x_\rho$ are the multiplicities, at a set of ρ points in the plane, of a *complete* and *regular* linear system of plane curves of dimension d and genus p. Inequalities of the following type are established. If x is a solution of Cremona's equations with $x_0>0$, $p\geqq 0$, $d\geqq 0$ and c is a proper characteristic for $p=0$, $d=2$, then $c_0x_0-c_1x_1-\cdots-c_\rho x_\rho\geqq 0$. If f is a proper characteristic for $p=d=0$ and x is a characteristic of $x_0>0$ and p, d non-negative and not both zero and $x_0\leqq f_0$, then $f_0x_0-f_1x_1-\cdots-f_\rho x_\rho\geqq 0$. These inequalities lead to criteria concerning proper solutions. (Received October 16, 1945.)

35. Edward Kasner and John DeCicco: *Problems on perspective maps.*

The authors present some theorems in the perspective mapping of a surface upon a plane from a given point. The only perspective conformalities upon a plane are Ptolemy's stereographic projection of a sphere, and the obvious limiting case of a parallel plane. If more than $3\infty^1$ geodesics are projected into straight lines, then all geodesics become straight lines, and the surface is a sphere, or any plane, parallel or not; furthermore the point of perspectivity is at the center of the sphere. The surfaces are classified with respect to the maximum number of geodesics which project into straight lines. There are four distinct classes. The class for which area preserving perspectivities exist is discussed. The complete integrals are cylinders whose directorial curves are defined by a Tchebycheff integral of non-elementary character. Other solutions may be found as envelopes of these cylinders. The sphere is not a solution. (November 20, 1945.)

36. C. E. Springer: *Rectilinear congruences whose developables intersect a surface in its lines of curvature.*

A congruence of lines is referred to a surface S, and the equation of the net of curves in which the developables of the congruence intersect the surface is obtained. This net is called the *intersector net* on S. It is shown that the exhibition of a congruence relative to S for which the intersector net coincides with the lines of curvature net on S requires a solution of a partial differential equation of Laplace. It is also demonstrated that the specification of a congruence for which the intersector net coincides with the asymptotic net on S requires a solution of two partial differential equations of parabolic type. (Received October 11, 1945.)

37. T. Y. Thomas: *Absolute scalar invariants and the isometric correspondence of Riemann spaces.*

Necessary and sufficient conditions for the isometric correspondence of Riemann spaces R_n and $\bar{R}_n$ are given in terms of the equality of absolute scalar invariants of the spaces. In the general case for which the spaces admit a complete set of n functionally independent scalars it is proved that these and a certain derived set of scalars suffice for the solution of the problem. The solution of the correspondence problem is given for spaces of two dimensions which do not admit two functionally independent scalars. (Received October 10, 1945.)

38. T. Y. Thomas: *Topological theory of dynamical systems.*

The projective or topological theory of dynamical systems is concerned with the study of the trajectories independently of their time parameterization. This paper deals with the possible changes in the invariants determining the system which leave the trajectories unaltered. The case of the conservative system is of especial interest and is treated in detail. Under the assumption that the dynamical systems admit essentially a single quadratic or energy integral it is proved that the most general transformation on the coefficients $g_{\alpha\beta}$ of the expression for the kinetic energy and the potential function V is given by $g_{\alpha\beta}=(cw+d)h_{\alpha\beta}$ and $V=(aw+b)/(cw+d)$ where the a, b, c, d are constants. It is shown, moreover, that the property of a conservative system of possessing essentially only one energy integral is invariant under this transformation. The methods can be applied to systems which are not conservative. (Received October 10, 1945.)

LOGIC AND FOUNDATIONS

39. A. R. Schweitzer: *On the genesis of number systems.* I.

This paper aims to effect a gradual transition from the foundations of geometry to postulates for number systems in terms of undefined relations (operations) analogous to concepts previously developed by the author, Amer. J. Math. vol. 31. The first of these sets uses the undefined operation "replacement" (transformation) analogous to that employed in Chapter II of the preceding article, p. 373. A set S of elements (α) is combined into dyads $(\alpha\beta)$ of a set T assumed subject to two types of replacement of dyads by elements, symbolized by $R(\alpha\beta)=\gamma$ and $P(\kappa\lambda)=\mu$. These relations are also expressed, $\gamma R(\alpha\beta)$ and $\mu P(\kappa\lambda)$ or $\gamma=\alpha+\beta$ and $\mu=\kappa\times\lambda$. A closer analogy is attained by assuming, instead of $R(\alpha\beta)=\gamma$, that if $\alpha\xi$, $\beta\xi(\xi\alpha, \xi\beta)$ are in T, then γ exists in S such that $\gamma\xi(\xi\gamma)$ in T replaces $\alpha\xi$, $\beta\xi(\xi\alpha, \xi\beta)$; in symbols, $R(\alpha\xi, \beta\xi)=\gamma\xi$ or $\gamma\xi R(\alpha\xi, \beta\xi)$ or $\gamma\xi=\alpha\xi+\beta\xi$; and so on. The relation $\gamma=\alpha+\beta$ then holds if and only if $\gamma\xi=\alpha\xi+\beta\xi(\xi\gamma=\xi\alpha+\xi\beta)$ for any ξ in S. Postulates in terms of $\gamma R(\alpha\beta)$ and $\mu P(\kappa\lambda)$ are also interpreted as analogous to the author's system ${}^2R_2$ (ibid. p. 382). (Received October 19, 1945.)

40. A. R. Schweitzer: *On the genesis of number systems*. II.

In continuation of the preceding paper, other developments of number systems are: (2) in terms of reflexive, symmetric and transitive relations $\alpha\beta R\gamma\delta(\alpha+\beta=\gamma+\delta)$ and $\kappa\lambda P\mu\nu(\kappa\times\lambda=\mu\times\nu)$ in analogy with the author's relation $\alpha\beta K\gamma\delta$ (ibid. p. 394) and (3) as an elaboration of linear order based on the author's system ${}^1R_1$ (ibid. p. 378). In system (2) the author employs the binary relational symbol $\alpha\beta RP\gamma\delta$ to express $R\alpha\beta=P\gamma\delta$ or $\alpha+\beta=\gamma\times\delta$; $\alpha\beta RR\gamma\delta$ and $\kappa\lambda PP\mu\nu$ are assumed respectively equivalent to $\alpha\beta R\gamma\delta$ and $\kappa\lambda P\mu\nu$. Further assumptions are: 1. $\alpha\beta RP\gamma\delta$ implies $\gamma\delta PR\alpha\beta$. 2. $\alpha\beta XY\lambda\mu$, $\lambda\mu YZ\gamma\delta$ imply $\alpha\beta XZ\gamma\delta$, where X, Y, Z are on the set (R, P). 3. There exist uniquely in S the distinct elements ω (zero) and ϵ (unity) such that $\alpha\omega RP\alpha\epsilon$ for any α in S. 4. $\alpha\omega RP\beta\epsilon$ implies $\alpha=\beta$. 5. For any α, β in S there exist uniquely γ, δ such that $\alpha\beta R\gamma\omega$ and $\alpha\beta P\delta\epsilon$. 6. $\alpha\beta R\xi\omega$ and $\beta\gamma R\eta\omega$ imply $\alpha\eta R\xi\gamma$; and similarly for the relation P. Correspondingly, distributive, inversive and commutative properties are stated. The extension of the preceding system S in analogy with the author's systems ${}^nR_n$ and ${}^nK_n$ $(n=1, 2, 3, \cdots)$ is discussed. (Received October 19, 1945.)

STATISTICS AND PROBABILITY

41. Will Feller: *Note on the law of large numbers and "fair" games.*

An example is exhibited to show that a game can be "fair" in the sense that the expectation of loss vanishes, and nevertheless such that the probability tends to one that after n trials there will be a positive loss L_n; the ratio of L_n to the accumulated entrance fees tends to zero as slowly as one pleases. On the other hand, in the classical Petersburg game entrance fees can be determined so that the game becomes fair in the sense that the probable loss or gain will be of smaller order of magnitude than the accumulated entrance fees. (To appear in the Annals of Mathematical Statistics.) (Received October 4, 1945.)

42. Will Feller: *On the normal approximation to the binomial.*

The goodness of the normal approximation to $T_{\lambda,\nu}=\sum_{k=\lambda}^{\nu} C_{n,k}p^k(1-p)^{n-k}$ is studied with particular reference to the practically important cases of small $np(1-p)$ and of comparatively large λ and ν. Limits of the integral are determined which depend quadratically on λ and ν and are such that the integral will approximate $T_{\lambda,\nu}$ from above or from below. The relative error is also studied. In a sense this paper continues a well known series of studies by Serge Bernstein (the latest in Izvestia Akademiia Nauk SSR, 1943). By the departure from the classical, but arbitrary, use of moments unexpected simplifications are obtained which render S. Bernstein's results more accurate and valid under less stringent conditions. (To appear in the Annals of Mathematical Statistics.) (Received October 4, 1945.)

43. P. R. Halmos: *The theory of unbiased estimation.*

Let $F(P)$ be a real-valued function defined on a subset E of the set D of all probability distributions on the real line. A function f of n real variables is an unbiased estimate of F if for every system $X_1, \cdots, X_n$ of independent random variables with the common distribution P the expectation of $f(X_1, \cdots, X_n)$ exists and equals $F(P)$, for all P in E. Under the assumption that E contains all purely discontinuous distributions, the class of all functions $F(P)$ which possess an unbiased estimate is characterized and all unbiased estimates of each such F are described. It is shown that there is in every case a unique symmetric unbiased estimate which coincides also with the unbiased estimate of least variance. Thus the classical estimates of the mean and the variance are justified from a new point of view, and also computable estimates of all higher moments are presented. It is interesting to note that for n greater than 3 neither the sample nth moment about the sample mean nor any constant multiple thereof is an unbiased estimate of the nth moment about the mean. (Received October 6, 1945.)

44. Isaac Opatowski: *Markoff chains and Tchebychev polynomials.*

Let the possible states be $0, 1, \cdots, n+1$ and the only transitions possible during any time dt $(i-1 \to i)$ for $1 \leqq i \leqq n+1$ and $(i+1 \to i)$ for $0 \leqq i \leqq n-1$. Let the conditional probabilities for these transitions be respectively $k_i dt + o(dt)$ and $g_i(dt) + o(dt)$, where $k_i = k$ for $1 \leqq i \leqq n$, $k_{n+1} = k$ or 0, $g_i = g$ or 0, k and g being two positives constants. The probability $P(t)$ of the existence of the state n at a time t if the state 0 existed at $t=0$ is in the general case a convolution of particular functions $P(t)$ corresponding to the following cases: (i) $k_{n+1}=0$, $g_i = g$ $(i \leqq n-1)$; (ii) $k_{n+1}=0$, $g_i=g$ $(i \leqq n-2)$, $g_{n-1}=0$; (iii) $k_{n+1}=k$, $g_i = g$ $(i \leqq n-1)$. In (i), $p(s) = \int_0^\infty e^{-st} P(t) dt = (k/g)^{n/2} / [sU_n(x)]$, where $U_n(x)$ is the Tchebychev polynomial of second kind and x is a linear function of s. The roots of U_n give an explicit expression of $P(t)$ as a linear combination of n exponentials whose coefficients are calculated in a form convenient for computations. In cases (ii) and (iii), $[p(s)]^{-1}$ is a linear combination of two U_i's and the roots of $[p(s)]^{-1}$ are located within narrow ranges, which makes the calculation of $P(t)$ possible within any accuracy desired. These chain processes occur in some biophysical phenomena and the paper will appear in Proc. Nat. Acad. Sci. U.S.A. under a slightly different title. (Received October 11, 1945.)

TOPOLOGY

45. Lipman Bers and Abe Gelbart: *A remark on the Lebesgue-Sperner covering theorem.*

A new and elementary proof is given of a somewhat stronger form of the well known Lebesgue-Sperner covering theorem (Math. Ann. vol. 70 p. 166; Abh. Math. Sem. Hamburgischen Univ. vol. 6 p. 265). Some corollaries are discussed. (Received October 19, 1945.)

46. R. H. Bing: *Solution of a problem of J. R. Kline.*

It is shown that a locally connected, compact, metric continuum S is topologically equivalent to the surface of a sphere provided no pair of points separates S but every simple closed curve separates S. On the assumption that an arc separates S, a simple closed curve is constructed that does not separate S. (Received October 10, 1945.)

47. O. G. Harrold: *The* ULC *property of certain open sets.* I. *Euclidean domains.*

Let M be a compact continuum which separates Euclidean 3-space. If M is deformation-free into a complementary domain A and $p^1(M)=0$, then the fundamental group of A vanishes. By means of this: if M^* is a compact continuum separating 3-space which is deformation-free into a complementary domain A, then A is ULC. If, in addition, $p^1(M^*)=0$ and A is bounded, this implies $\bar{A}$ is a singular 3-cell by a result of S. Eilenberg and R. L. Wilder (Amer. J. Math. vol. 64 (1942) pp. 613–622).

This yields an affirmative solution to a problem of that paper for $n=3$. (Received October 19, 1945.)

48. F. B. Jones: *Concerning the separability of certain locally connected metric spaces.*

The main purpose of this paper is to settle in the affirmative the question: Is a metric space satisfying Axioms 0–4 of R. L. Moore's *Foundations* completely (perfectly) separable? In other words: Must every locally connected complete metric space which contains no cut point and in which the Jordan curve theorem holds true be separable? The answer is obtained rather easily after establishing the following more general theorem: Let S denote a locally connected complete metric space such that no subset of S containing exactly two points cuts S. If S contains no skew curve of type 1, then S is separable. (A skew curve of type 1 is a set formed by the sum of three triods such that no point except an end point of one of them belongs to more than one of them but each such end point belongs to all three of them.) (Received October 15, 1945.)

49. J. L. Kelley and Everett Pitcher: *Applications of natural homomorphism sequences.*

The systematic framework of natural homomorphism sequences introduced by Hurewicz (Bull. Amer. Math. Soc. Abstract 47-7-329) is elaborated and several applications are made. For instance the theory is used to introduce a new set of invariants of a class of continuous maps which includes simplicial maps (compare Bull. Amer. Math. Soc. Abstract 46-5-216 by one of the authors). It is likewise used to prove the duality theorem of Alexander-Pontrjagin. The argument used shows in particular that if X is a compact space whose $(r-1)$- and r-dimensional homology groups vanish and Y is a closed subset, then the $(r-1)$-dimensional homology group of Y is a function of the space $X-Y$. The theory is applied also to obtain principal results about characteristic groups at critical levels in the critical point theory of Morse. (Received October 20, 1945.)

50. S. B. Myers: *Equicontinuous sets of mappings.*

Let X^Y be the set of all continuous mappings of a connected, separable, first countable T_1-space Y into a locally compact, complete metric space X. The author proves that the following three conditions on a subset T of X^Y are equivalent. (1) T is closed in X^Y under the compact-open topology (see R. H. Fox, Bull. Amer. Math. Soc. vol. 51 (1945) p. 429), T is equicontinuous, $T(y)$ is compact for at least one $y\in Y$. (2) T is compact under the compact-open topology. (3) There exists an admissible topology on T under which T is compact (an admissible topology is one in which $t(y)$ is simultaneously continuous in y and t for all $y\in Y$ and all $t\in T$). (1) and (3) are equivalent without the first countability assumption on Y. Now let H be the group (under superposition) of all homeomorphisms of a connected, locally compact, complete metric space X onto itself. Then it is shown that an abstract group can be topologized to become an effective, compact, transformation group on X (see Montgomery and Zippin, Duke Math. J. vol. 4 (1938) p. 363) if and only if it is isomorphic to an equicontinuous subgroup G of H, such that $G(x)$ is compact for at least one $x\in X$ and G is closed in H under the compact-open topology. (Received October 18, 1945.)

51. R. H. Sorgenfrey: *Concerning continua irreducible about n points.*

In this note it is shown that for a (compact) continuum M to be irreducible about n points it is necessary and sufficient that for every proper decomposition of M into $n+1$ subcontinua the sum of some n of these fails to be connected. (Received October 5, 1945.)

52. F. A. Valentine: *Set properties determined by conditions on linear sections.*

Let R_n be an n-dimensional Euclidean space. A linear section of a set $S \subset R_n$ with an $(n-1)$-dimensional hyperplane L_{n-1} is the set $S \cdot L_{n-1}$. The following theorems are the principal ones proved: (1) Let $S \subset R_n$ $(n \geqq 2)$. If each $(n-1)$-dimensional linear section of S is connected and closed, then S is closed; (2) Let $S \subset R_n$ $(n \geqq 3)$. Suppose that relative to each $(n-1)$-dimensional hyperplane L_{n-1}, the set $S \cdot L_{n-1}$ is an open one with a connected complement. Then S is open; (3) If each $(n-1)$-dimensional linear section of $S \subset R_n$ $(n \geqq 2)$ is bounded and connected, then S is bounded and connected. The following theorem is one which is related to those of Liberman: (4) Let $S \subset R_n$ $(n \geqq 3)$. If each two-dimensional plane section $S \cdot L_2$ is a continuum (compact) with a connected complement relative to L_2, then S is convex. This last theorem can be generalized so that S need not be bounded. It should be observed that in the above theorems no hypotheses are placed on S itself. Hypotheses are placed only on linear sections. (Received October 7, 1945.)

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POLARITIES IN FINITE PROJECTIVE PLANES

REINHOLD BAER

Introduction. If π is a finite projective plane,[1] then every line in π carries the same number $n+1$ of points and through every point there pass $n+1$ lines. The total number of points (lines) in π is n^2+n+1. This integer n ($\geqq 2$) shall retain its significance throughout this note. No further requirement is imposed upon π. In particular it is not necessary that the Theorem of Desargues is valid in π.

We are going to use the arithmetical properties of n, and of other suitably selected numbers, for an investigation of the polarities[2] in π. In this fashion we prove the rather surprising fact that there exist always at least $n+1$ absolute points (=points lying on their polars). This lower limit is reached for even n if, and only if, all the absolute points are collinear, and for odd n if, and only if, no line carries more than two absolute points. If the number M of absolute points is greater than $n+1$, then n is a square, and every prime divisor of n is a divisor of $M-1$.

A polarity will be termed regular, if any two lines which carry absolute points, but do not carry their own poles, carry the same number of absolute points. Denoting this number by $i+1$, we find $0<i^2\leqq n$, $i\equiv n$ modulo 2 and $n-1\equiv 0$ modulo $i+1$. If n is even, then $n=i^2$; if n is odd and π Desarguesian, then $i=1$ or $n=i^2$. These results are corollaries to theorems describing the distribution of elliptic and hyperbolic elements[3] for a regular polarity.

As an application of these results we prove the nonexistence of hyperbolic polarities, using this term in a rather comprehensive fashion.[4] This theorem constitutes a generalization of Topel's theorem[5] asserting that every geometry of Bolyai-Lobachevskiĭ is infinite.

1. **The absolute elements.** It is well known[6] that an absolute line[7]

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[1] See, for example, Veblen-Young [5] or Baer [1] for the elementary concepts used. Numbers in brackets refer to the Bibliography at the end of the paper.

[2] For a definition of "polarity" see Veblen-Young [5, p. 263].

[3] We follow Liebmann [3, pp. 36, 37] in introducing these concepts. For their definition, as used here, see §2 below.

[4] The definition of the term will be found in §3. For a more detailed discussion of this new concept, see a forthcoming publication of the author.

[5] Topel [4].

[6] Baer [1, lemma].

[7] Of the polarity under consideration. We are going to omit this phrase in the future.

(= line passing through its pole) carries one and only one absolute point.

Lemma. *If the line h is not an absolute line, then the number of points on h which are not absolute is even.*

Proof. The pole H of h is not on h, since h is not absolute. If P is a point on h, then denote by P' the intersection of h and of the polar of P. The polar of P is then just the line[8] $H+P'$. Thus it follows that P is absolute if, and only if, $P=P'$, and that $P''=P$. The points on h which are not absolute occur therefore in pairs, proving our contention.

Theorem 1. *If n is even, then every line carries an odd number of absolute points.*

Proof. If n is even, then every line carries an odd number of points. It is a consequence of the lemma that an even number of these is not absolute, and the number of absolute points is therefore odd.

Theorem 2. *Assume that n is odd. Then the line h is absolute if, and only if, h carries one and only one absolute point.*

Proof. That absolute lines carry one and only one absolute point has been remarked before. If the line h is not absolute, then it follows from the lemma that the line h carries an even number of absolute points, since h carries an even number of points for odd n. This proves our contention, since 1 is not even.

Remark 1. If n is even, then it may happen that the absolute points are just the $n+1$ points of a suitable line. But it follows from Theorem 1 that every line carries absolute points. Thus there will exist lines which are not absolute, though they carry one and only one absolute point.

We denote by *M the number of absolute points.* This integer M and the invariant n are connected by the following fundamental congruences.

Theorem 3. (a) $M\equiv n+1$ modulo 2.

(b) $(M-n-1)n^{(p^j-1)/2}(n^{(p-1)p^j/2}-1)\equiv 0$ modulo p^{j+1} *for every odd prime number p and every $j\geqq 0$.*

Proof. If m is a positive integer, then we term an *m-cycle* every ordered set[9] of m points $P(1), \cdots, P(m)$ with the property:

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[8] If P and Q are two different points, then $P+Q$ designates the uniquely determined line passing through P and Q.

[9] Note that these points need not be different.

(*) *$P(i)$ is on the polar of $P(i+1)$ for $0<i<m$ and $P(m)$ is on the polar of $P(1)$.*

The m-cycles $P(1), \cdots, P(m)$ and $Q(1), \cdots, Q(m)$ are said to be equal if, and only if, $P(1)=Q(1), \cdots, P(m)=Q(m)$. Finally we denote by *$N(m)$ the number of distinct m-cycles.*

(i) *If $2<m$, then $N(m)=(n^2+n+1)(n+1)^{m-2}+nN(m-2)$.*

To prove this recursion formula we say that the $m-1$ points $P(1), \cdots, P(m-1)$ form an m-chain if $P(i)$ is on the polar of $P(i+1)$ for $0<i<m-1$. Since the point $P(1)$ of an m-chain may be any point in the plane, $P(1)$ may be selected in n^2+n+1 different fashions. If $P(1), \cdots, P(i)$ are already selected, then $P(i+1)$ may be any point on the polar of $P(i)$, and thus $P(i+1)$ may be chosen in $n+1$ different ways. Consequently we have shown that:

(i′) the number $N'(m)$ of different m-chains is $(n^2+n+1)(n+1)^{m-2}$.

If $P(1)=P(m-1)$, then we term the m-chain $P(1), \cdots, P(m-1)$ an m-chain of first kind; and it is an m-chain of second kind if $P(1)\neq P(m-1)$. If this m-chain is of the first kind, then $P(m-2)$ is on the polar of $P(m-1)=P(1)$, proving that the $m-2$ points $P(1), \cdots, P(m-2)$ form an $(m-2)$-cycle. Thus every m-chain of the first kind "begins" with an $(m-2)$-cycle, and every $(m-2)$-cycle is the beginning of one and only one m-chain of the first kind. Hence we have shown that:

(i″) $N(m-2)$ is the number of m-chains of the first kind.

If $P(1), \cdots, P(m-2), P(1)$ is an m-chain of the first kind, then $P(1), \cdots, P(m-2), P(1), P$ is an m-cycle if, and only if, P is on the polar of $P(1)$. Hence there exist exactly $(n+1)N(m-2)$ m-cycles which begin with an m-chain of the first kind.

If $P(1), \cdots, P(m-1)$ is an m-chain of the second kind, then $P(1)\neq P(m-1)$, and the polars of $P(1)$ and $P(m-1)$ meet in a uniquely determined point P. The only m-cycle beginning with the given m-chain of the second kind is, therefore, $P(1), \cdots, P(m-1), P$. Thus the number of m-chains of the second kind equals the number of m-cycles which begin with an m-chain of the second kind. It is a consequence of (i′) and (i″) that this number is

$$(n^2+n+1)(n+1)^{m-2} - N(m-2).$$

Combining the results of the last two paragraphs we find that

$$\begin{aligned} N(m) &= (n+1)N(m-2) + (n^2+n+1)(n+1)^{m-2} - N(m-2) \\ &= (n^2+n+1)(n+1)^{m-2} + nN(m-2), \end{aligned}$$

completing the proof of (i).

(ii) $N(2m+1)=(n+1)^{2m+1}+n^m(M-n-1)$.

If $m=0$, then we have to determine the number $N(1)$ of 1-cycles. But the point $P(1)$ is a 1-cycle if, and only if, $P(m)=P(1)$ is on the polar of $P(1)$, and this is equivalent to saying that $P(1)$ be an absolute point. The number $N(1)$ of 1-cycles is, therefore, just the number M of absolute points. Since the right side of (ii) reduces to M if $m=0$, we have verified (ii) for $m=0$.

We now prove (ii) by complete induction with regard to m. We infer from (i) and the induction hypothesis that

$$\begin{aligned} N(2m+1) &= (n^2+n+1)(n+1)^{2m-1} + nN(2m-1) \\ &= (n^2+n+1)(n+1)^{2m-1} + n(n+1)^{2m-1} \\ &\quad + n^m(M-n-1) \\ &= (n+1)^{2m-1}(n^2+2n+1) + n^m(M-n-1) \\ &= (n+1)^{2m+1} + n^m(M-n-1), \end{aligned}$$

completing the proof of (ii).

(iii) $N(2m)=(n+1)^{2m}+(n+1)n^{m+1}$.

If $m=1$, then the right side of (iii) reduces to $(n+1)(n+1+n^2)$. But this is just the number of 2-cycles $P(1)$, $P(2)$, since $P(1)$ may be selected in n^2+n+1 ways, since $P(2)$ has to be a point on the polar of $P(1)$, and since $P(1)$ is on the polar of $P(2)$ if, and only if, $P(2)$ is on the polar of $P(1)$, so that $P(2)$ may be any one of the $n+1$ points on the polar of $P(1)$.

To prove (iii) now by complete induction with regard to m, we deduce from (i) and the induction hypothesis that

$$\begin{aligned} N(2m+2) &= (n^2+n+1)(n+1)^{2m} + nN(2m) \\ &= (n^2+n+1)(n+1)^{2m} + n(n+1)^{2m} + (n+1)n^{m+2} \\ &= (n+1)^{2m+2} + (n+1)n^{m+2}, \end{aligned}$$

completing the proof of (iii).

(iv) *$N(p^j)\equiv N(p^{j-1})$ modulo p^j for every prime p and $0<j$.*

If $P(1), \cdots, P(p^{j-1})$ is a p^{j-1} cycle, then we may derive from this p^{j-1}-cycle a p^j-cycle by repeating it p times:

$$P(1), \cdots, P(p^{j-1}), \cdots, P(1), \cdots, P(p^{j-1}), \cdots, P(1), \cdots, P(p^{j-1}).$$

Consider now one of the $N(p^j)-N(p^{j-1})$ p^j-cycles which cannot be derived from a p^{j-1}-cycle in the indicated fashion, say the cycle $P(1), \cdots, P(p^j)$. Suppose that i is an integer such that $0\leq i<p^j$ and such that $P(k)=P(k+i)$ for every k where if necessary we reduce $k+i$ modulo p^j. If $i\neq 0$, then $i=i'p^h$ where i' is prime to p and $0\leq h<j$. The permutation: $k\rightarrow k+i$ is a cyclic permutation which

leaves invariant the cycle $P(1), \cdots, P(p^j)$. Every power of this permutation leaves this cycle invariant; and thus this cycle is left invariant by the permutation: $k \rightarrow k+p^h$. But this is impossible, since it would imply that the cycle under consideration could be derived from a p^{j-1}-cycle. Hence our cycle is left invariant by the identity permutation only. But the point sets $P(i), \cdots, P(p^j), P(1), \cdots, P(i-1)$ are all p^j-cycles, and from what we showed just now, it follows that they are all different. Consequently we can divide the set of those p^j-cycles which cannot be derived from p^{j-1}-cycles into mutually exclusive sets containing p^j cycles each. Thus the number of these cycles which is $N(p^j)-N(p^{j-1})$ is divisible by p^j, proving (iv).

From (ii), (iii) and (iv) we infer that

$$M = N(1) \equiv N(2) = (n+1)(n^2+n+1) \text{ modulo } 2.$$

But n^2+n+1 is always an odd number, proving (a).

From (ii) and (iv) we infer that we have for odd primes p

$$(n+1)^{p^j} + n^{(p^j-1)/2}(M-n-1) \equiv N(p^j) \equiv N(p^{j-1})$$
$$\equiv (n+1)^{p^{j-1}} + n^{(p^{j-1}-1)/2}(M-n-1) \text{ modulo } p^j.$$

Since $j \leqq p^{j-1}$ for $0<j$, it follows that the p^{j-1}th and the p^jth power of $n+1$ are divisible by p^j, if $n+1$ is divisible by the prime p. If $n+1$ is prime to p, then these powers of $n+1$ are congruent modulo p^j, since $p^j - p^{j-1}$ is the order of the multiplicative group of classes of residues modulo p^j which are prime to p. Hence it follows from the above congruence that

$$(M-n-1)(n^{(p^j-1)/2} - n^{(p^{j-1}-1)/2}) \equiv 0 \text{ modulo } p^j,$$

and (b) is an immediate consequence of this congruence.

Remark 2. The reader might wonder why we did not state the congruences modulo 2^j which could be deduced from (iii) and (iv) with the exception of the case $j=1$, stated in (a). The reason is that these congruences are satisfied for every integer n—they do not involve M at all—so that they do not give any new information. As a matter of fact for most of our purposes it would suffice to state the case $j=0$ of the congruences (b). But no simplification or abbreviation of our proofs could be obtained by the restriction to the case $j=0$.

The same method which we employed when proving (iv) could be used for a proof of the following generalization of (iv).

(iv′) $N(m) \equiv \sum_d \bar{d} N(m/d)$ modulo m, where $1<m$ and where the summation ranges over all squarefree divisors d of m, the number $\bar{d}$ being $+1$ or -1 according as the number of prime factors of d is odd or even.

But so far we have not made any applications of (iv′) that could not be deduced from Theorem 3, (a) and (b).

THEOREM 4. *If p is a divisor of n, then $M \equiv 1$ modulo p.*

PROOF. For $p=2$ we deduce this from Theorem 3, (a). If p is odd, then considering the special case $j=0$ of Theorem 3, (b) we obtain $0 \equiv (M-n-1)(n^{(p-1)/2}-1) \equiv 1-M$ modulo p, proving our contention.

Remark 3. From $1<n$ it follows that n is divisible by certain primes; and Theorem 4 shows, therefore, the existence of absolute points.[10] We improve upon this statement in the next theorem.

THEOREM 5. *There exist at least $n+1$ absolute points.*

PROOF. If n is even, then we deduce from Theorem 1 the existence of absolute points on every line. Consider a point P which is not absolute (every absolute line carries n of them). Then each of the $n+1$ lines through P carries an absolute point, but no two of these lines through P have an absolute point in common. Thus there exist at least $n+1$ absolute points.

If n is odd, then there exists by Theorem 4—as has been pointed out in Remark 3—an absolute point H. Denote by h the polar of H. Then h passes through H. If k is a line through H, $k \neq h$, then it follows from Theorem 2 that k carries an absolute point, different from H. Since there pass through H exactly n lines apart from h, we have shown the existence of the absolute point H and of n further absolute points.

COROLLARY 1. *No line carries more than two absolute points if, and only if, n is odd and the number M of absolute points is $n+1$.*

PROOF. Suppose first that no line carries more than two absolute points. Since there exist by Theorem 5 at least $n+1$ absolute points, there exist lines carrying exactly two absolute points. It follows from the lemma that n is odd. If H is some fixed absolute point, then we infer from Theorem 2 that H is the one and only absolute point on the polar h of H, and that every line $k \neq h$ passing through H carries one and only one absolute point different from H. Since every absolute point may be connected with H by a line, this implies that the number of absolute points is $n+1$.

Assume conversely that n is odd and $M=n+1$. If the line h carries an absolute point P, then either h is an absolute line and P its pole or else the polar w of P is an absolute line, passing through P, and $w \neq h$. But each of the lines different from w and passing through P carries by Theorem 2 at least one further absolute point. Since there pass through P just n lines different from w, since different lines through P meet in P, and since there exist exactly n absolute points different from P, it follows that the lines through P which are different from w carry just one absolute point different from P. This result applied to h proves that h carries at most two absolute points.

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[10] The corresponding theorem for projectivities is trivial, since every involutorial projectivity possesses fixed points.

COROLLARY 2. *The following properties imply each other:*

(i) *n is even and $M=n+1$.*

(ii) *There exists a line all of whose points are absolute.*

(iii) *All the absolute points are collinear.*

PROOF. Suppose that (i) is satisfied and that the line h carries a point P which is not absolute. It is a consequence of Theorem 1 that every line carries absolute points. The $n+1$ lines through P have no absolute point in common and carry each at least one absolute point which is necessarily different from P. Hence every line through P carries one and only one absolute point, since only $n+1$ absolute points are available. Thus we have shown that a line carries one and only one absolute point if, and only if, not all of its points are absolute. Since there exist, by Theorem 5, at least two different absolute points, and since these may be connected by a line, this line carries absolute points only, and we have shown that (ii) is a consequence of (i).

If h is a line all of whose points are absolute, then it is impossible that a point outside h is absolute,[11] proving that (iii) is a consequence of (ii).

Assume finally the collinearity of the absolute points. Then there exists a line h which carries all the absolute points. Since there are $n+1$ points on h, and since by Theorem 5 there exists at least $n+1$ absolute points, it follows that every point on h is absolute and that $M=n+1$. If k is a line different from h which does not pass through the pole of h, then k is not absolute, though its one and only absolute point is its intersection with h. Hence n is even by the lemma, proving that (i) is a consequence of (iii).

Remark 4. Conditions (ii) and (iii) may be combined into the following condition which is equivalent to them.

(iv) There exists one and only one line all of whose points are absolute.

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[11] See Baer [**1**, lemma].

Remark 5. Corollaries 2 and 1 show the impossibility of improving Theorem 5, since it is easy to construct polarities in finite Desarguesian planes which meet the requirements of these corollaries; see the Appendix to §2.

THEOREM 6. *If there exist more than* $n+1$ *absolute points, then* n *is a square.*

PROOF. If n is not a square, then there exists[12] an infinity of odd prime numbers p such that n is not a quadratic residue modulo p. If p is such a prime number, then n is prime to p and[13] $n^{(p-1)/2} \not\equiv 1$ modulo p. Since $n^{(p-1)/2}-1$ is prime to p, it follows from Theorem 3, (b) that $M \equiv n+1$ modulo p for each odd prime number p such that n is not a quadratic residue modulo p. Since there exists an infinity of such primes, we have $M=n+1$, if n is not a square, as we claimed.

The following alternative proof may be of interest. There exists, if n is not a square, at least one odd prime p such that n is not a quadratic residue modulo p. Then n and $n^{(p-1)p^j/2}-1$ for $0 \leqq j$ are prime to p. Hence it follows from Theorem 3, (b) that $M-n-1 \equiv 0$ modulo p^j for every positive j, proving that $M=n+1$.

THEOREM 7. *There exist at least* $n(n+1)/2$ *points which are not absolute, but are on absolute lines.*[14]

PROOF. There exist by Theorem 4 at least $n+1$ absolute lines. If $h(1), \cdots, h(n+1)$ are $n+1$ distinct absolute lines, then each of them carries n points which are not absolute. The line $h(j)$ meets each of the lines $h(1), \cdots, h(j-1)$ in a point which is not absolute. Thus there are on $h(j)$ at least $n-j+1$ points which are not absolute and which are not on the lines $h(1), \cdots, h(j-1)$. This assures us of the existence of at least $n+(n-1)+ \cdots +1=n(n+1)/2$ distinct points on the absolute lines $h(1), \cdots, h(n+1)$ which are not absolute, proving our contention.

Remark 6. It is a consequence of Theorem 1 of the next section that the lower limit given in the present theorem may actually be reached.

COROLLARY 3. *There exist at least* $(n+1)(n+2)/2$ *points which are on absolute lines.*

This is an immediate consequence of Theorems 4 and 7.

It is of interest to note that the limit given in Theorem 7 accounts for just less than half of the points in the plane, and that more than half of the points of the plane are accounted for by Corollary 3.

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[12] See, for example, Hecke [**2**, p. 199, Satz 147].

[13] See, for example, Hecke [**2**, p. 57, (31)].

[14] Such points will be termed hyperbolic points.

2. **Regular polarities.** It will be convenient to introduce the following notation:[15] If the line h is not absolute, but carries absolute points, then h is *an elliptic line*; dually the point H is termed *a hyperbolic point*, if H is not absolute, though absolute lines pass through H. Clearly the pole of an elliptic line is hyperbolic and the polar of a hyperbolic point is elliptic.

It is a consequence of Theorem 5, §1, that there always exist two different absolute points. The line connecting them is certainly not absolute, since it carries at least two absolute points; and thus there exist always elliptic lines and hyperbolic points. The lines carrying more than one absolute point are all elliptic lines. The converse is true, if n is odd (Theorem 2, §1); but if n is even, then there may exist elliptic lines carrying one (or $n+1$) absolute points (Corollary 2, §1).

The polarity under consideration shall be termed *regular* if any two elliptic lines carry the same number of absolute points. The existence of irregular polarities may be inferred from Corollary 2, §1. On the other side it is easy to show that the polarity is regular whenever n is odd and the Theorem of Desargues (or an even weaker hypothesis) is valid. Whether or not it suffices to assume that n be odd or that the absolute points are not collinear, in order to assure regularity, is still an open question.

As has been pointed out before, there exist always two different absolute points. Assuming now, as we shall do throughout this section, the regularity of our polarity, it follows that the number of absolute points on an elliptic line cannot be less than 2. Thus there exists a positive number i such that *every elliptic line carries exactly* $i+1$ *absolute points*; and it is an immediate consequence of the lemma of §1 that $i \equiv n$ modulo 2.

THEOREM 1. *The number of absolute points is* $in+1$, *the number of hyperbolic points is* $(in+1)n(i+1)^{-1}$, *and the number of points that are neither absolute nor hyperbolic is* $n(n-i^2)(i+1)^{-1}$.

PROOF. The existence of absolute points is a consequence of Theorem 3, §1. If H is an absolute point, h its polar and $k \neq h$ a line through H, then k is not absolute, since absolute lines carry one and only one absolute point. Consequently k is elliptic and carries, therefore, exactly i absolute points which are different from H. Since there pass exactly n lines through H which are different from h, since different lines through H have only H in common, and since every absolute point, not H, is on a line through H which is different from h, it follows that $in+1$ is the number of absolute points.

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[15] Liebmann [3, pp. 36, 37].

If h is some absolute line, then its pole H is the only absolute point on h, and all the other n points on h are hyperbolic. Since every hyperbolic point lies on exactly $i+1$ absolute lines, and since the number of absolute lines has been shown to be $in+1$, it follows that $(in+1)n(i+1)^{-1}$ is the number of hyperbolic points.

There exist altogether n^2+n+1 points of which $in+1$ are absolute and $(in+1)n(i+1)^{-1}$ are hyperbolic. The number of points that are neither absolute nor hyperbolic is therefore

$$\begin{aligned} n^2 + n + 1 - (in + 1) - (in + 1)n(i + 1)^{-1} \\ &= n^2 + n - in - (in + 1)n(i + 1)^{-1} \\ &= [(n + 1 - i)(i + 1) - (in + 1)]n(i + 1)^{-1} \\ &= (n - i^2)n(i + 1)^{-1}. \end{aligned}$$

THEOREM 2. *Every elliptic line carries* $(n-1)i(i+1)^{-1}$ *hyperbolic points and* $(n-i^2)(i+1)^{-1}$ *points that are neither absolute nor hyperbolic.*

PROOF. Denote by j the number of hyperbolic points on the elliptic line h. Each of these lies on exactly $i+1$ absolute lines. Of all the $in+1$ absolute lines (Theorem 1) exactly $i+1$ meet the elliptic line h in absolute points; and thus we have

$$in + 1 - (i + 1) = j(i + 1) \quad \text{or} \quad j = (n - 1)i(i + 1)^{-1},$$

as we claimed.

Since every elliptic line carries altogether $n+1$ points of which $i+1$ are absolute and $(n-1)i(i+1)^{-1}$ are hyperbolic, it follows that the number of points on an elliptic line that are neither absolute nor hyperbolic is exactly

$$\begin{aligned} n + 1 - (i + 1) - (n - 1)i(i + 1)^{-1} \\ &= [(n - i)(i + 1) - (n - 1)i](i + 1)^{-1} = (n - i^2)(i + 1)^{-1}. \end{aligned}$$

COROLLARY 1. (a) $i+1$ *is a divisor of* $n-1$.

(b) $i^2 \leqq n$.

(c) $i^2 \neq n$ *implies* $i^2+i+1 \leqq n$.

(d) $i^2=n$ *if, and only if, every point is either absolute or hyperbolic.*

(e) $i^2=n$ *if* n *is even.*

(f) n *is a square if* $i \neq 1$.

PROOF. It is a consequence of the first statement of Theorem 2 that $i+1$ is a divisor of $(n-1)i$. But i and $i+1$ are relatively prime, showing that $i+1$ is a divisor of $n-1$.

It is a consequence of the second statement of Theorem 2 that $(n-i^2)(i+1)^{-1}$ is a not negative integer. Hence $0 \leqq n-i^2$, proving (b); and $n \neq i^2$ implies $1 \leqq (n-i^2)(i+1)^{-1}$ or $i+1 \leqq n-i^2$, proving (c).

It is a consequence of the last statement of Theorem 1 that $i^2=n$ is equivalent to saying that the number of points that are neither absolute nor hyperbolic is 0, proving (d).

If n is even, then we infer from Theorem 1, §1, that every point is on an absolute line so that every point is either absolute or hyperbolic. Hence (e) is a consequence of (d). (f) is an immediate consequence of Theorem 1 and Theorem 6, §1.

THEOREM 3. *On a line that is neither absolute nor elliptic there are* $(in+1)(i+1)^{-1}$ *hyperbolic points and* $(n+i)(i+1)^{-1}$ *points that are neither absolute nor hyperbolic.*

PROOF. If the line h is neither absolute nor elliptic, then none of the points on h is absolute. Denote by m the number of hyperbolic points on h. Since each hyperbolic point is on exactly $i+1$ absolute lines, and since the number of absolute lines is $in+1$ (Theorem 1), it follows that $in+1=m(i+1)$, proving our first contention.

The number of points on h that are neither absolute nor hyperbolic is therefore

$$\begin{aligned} n + 1 - (in + 1)(i + 1)^{-1} &= [(n + 1)(i + 1) - (in + 1)](i + 1)^{-1} \\ &= (n + i)(i + 1)^{-1}. \end{aligned}$$

It is customary to term lines h and k perpendicular if h passes through the pole of k and k passes, therefore, through the pole of h. A *point* may be termed[16] *elliptic* if it is the intersection of two perpendicular elliptic lines. Dually the polars of elliptic points are termed *hyperbolic lines.*

THEOREM 4. *Every elliptic line carries at least* $(n-1)i(i+1)^{-1}$ *elliptic points.*

PROOF. If the line h is elliptic, then the pole H of h is hyperbolic. Dualizing Theorem 2 we find that there pass exactly $(n-1)i(i+1)^{-1}$ elliptic lines through H. All these are perpendicular to h, and they meet h in $(n-1)i(i+1)^{-1}$ different points which are clearly elliptic, proving our contention.

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[16] This is a change from the terminology used by Liebmann [3, pp. 36, 37].

COROLLARY 2. *There are at least* $(i-1)(i+n)(i+1)^{-1}$ *points on every elliptic line which are at the same time elliptic and hyperbolic.*

PROOF. We make first the obvious remark that absolute points are never elliptic. If h is an elliptic line, then it carries $n+1-(i+1)=n-i$ points that are not absolute, and of these $(n-1)i(i+1)^{-1}$ are hyperbolic (Theorem 2) and at least $(n-1)i(i+1)^{-1}$ are elliptic (Theorem 4). Thus the number of points on h that are at the same time elliptic and hyperbolic is not less than

$$\begin{aligned}-(n-i)+2(n-1)i(i+1)^{-1}\\ &= -[(n-i)(i+1)-2(n-1)i](i+1)^{-1}\\ &= (i^2-i+ni-n)(i+1)^{-1}=(i-1)(i+n)(i+1)^{-1}.\end{aligned}$$

Note that the lower limit given in Corollary 2 is 0 if, and only if, $i=1$.

THEOREM 5. *If* $1<i$, *then every point is hyperbolic or absolute or elliptic.*[17]

PROOF. If the point P is neither absolute nor hyperbolic, then there pass, by Theorem 3, exactly $(in+1)(i+1)^{-1}$ elliptic lines through P. But

$$\begin{aligned}(in+1)(i+1)^{-1}-(n+1)/2\\ &= (i+1)^{-1}[2(in+1)-(i+1)(n+1)]/2\\ &= (i+1)^{-1}(in+1-i-n)/2=(i+1)^{-1}(n-1)(i-1)/2\end{aligned}$$

is certainly a positive number, since $1<i$. Hence $(n+1)/2<(in+1)(i+1)^{-1}$, proving that more than half of the lines through P are elliptic. Since P is not situated on any absolute line, there exist exactly $(n+1)/2$ pairs of perpendicular lines, meeting in P. Since more than half of the lines through P are elliptic, one of these pairs of perpendicular lines must consist of elliptic lines. Hence P is elliptic, proving our theorem.

Remark. If n is even, then it follows from Theorem 1, §1, that every point is on an absolute line. Thus for even n every point is absolute or hyperbolic. Likewise every line is absolute or elliptic, proving that for even n hyperbolic and elliptic points are the same.

THEOREM 6. *The following three conditions are necessary and sufficient for the absence of points that are at the same time hyperbolic and elliptic.*

(a) $i=1$.

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[17] The word "or" used here should not be understood in the exclusive sense.

(b) *If P is an elliptic point on the elliptic line h, then the line through P perpendicular to h is elliptic.*

(c) *Every point is elliptic or absolute or hyperbolic.*

PROOF. Assume first the absence of points that are at the same time hyperbolic and elliptic. Then we infer $i=1$ from Corollary 2. If h is an elliptic line, then its pole H is a hyperbolic point. From $i=1$ and Theorem 2 we infer now that the number of absolute points on h is 2, and that there are exactly $(n-1)/2$ hyperbolic points on h. We deduce from Theorem 4 the existence of not less than $(n-1)/2$ elliptic points on h. Since the classes of elliptic, absolute and hyperbolic points are mutually exclusive, and since there are $n+1=2+(n-1)/2+(n-1)/2$ points on h, we have shown that $(n-1)/2$ is the number of elliptic points on h. It follows from Theorem 2 that there pass through H exactly $(n-1)/2$ elliptic lines. Each of them is perpendicular to h, and they meet h in $(n-1)/2$ different elliptic points. Thus a point P on h is elliptic if, and only if,[18] $P+H$ is elliptic, showing the necessity of (b). If finally the point Q is neither absolute nor hyperbolic, then no absolute line passes through Q. There exist absolute points. Thus, connecting Q to an absolute point, we obtain an elliptic line through Q. But we have shown that the $(n-1)/2$ points on an elliptic line, that are neither absolute nor hyperbolic, are elliptic, proving that Q is elliptic. This completes the proof of the necessity of the three conditions (a) to (c).

We assume conversely the validity of the conditions (a) to (c). If the point P is elliptic, then there exists, by definition, a pair of perpendicular elliptic lines h, k which meet in P. The pole H of h is a hyperbolic point, and we infer from (b) that a point Q on h is elliptic if, and only if, $H+Q$ is an elliptic line. We infer from (a) that there are on h exactly two absolute points, and it follows from Theorem 2 that the number of hyperbolic points on h is $(n-1)/2$, and that the number of elliptic lines through H is $(n-1)/2$. Since the elliptic points on h are just the intersections of h with elliptic lines through H, it follows that $(n-1)/2$ is the number of elliptic points on h. Since there are on h just $n+1=2+(n-1)/2+(n-1)/2$ points, we infer from (c) the impossibility of the existence of an elliptic point on h which is at the same time hyperbolic, completing the proof.

Remark. That (c) is not a consequence of (a) and (b), and that (a) is not a consequence of (b) and (c), will be shown in the Appendix. Whether or not (b) is a consequence of (a) and (c) is still an open question.

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[18] See footnote 8.

COROLLARY 3. *If no point is at the same time hyperbolic and elliptic, and if the two perpendicular lines h and k meet in the point P, then* (i) *P is elliptic, provided both lines h and k are elliptic or h and k are both hyperbolic; and* (ii) *P is hyperbolic, provided one of the lines h and k is elliptic and the other one is hyperbolic.*

PROOF. That P is elliptic if h and k are elliptic is the definition of elliptic points, and that P is hyperbolic if one of the lines h and k is elliptic, though the other one is hyperbolic, may be inferred from Theorem 6, (b). Assume finally that P is a hyperbolic point. Then it follows from Theorem 6, (a) that exactly two absolute lines pass through P; and it follows from Theorem 2 that $(n-1)/2$ is the number of elliptic lines passing through P. Since hyperbolic lines cannot be absolute or elliptic, the number of hyperbolic lines through P is $(n-1)/2$ too. Every elliptic line through P is perpendicular to a hyperbolic line through P, since P is not elliptic, and since lines that are neither absolute nor elliptic are hyperbolic by Theorem 6, (c). Thus every hyperbolic line through P is perpendicular to an elliptic line through P, since elliptic and hyperbolic lines through P are equal in number. The condition, stated under (ii), is therefore necessary and sufficient for P to be hyperbolic, and (i) is an immediate consequence of this fact.

COROLLARY 4. *4 is a divisor of $n+1$ if no point is at the same time elliptic and hyperbolic.*

PROOF. If P is an elliptic point, then we infer from Theorem 6, (a), and Theorem 2 that $(n+1)/2$ elliptic lines pass through P, and we deduce from Corollary 3, (i), that the elliptic lines passing through P occur in pairs of perpendicular lines. Thus $(n+1)/2$ is even and $n+1$ is divisible by 4.

Appendix. The Desarguesian case. To illustrate the preceding considerations we collect here without proofs[19] the main facts for the special case that the Theorem of Desargues holds in the finite projective plane under consideration. Then it is well known that the projective plane may be represented by means of coordinates from a finite field F. This field F is commutative and contains p^m elements for p a suitable prime. Every line carries p^m+1 points so that $n=p^m$.

A. $p=2$.

Then there arise two possibilities. Either all the absolute points are collinear or else the polarity is regular with $i^2=n$.

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[19] Most of the proofs are simple and immediate. Some of them will be given in a forthcoming study of polarities in Desarguesian projective planes.

B. p is odd.

Then the polarity is always regular, and the point P on the elliptic line h is elliptic if, and only if, the line through P perpendicular to h is elliptic. The number i is either 1 or $i^2=n$. If $i=1$, then points are at the same time elliptic and hyperbolic if, and only if, -1 is a square in F, and this is the case if, and only if, p^m-1 is divisible by 4. Thus $n+1$ is divisible by 4 if, and only if, -1 is not a square in F, and we have shown, by Theorem 6, §2:

No point is at the same time elliptic and hyperbolic if, and only if, $i=1$ *and* 4 *is a divisor of* $n+1$.

3. **Hyperbolic polarities.** The polarity under consideration shall be termed *hyperbolic*,[20] if there exists a nonvacuous set of points, called the *interior* points, subject to the following requirements:

(i) *The point P is an interior point if, and only if, the points on its polar are neither interior nor absolute points.*

(ii) *The point P is neither interior nor absolute if, and only if, there exist interior points on its polar.*

(iii)[21] *If the line h passes through the interior point P, then there exists an interior point $Q\neq P$ on h.*

(iv) *If h and k are perpendicular lines, and if each of these lines carries interior points, then h and k meet in an interior point.*

It is known[22] that, in general, (iv) is not a consequence of the first three postulates. Whether or not (iv) may be deduced from (i) to (iii) in case the projective plane is finite is still an open question.

It will be convenient to term a line interior if it carries interior points, and to term elements exterior if they are neither absolute nor interior. It is readily seen that the polar of an exterior point is an interior line, that the pole of an exterior line is an interior point, that elliptic points and lines are interior, and that hyperbolic points and lines are exterior.

The main result of this section is the following theorem.

THEOREM.[23] *There does not exist a hyperbolic polarity in a finite plane.*

The proof will be effected in several steps.

I. *Any two interior lines carry the same number of interior points.*

PROOF. Assume first that the interior lines h and k are not perpendicular. We may assume without loss in generality that the number m of interior points on h is not less than the number m' of interior points on k. Denote by K the pole of k. If J is an interior point on h, then $J \neq K$, since K is exterior, $J+K$ is an interior line perpendicular to k; and it follows from (iv) that $J+K$ and k meet in an interior point J'. If J and Q are different interior points on h, then K is not on $J+Q=h$ so that the interior lines $K+J$ and $K+Q$, perpendicular to k, meet k in different points J' and Q' respectively. Thus there exist at least m interior points on k, proving $m' \leqq m \leqq m'$ or $m=m'$. This proves our contention for interior lines that are not perpendicular.

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[20] For a justification and discussion of this term, see a forthcoming publication of the author.

[21] No use will be made of this condition in the proof of the theorem below.

[22] A proof of this fact will be given in a forthcoming publication.

[23] This is an extension of Topel's theorem that the planes of Bolyai-Lobachevskiĭ are infinite; see Topel [4].

Assume now that h and k are perpendicular interior lines. They meet by (iv) in an interior point P. Every line through P is interior, and there exist $n+1$ lines through P. Hence there exists an interior line q through P which is different from h as well as k. Since h, k, q are copunctual, and since h and k are perpendicular, q is not perpendicular to either h or k. Hence it follows from what we showed in the first paragraph of this proof that the number of interior points on h is the same as the number of interior points on q is the same as the number of interior points on k, and this completes the proof of I.

We infer from I the existence of an integer m ($\geqq 0$) such that:

I'. *Every interior line carries exactly* $m+1$ *interior points.*

Next we prove:

II. *Every interior line carries exactly* $m+1$ *exterior points.*

Proof. If P is an interior point on the interior line h, then the polar of P carries only exterior points by (i). Thus the polar of P meets h in the exterior point P'. If H is the pole of h, then H is not on h, and $H+P'$ is the polar of P. Since the polars of points on h have only the pole H of h in common, one verifies now that h carries at least $m+1$ exterior points.

If Q is an exterior point on h, then its polar is an interior line (by definition of exterior point) which passes through the pole H of h. Thus the polar of Q is perpendicular to h and interior. It follows from (iv) that the polar of Q meets h in an interior point Q'. One verifies as before that the number $m+1$ of interior points on h cannot be less than the number of exterior points on h. Our contention II is an immediate consequence of the results of the two paragraphs of this proof.

III. *Every interior line carries* $n-2m-1$ *absolute points.*

Proof. Every line carries $n+1$ points. Every point on an interior line is either interior or absolute or exterior, and these three classes are mutually exclusive. Now III is an immediate consequence of I′ and II.

III′. *The polarity is regular with* $i=n-2m-2$.

This is an almost immediate consequence of III, since exterior lines do not carry absolute points.

It has been pointed out before that elliptic points are interior and hyperbolic points are exterior. We infer from (i) that interior points cannot be exterior. Hence elliptic points cannot be hyperbolic. Thus it follows from III′ and Theorem 6, (a), §2, that $n-2m-2=i=1$, and it follows from III that every interior line carries exactly two absolute points. Since all the $n+1$ lines passing through some interior point are interior, and since these have only an interior point in common, this implies the existence of $2(n+1)$ absolute points. On the other hand it follows from Theorem 1, §2, that there exist exactly $in+1=n+1<2(n+1)$ absolute points, a contradiction proving our theorem.

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CYLINDERS IN A CONE

B. M. STEWART AND F. HERZOG

1. **The two problems.** Let B_0 be the $(k-1)$-volume[1] of a figure that lies in a $(k-1)$-dimensional hyperplane of the k-dimensional euclidean space R_k. Throughout this paper k will be a fixed integer greater than unity. Let Q be any point in R_k, not a point of the hyperplane containing B_0, and let h be the length of the altitude drawn from Q to the hyperplane containing B_0. If Q is joined to each point of B_0 by a line, the resulting figure is a k-dimensional cone whose k-volume V is given by $V=B_0h/k$. If P_0 is the foot of the above altitude, choose n points $P_1, P_2, \cdots, P_n$ on P_0Q in the natural order $P_0, P_1, P_2, \cdots, P_n, Q$. Through P_i $(i=1, 2, \cdots, n)$ draw a hyperplane parallel to B_0 cutting the cone in a $(k-1)$-dimensional figure B_i which is similar to B_0. Let V_{in} be the k-volume of the right cylinder one of whose bases is B_i while the opposite base lies in the hyperplane containing B_{i-1} $(i=1, 2, \cdots, n)$. Let $X_n=V_{1n}+V_{2n}+\cdots+V_{nn}$ and let $x_{1n}, x_{2n}, \cdots, x_{nn}$ be the altitudes of the cylinders $V_{1n}, V_{2n}, \cdots, V_{nn}$, respectively. (Here x_{in} is the length of $P_{i-1}P_i$.)

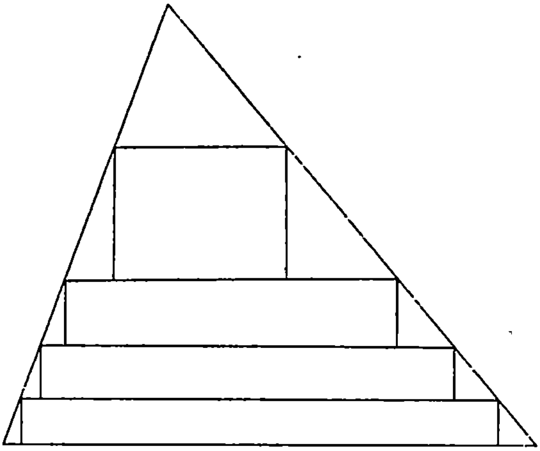

FIG. 1.

Figures 1 and 2 illustrate the cases $k=2$, $n=4$ and $k=3$, $n=3$, respectively.

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[1] By "m-volume" we mean the m-dimensional content; thus 1-volume = length, 2-volume = area, and so on.

Our first problem is to obtain the maximum of X_n for fixed n when no restrictions other than the above are placed upon the V_{in}. Our second problem is to obtain the maximum of X_n for fixed n under the added condition that $V_{1n} = V_{2n} = \cdots = V_{nn}$. We shall refer to these problems hereafter as the *first* and *second problems*, respectively.[2] In order to avoid ambiguity in notation we shall denote those values of the variables $X_n, x_{1n}, x_{2n}, \cdots, x_{nn}$ which correspond to the solution of the first problem by $S_n, s_{1n}, s_{2n}, \cdots, s_{nn}$, respectively, and those values which correspond to the solution of the second problem by $T_n, t_{1n}, t_{2n}, \cdots, t_{nn}$, respectively.

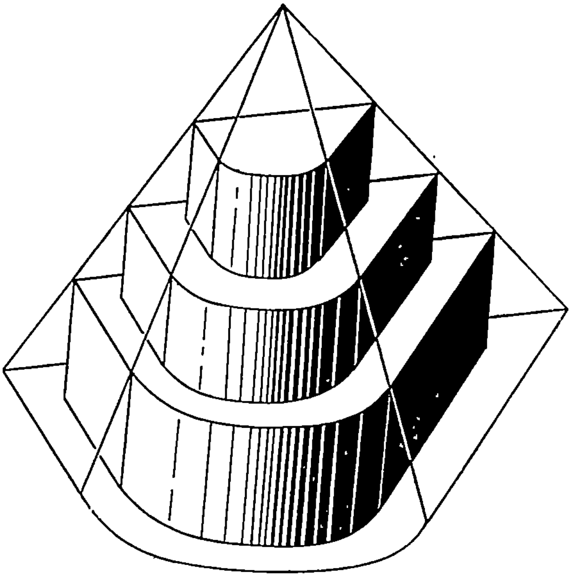

FIG. 2.

In the first problem we show that

$$S_n = y_n^{k-1} V, \tag{1}$$

$$\begin{aligned} s_{1n} &= (1 - y_n)h, \\ s_{in} &= y_n y_{n-1} \cdots y_{n-i+2}(1 - y_{n-i+1})h, \qquad i = 2, 3, \cdots, n; \end{aligned} \tag{2}$$

where the numbers y_n are defined by the recursion formula

$$y_0 = 0, \qquad y_n = (k-1)/(k - y_{n-1}^{k-1}), \qquad n = 1, 2, \cdots . \tag{3}$$

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[2] The second problem for the case $k=2$, $n=2$, but for a quarter-circle rather than a triangle, was treated in the following paper: B. M. Stewart, *Two rectangles in a quarter-circle*, Amer. Math. Monthly vol. 52 (1945) pp. 92–94.

In the second problem we show that

$$T_n = knu_nV, \tag{4}$$

$$t_{in} = u_n[(1 + u_{n-1})(1 + u_{n-2}) \cdots (1 + u_{n-i})]^{k-1}h, \quad i = 1, 2, \cdots, n; \tag{5}$$

where the numbers u_n are defined by the recursion formula

$$u_0 = 1/(k - 1), \qquad u_n = u_{n-1}/(1 + u_{n-1})^k, \qquad n = 1, 2, \cdots. \tag{6}$$

2. **Proof.** The following formulas which will be needed in the proof follow easily from the proportion $B_1/(h-x_{1n})^{k-1} = B_0/h^{k-1}$:

$$V_{1n} = B_0x_{1n}(h - x_{1n})^{k-1}/h^{k-1}, \tag{7}$$

$$V' = B_0(h - x_{1n})^k/kh^{k-1}, \tag{8}$$

where V' is the k-volume of the cone whose base is B_1 and whose altitude P_1Q has the length $h-x_{1n}$.

The proofs of the results for both problems are by induction with respect to n. The two problems are identical when $n=1$, and by elementary calculus it follows readily from (7) together with $X_1 = V_{11}$ that $s_{11} = t_{11} = h/k$ and that $S_1 = T_1 = [(k-1)/k]^{k-1}V$. These results agree with (1), (2), (4), and (5) when $n=1$.

In the first problem we assume that formulas (1) and (2) are correct for $n-1$ where $n \geqq 2$. Let x_{1n} be chosen arbitrarily $(0 < x_{1n} < h)$ and, depending on the choice of x_{1n}, let $x_{2n}, x_{3n}, \cdots, x_{nn}$ be chosen so as to maximize the combined k-volume of the remaining $n-1$ cylinders, namely:

$$X_{n-1}' = V_{2n} + V_{3n} + \cdots + V_{nn}.$$

By the induction hypothesis (see (1)) the maximum of X_{n-1}' is given by $S_{n-1}' = y_{n-1}^{k-1}V'$. We thus obtain, for each choice of x_{1n}, a maximum value of X_n, namely, $X_n = V_{1n} + S_{n-1}'$. Considering this X_n as a function of x_{1n}, namely (see (7) and (8)),

$$X_n = B_0x_{1n}(h - x_{1n})^{k-1}/h^{k-1} + y_{n-1}^{k-1}B_0(h - x_{1n})^k/kh^{k-1}, \tag{9}$$

we see that X_n reaches its maximum value S_n when x_{1n} has the value

$$s_{1n} = \frac{1 - y_{n-1}^{k-1}}{k - y_{n-1}^{k-1}}\,h = \left(1 - \frac{k-1}{k - y_{n-1}^{k-1}}\right)h,$$

or by (3) we may write

$$s_{1n} = (1 - y_n)h, \tag{10}$$

in agreement with (2). Substituting this value of s_{1n} for x_{1n} in (9) and using (3) we obtain (1).

It remains to show (2) for $i = 2, 3, \cdots, n$. Note that S_{n-1}' represents the solution of the first problem for V'. Hence if the altitudes of these $n-1$ cylinders are denoted by $s_{1,n-1}'$, $s_{2,n-1}'$, $\cdots$, $s_{n-1,n-1}'$, we have $s_{in}=s_{i-1,n-1}'$ for $i=2, 3, \cdots, n$ and hence by the induction hypothesis (see (2)) $s_{in}=y_{n-1}y_{n-2}\cdots y_{n-i+2}(1-y_{n-i+1})(h-s_{1n})$. By (10) this establishes (2) for $i=2, 3, \cdots, n$. This completes the proof of (1) and (2).

In the second problem, the case $n=1$ having been disposed of above, we now assume that (4) and (5) are correct for $n-1$, where $n\geqq 2$. We shall choose x_{1n} arbitrarily but such that it is possible to inscribe $n-1$ cylinders of k-volume equal to V_{1n} in V'. By the induction hypothesis (see (4)) this is possible if and only if

$$(n-1)V_{1n} \leqq k(n-1)u_{n-1}V'. \tag{11}$$

By virtue of (7) and (8) and the fact that $0<x_{1n}<h$, the inequality (11) is equivalent to

$$0 < x_{1n} \leqq u_{n-1}h/(1+u_{n-1}). \tag{12}$$

Our problem is therefore to maximize the quantity

$$X_n = nV_{1n} = nB_0x_{1n}(h-x_{1n})^{k-1}/h^{k-1} \tag{13}$$

within the interval (12). But the function X_n increases over the interval $0<x_{1n}\leqq h/k$ and the right-hand end point of the interval (12) is less than h/k. (This follows easily from (6): the u_n form a decreasing sequence of positive numbers, hence $u_{n-1}/(1+u_{n-1})<u_{n-1}\leqq u_1=(k-1)^{k-1}/k^k<1/k$, $n=2, 3, \cdots$.) Therefore the function X_n assumes its maximum in the interval (12) at the right-hand end point, so that we obtain $t_{1n}=u_{n-1}h/(1+u_{n-1})$ and by (6) we may write

$$t_{1n} = u_n(1+u_{n-1})^{k-1}h, \tag{14}$$

$$h - t_{1n} = h/(1+u_{n-1}). \tag{15}$$

Substituting the values of t_{1n} and $h-t_{1n}$ from (14) and (15) for x_{1n} and $h-x_{1n}$ in (13) we obtain (4).

When x_{1n} assumes the value t_{1n} given in (14), the inequality (11) becomes an equation. Consequently, the $n-1$ cylinders V_{2n}, V_{3n}, $\cdots$, V_{nn} must be the solution of the second problem for V'. Hence if the altitudes of these cylinders are denoted by $t_{1,n-1}'$, $t_{2,n-1}'$, $\cdots$, $t_{n-1,n-1}'$, we have $t_{in}=t_{i-1,n-1}'$ for $i=2, 3, \cdots, n$ and hence, by the induction hypothesis (see (5)),

$$t_{in} = u_{n-1}[(1+u_{n-2})(1+u_{n-3})\cdots(1+u_{n-i})]^{k-1}(h-t_{1n}).$$

By (15) and (6) this establishes (5) for $i=2, 3, \cdots, n$. Since (14) agrees with (5) for $i=1$, this completes the proof.

3. Asymptotic formulas for S_n and T_n. The problem arises whether the quantities S_n and T_n, given by (1) and (4), respectively, can be expressed directly in terms of n. This seems possible only for the S_n in the case $k=2$, that is, for the problem of maximizing the combined area of n rectangles inscribed in a triangle. Indeed in this case (3) becomes $y_n=1/(2-y_{n-1})$, which together with $y_0=0$ yields easily by induction that $y_n=n/(n+1)$. Hence from (2) and (1) we obtain $s_{in}=h/(n+1)$ for $i=1, 2, \cdots, n$ and $S_n=nV/(n+1)$ or

$$S_n/V = 1 - 1/(n+1).$$

Thus the problem arises to give at least an asymptotic formula for S_n/V when $k\geqq 3$, as well as an asymptotic formula for T_n/V when $k\geqq 2$.

We begin by establishing an asymptotic formula for the y_n (in the case $k\geqq 3$). We put

$$z_n = 1 - y_n \tag{16}$$

and obtain from (3)

$$z_0 = 1, \qquad 1/z_n = F(z_{n-1}), \qquad n = 1, 2, \cdots, \tag{17}$$

where

$$F(z) = \frac{k-(1-z)^{k-1}}{1-(1-z)^{k-1}}. \tag{18}$$

It is easily established from (3) by mathematical induction that $(k-1)/k\leqq y_n<1$ so that by (16)

$$0 < z_n \leqq 1/k, \qquad n = 1, 2, \cdots. \tag{19}$$

We shall need the two following facts about $F(z)$, defined in (18). In the first place,

$$F(z) = 1 + \frac{k-1}{C_{k-1,1}z - C_{k-1,2}z^2 + \cdots + (-1)^k z^{k-1}}.$$

The terms in the denominator on the right are decreasing in absolute value when $0<z<2/(k-2)$, hence for such values we have $F(z)>1+1/z$, so that in particular by (19), since $1/k<2/(k-2)$,

$$F(z_j) > 1 + 1/z_j, \qquad j = 1, 2, \cdots. \tag{20}$$

Secondly, we conclude from (18) that $F(z)$ is a regular function except for poles at the points $z=0$ and $z=1-\exp[2\pi im/(k-1)]$ with $m=1, 2, \cdots, k-2$, so that $F(z)$ admits of a Laurent expansion in the region $0<|z|<|1-\exp[2\pi i/(k-1)]|=2\sin[\pi/(k-1)]$. This Laurent series is easily seen to be

$$(21) \qquad F(z) = 1/z + k/2 + G(z), \qquad G(z) = \sum_{m=1}^{\infty} a_m z^m,$$

where the power series $G(z)$ converges for $|z|<2\sin[\pi/(k-1)]$. Since $1/k<4/(k-1)\leqq 2\sin[\pi/(k-1)]$ for $k\geqq 3$, we conclude that $G(z)/z$ is bounded for $|z|\leqq 1/k$. The latter fact together with (19) yields

$$(22) \qquad G(z_n) = O(z_n).^3$$

From (17) and (20) we have $1/z_j>1+1/z_{j-1}, j=2, 3, \cdots$. By adding these inequalities from $j=2$ to $j=n$, we obtain $1/z_n>n-1+1/z_1$; hence

$$(23) \qquad z_n = O(1/n).$$

From (17) and (21) we conclude that $1/z_j=1/z_{j-1}+k/2+G(z_{j-1})$, $j=2, 3, \cdots$. By adding these equations from $j=2$ to $j=n$, we obtain

$$(24) \qquad 1/z_n = 1/z_1 + k(n-1)/2 + \sum_{i=1}^{n-1} G(z_i).$$

Applying (22) and (23) to the $G(z_i)$ in (24), we thus obtain

$$1/z_n = kn/2 + O(\log n) = (kn/2)[1 + O(n^{-1}\log n)],$$
$$z_n = (2/kn)[1 + O(n^{-1}\log n)] = 2/kn + O(n^{-2}\log n).$$

Therefore by substituting this result in (1), we obtain the following asymptotic formula, valid for $k\geqq 3$:

$$S_n/V = y_n^{k-1} = (1-z_n)^{k-1} = [1 - 2/kn + O(n^{-2}\log n)]^{k-1},$$
$$S_n/V = 1 - 2(k-1)/kn + O(n^{-2}\log n).$$

To establish an asymptotic formula for T_n/V when $k\geqq 2$, we begin with an asymptotic formula for the u_n. We write (6) in the form $1/u_j=(1+u_{j-1})^k/u_{j-1}$ or

$$(25) \qquad 1/u_j = 1/u_{j-1} + k + C_{k,2}u_{j-1} + \sum_{m=3}^{k} C_{k,m}u_{j-1}{}^{m-1},$$

where the $\sum$ in (25) is to mean zero when $k=2$. By adding these equations from $j=2$ to $j=n$, we obtain

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[3] The notation $f(n)=O(\phi(n))$ is used here to mean that $|f(n)|<A\phi(n)$ for sufficiently large n, where A is independent of n but may depend on k. In particular we shall use the fact that $\sum_{i=1}^{n-1}(1/i)^m$ is of the form $\log n+O(1)$ when $m=1$ and of the form $O(1)$ when $m>1$. Also if $\phi(n)\to 0$ the reciprocal value of $1+O(\phi(n))$ is again of the form $1+O(\phi(n))$.

$$(26)\qquad 1/u_n = 1/u_1 + k(n-1) + C_{k,2}\sum_{i=1}^{n-1} u_i + \sum_{m=3}^{k} C_{k,m}\sum_{i=1}^{n-1} u_i^{m-1}$$

for $n=2, 3, \cdots$. Hence since $u_i>0$ for $i=1, 2, \cdots$, we have $1/u_n>k(n-1)$ and

$$(27)\qquad u_n = O(1/n).$$

Applying (27) to the u_i on the right of (26), we have (see footnote 3)

$$1/u_n = kn + O(\log n) = kn[1 + O(n^{-1}\log n)],$$

$$(28)\qquad u_n = (1/kn)[1 + O(n^{-1}\log n)] = 1/kn + O(n^{-2}\log n).$$

Using (28) in (26) we have, since $\sum_{i=1}^{n-1} i^{-2}\log i = O(1)$,

$$\begin{aligned} 1/u_n &= kn + 2^{-1}(k-1)\log n + O(1) \\ &= kn\left[1 + \frac{k-1}{2k}\,\frac{\log n}{n} + O\left(\frac{1}{n}\right)\right], \\ u_n &= \frac{1}{kn}\left[1 - \frac{k-1}{2k}\,\frac{\log n}{n} + O\left(\frac{1}{n}\right)\right]. \end{aligned}$$

Finally, substituting this result in (4), we obtain the following asymptotic formula:

$$T_n/V = 1 - \frac{k-1}{2k}\,\frac{\log n}{n} + O\left(\frac{1}{n}\right).$$

4. **Table.** In conclusion we append a brief table which indicates how involved the numbers s_{in}, S_n, t_{in}, T_n become, even for small values of k and n.

First problem			Second problem		
	$k=2$	$k=3$		$k=2$	$k=3$
s_{11}/h	1/2	1/3	t_{11}/h	1/2	1/3
S_1/V	1/2	4/9	T_1/V	1/2	4/9
s_{12}/h	1/3	5/23	t_{12}/h	1/5	4/31
s_{22}/h	1/3	6/23	t_{22}/h	2/5	9/31
S_2/V	2/3	324/529	T_2/V	16/25	17496/29791

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COUNTABLE CONNECTED SPACES

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Introduction. Let $\mathfrak{S}$ be the class of all countable and connected perfectly separable Hausdorff spaces containing more than one point. It is known that an $\mathfrak{S}$-space cannot be regular or compact. Urysohn, using a complicated identification of points, has constructed the first example of an $\mathfrak{S}$-space.[1] Two $\mathfrak{S}$-spaces, X and X^*, more simply constructed and not involving identifications, are presented here. The space X^* is a connected subspace of X and contains a dispersion point; that is, the subspace formed from X^* by removing this one point is totally disconnected.

1. **Sequences.** The null sequence or any finite sequence of positive integers will hereafter be called more briefly a sequence. The null sequence or a sequence having an even number of elements is said to be even and a sequence having an odd number of elements is said to be odd. A sequence will usually be denoted by a lower case Greek letter: an arbitrary sequence by α, β, or γ; an arbitrary even sequence by λ, μ, or ν; the null sequence by o. A positive integer will be denoted by a lower case italic letter (not x, y, or z), which may also serve to represent the sequence consisting of that single integer.

The relation $\alpha \geqq i$ signifies that $a \geqq i$ for every element a of α, and $\alpha < i$ that $a < i$ for every element a of α. The null sequence vacuously satisfies both $o \geqq i$ and $o < i$.

The sequence formed by adjoining β to the end of α is denoted by $\alpha\beta$.

DEFINITION. *The relation $\beta \supset_i \alpha$ is to mean that a sequence α' exists such that $\beta = \alpha\alpha'$ and $\alpha' \geqq i$.*

Some immediate consequences of the preceding definitions are:

1.1. $\alpha \supset_i \alpha$.

1.2. *If $\beta \supset_j \alpha$ and $j \geqq i$, then $\beta \supset_i \alpha$.*

1.3. *If $\gamma \supset_i \beta$ and $\beta \supset_i \alpha$, then $\gamma \supset_i \alpha$.*

1.4. *If $\gamma \supset_a \alpha$ and $\gamma \supset_b \beta$, then $\beta \supset_a \alpha$ or $\alpha \supset_b \beta$.*

PROOF. Let $\gamma \supset_a \alpha$ and $\gamma \supset_b \beta$; then sequences α', β' exist such that

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[1] Paul Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. vol. 94 (1925) pp. 262–295; see pp. 274–283 for the example.

$\gamma=\alpha\alpha'$, $\alpha'\geqq a$, and $\gamma=\beta\beta'$, $\beta'\geqq b$. Since $\alpha\alpha'=\beta\beta'$, there exists a sequence α'' such that $\beta=\alpha\alpha''$ or a sequence β'' such that $\alpha=\beta\beta''$. If $\beta=\alpha\alpha''$, then $\alpha\alpha'=\beta\beta'=\alpha\alpha''\beta'$; hence $\alpha'=\alpha''\beta'$. But $\alpha'\geqq a$, so $\alpha''\geqq a$ and consequently $\beta\supset_a\alpha$. Similarly, if $\alpha=\beta\beta''$, then $\alpha\supset_b\beta$.

2. **Points in** X. The space X shall consist of two disjoint subsets: Y, the set of all *even* sequences; and Z, the set of all ordered pairs $\{k, (\mu, \nu)\}$ composed of a positive integer k and a set $(\mu, \nu)=(\nu, \mu)$ of *even* sequences μ and ν. Hereafter a point μ in Y will be denoted by $y(\mu)$ and a point $\{k, (\mu, \nu)\}$ in Z by $z_k(\mu, \nu)$. Evidently X is countable.

The neighborhoods in X will be formed from certain subsets $Y_i(\alpha)$ of Y, defined for every positive integer i and every sequence α.

DEFINITION. *$Y_i(\alpha)$ is the set of all points $y(\mu)$ such that $\mu\supset_i\alpha$.*

Some properties of these sets are:

2.1. $y(\mu)\in Y_i(\mu)$.

2.2. *If $j\geqq i$, then* $Y_j(\alpha)\subset Y_i(\alpha)$.

2.3. *If $y(\mu)\in Y_i(\alpha)$, then* $Y_i(\mu)\subset Y_i(\alpha)$.

2.4. *$Y_a(\alpha)Y_b(\beta)\neq 0$ is equivalent to: $\beta\supset_a\alpha$ or $\alpha\supset_b\beta$.*

PROOF. If the set $Y_a(\alpha)Y_b(\beta)$ contains a point $y(\mu)$, then $\mu\supset_a\alpha$ and $\mu\supset_b\beta$; therefore, $\beta\supset_a\alpha$ or $\alpha\supset_b\beta$.

Now, if $\beta\supset_a\alpha$, define $m=\max(a, b)$, $\nu=\beta$ if β is even, $\nu=\beta m$ if β is odd. Thus ν is even and $\nu\supset_b\beta$ so $y(\nu)\in Y_b(\beta)$. Moreover $\nu\supset_a\beta\supset_a\alpha$, hence $\nu\supset_a\alpha$ so $y(\nu)\in Y_a(\alpha)$. Therefore $Y_a(\alpha)Y_b(\beta)\neq 0$. Similarly $Y_a(\alpha)Y_b(\beta)\neq 0$ if $\alpha\supset_b\beta$.

COROLLARY. *If $\alpha\neq\beta$ and $\alpha\beta<i$, then* $Y_i(\alpha)Y_i(\beta)=0$.

To every point $z=z_k(\mu, \nu)$ a unique positive integer $q(z)=q_k(\mu, \nu)$ is assigned as follows. The set of all sets (μ, ν) of even sequences μ and ν, being countable and infinite, can be mapped onto the set of positive integral primes by some 1-1 mapping $p(\mu, \nu)$. Define

$$q_k(\mu, \nu) = [p(\mu, \nu)]^k$$

According to the unique factorization theorem of arithmetic, q is a 1-1 mapping of the point set Z onto a subset of the positive integers. Moreover, since the infinite sequence of positive integers $q_k(\mu, \nu)$ for $k=1, 2, \cdots$ is strictly increasing, $q_k(\mu, \nu)\to\infty$ as $k\to\infty$.

3. **Neighborhoods in X.** For every point x in X and every positive integer i, a neighborhood V_ix of x is now defined.

DEFINITION.

$$V_iy(\mu) = Y_i(\mu);$$

$$V_iz_k(\mu, \nu) = z_k(\mu, \nu) + Y_i(\mu q) + Y_i(\nu q), \qquad q = q_k(\mu, \nu).$$

Under this definition of neighborhood X forms a Hausdorff topological space; that is, X satisfies the following neighborhood axioms.

Axiom 1. To every point x in X there corresponds at least one neighborhood of x; every neighborhood of x contains x by 2.1 or by definition.

Axiom 2. If V_ix and V_jx are two neighborhoods of x, a neighborhood V_mx of x exists such that $V_mx \subset V_ixV_jx$. Indeed, if $m = \max(i, j)$, then $V_mx = V_jxV_ix$ by 2.2.

Axiom 3. If V_ix contains a point $y(\mu)$, there exists a neighborhood of $y(\mu)$ contained in V_ix. By 2.3 such a neighborhood is $V_iy(\mu)$.

Axiom 4H. Every two distinct points x, x' in X are *Hausdorff- or H-separable*; that is, there exist neighborhoods V_ix of x and V_ix' of x' such that $V_ixV_ix' = 0$. The intersection V_ixV_ix' can be reduced to the sum of at most four intersections, each of the form $Y_i(\alpha)Y_i(\alpha')$. If α, α' are both even, then $\alpha \neq \alpha'$ since $x \neq x'$. And also $\alpha \neq \alpha'$, if α, α' are both odd; for then even sequences μ, μ' and positive integers q, q' exist such that $\alpha = \mu q$, $\alpha' = \mu' q'$, and, since $x \neq x'$, $q \neq q'$. Thus, according to the corollary of 2.4, $Y_i(\alpha)Y_i(\alpha') = 0$ when i is chosen so that $\alpha\alpha' < i$. An integer i then exists for which $V_ixV_ix' = 0$.

Thus X is a nondegenerate countable Hausdorff space. Evidently X is also perfectly separable.

4. **Connectedness of X.** Two distinct points x, x' in a space E are said to be *$\bar{H}$-separable* provided neighborhoods V of x and V' of x' exist such that $\bar{V}\bar{V}' = 0$; otherwise, the points x, x' are said to be *$\bar{H}$-inseparable.* A single point is also said to be $\bar{H}$-inseparable if it is $\bar{H}$-inseparable with every other point in E.

A space E containing an $\bar{H}$-inseparable point x is connected; for otherwise E could be covered by two non-null disjoint isolated (open and closed) sets V, V', one of which contains x; but this would imply the contradiction

$$0 = VV' = \bar{V}\bar{V}' \neq 0.$$

Moreover, if E is a Hausdorff space, then no point of E satisfies the regularity axiom, or, more briefly, is regular. For let x' be any point in E distinct from x. Since x, x' are H-separable in E, there exist disjoint neighborhoods V of x and V' of x'; consequently

$$V\overline{V}' = 0 = \overline{V}V'.$$

If x were a regular point of E, then a neighborhood U of x would exist such that $V \supset \overline{U}$, so

$$0 = V\overline{V}' \supset \overline{U}\overline{V}' \neq 0.$$

Similarly, if x' were a regular point of E, then a neighborhood U' of x' would exist such that $V' \supset \overline{U}'$, so

$$0 = \overline{V}V' \supset \overline{V}\overline{U}' \neq 0.$$

By considering the sets $Y_i(\alpha)$ every point in the space X is now shown to be $\overline{H}$-inseparable. Hence X is connected and no point of X is regular.

DEFINITION. *$Z_i(\alpha)$ is the set of all points $z_k(\mu, \nu)$ such that $\mu q \supset_i \alpha$ or $\nu q \supset_i \alpha$, $q = q_k(\mu, \nu)$.*

4.1. $\overline{Y}_i(\alpha) = Y_i(\alpha) + Z_i(\alpha)$.

PROOF. The following equivalent statements show that $Y\overline{Y}_i(\alpha) = Y_i(\alpha)$:

$y(\mu) \in \overline{Y}_i(\alpha)$.

For all j: $V_j y(\mu) Y_i(\alpha) \neq 0$.

For all j: $Y_j(\mu) Y_i(\alpha) \neq 0$.

For all j: $\alpha \supset_j \mu$ or $\mu \supset_i \alpha$.

$\mu \supset_i \alpha$.

$y(\mu) \in Y_i(\alpha)$.

The following equivalent statements show that $Z\overline{Y}_i(\alpha) = Z_i(\alpha)$, where $q_k(\mu, \nu)$ has been abbreviated to q:

$z_k(\mu, \nu) \in \overline{Y}_i(\alpha)$.

For all j: $V_j z_k(\mu, \nu) Y_i(\alpha) \neq 0$.

For all j: $[Y_j(\mu q) + Y_j(\nu q)]\, Y_i(\alpha) \neq 0$.

For all j: $Y_j(\mu q) Y_i(\alpha) \neq 0$ or $Y_j(\nu q) Y_i(\alpha) \neq 0$.

For all j: $\alpha \supset_j \mu q$ or $\mu q \supset_i \alpha$ or $\alpha \supset_j \nu q$ or $\nu q \supset_i \alpha$.

$\mu q \supset_i \alpha$ or $\nu q \supset_i \alpha$.

$z_k(\mu, \nu) \in Z_i(\alpha)$.

4.2. $Z_a(\alpha) Z_b(\beta) \neq 0$; *hence every two distinct points in X are $\overline{H}$-inseparable.*

PROOF. Evidently there exist even sequences μ, ν such that $\mu \supset_a \alpha$ and $\nu \supset_b \beta$. And since $q_k(\mu, \nu) \to \infty$ as $k \to \infty$ a positive integer k exists for which

$$q = q_k(\mu, \nu) \geqq \max (a, b).$$

Therefore $\mu q \supset_a \alpha$ and $\nu q \supset_b \beta$; so $z_k(\mu, \nu) \in Z_a(\alpha) Z_b(\beta)$.

Thus X is an $\mathfrak{S}$-space whose every point is $\overline{H}$-inseparable.

5. **The space X^*.** Let X^* be the relative subspace of X formed by removing from X all points $z_k(\mu, \nu)$ except those of the form $z_k(\mu, o)$, $\mu \neq o$. Notice that every X^*-neighborhood of a point in X^* is also an X-neighborhood of that point. The argument of 4.2 shows that the set $Z_a(\alpha) Z_b(o)$ contains a point of X^*. The point $y(o)$ is then an $\overline{H}$-inseparable point of X^*. Thus X^*, being a nondegenerate connected subspace of an $\mathfrak{S}$-space, is also an $\mathfrak{S}$-space.

6. **The space X^{**}.** Let X^{**} be the relative subspace of X^* formed by removing from X^* the single point $y(o)$. This point is a dispersion point of X^*; for the following recursive construction of isolated subsets in the space X^{**} shows that X^{**} is totally disconnected.

Definition. *For every non-null even sequence* λ *and every positive integer* i *such that* $\lambda < i$ *let*

$$X_i(\lambda) = \sum_{n=1}^{\infty} [Y_i^n(\lambda) + Z_i^n(\lambda)],$$

the sets $Y_i^n(\lambda)$ *and* $Z_i^n(\lambda)$ *being recursively defined as follows:*

$Y_i^n(\lambda)$ *is the set of all points* $y(\mu)$ *such that* $\mu \supset_i \alpha^n$, *where* $\alpha^n = \lambda$ *if* $n = 1$, *and* $\alpha^n = q(z)$ *for some* $z \in Z_i^{n-1}(\lambda)$ *if* $n > 1$;

$Z_i^n(\lambda)$ *is the set of all points* $z = z_k(\mu, o)$ *such that* $y(\mu) \in Y_i^n(\lambda)$ *and* $q(z) \geqq i$.

6.1. $V_i x \subset X_i(\lambda)$ *for all* $x \in X_i(\lambda)$; *hence* $X_i(\lambda)$ *is open in* X^{**}.

Proof. Let $y(\mu) \in Y_i^n(\lambda)$; then $\mu \supset_i \alpha^n$. If $y(\nu) \in V_i y(\mu)$, then $\nu \supset_i \mu \supset_i \alpha^n$, so $y(\nu) \in Y_i^n(\lambda)$.

Let $z = z_k(\mu, o) \in Z_i^n(\lambda)$; then $\mu \supset_i \alpha^n$ and $q(z) \geqq i$. If $y(\nu) \in V_i z$, then $\nu \supset_i \mu q(z)$ or $\nu \supset_i q(z)$. Now $\nu \supset_i \mu q(z)$, $q(z) \geqq i$, implies that $\nu \supset_i \mu \supset_i \alpha^n$ and hence that $y(\nu) \in Y_i^n(\lambda)$. And $\nu \supset_i q(z)$, $z \in Z_i^n(\lambda)$, implies that $y(\nu) \in Y_i^{n+1}(\lambda)$.

6.2. $V_i x X_i(\lambda) = 0$ *for all* $x \notin X_i(\lambda)$; *hence* $X_i(\lambda)$ *is closed in* X^{**}.

Proof. Let $y(\mu) \notin X_i(\lambda)$. Suppose the set $V_i y(\mu) X_i(\lambda)$ contains a point $y(\nu)$; then $\nu \supset_i \mu$ and $\nu \supset_i \alpha^n$. Therefore $\mu \supset_i \alpha^n$ or $\alpha^n \supset_i \mu$. Now $\alpha^n \not\supset_i \mu$, since $\alpha^1 = \lambda < i$ and since α^n is a single integer if $n > 1$. Hence $\mu \supset_i \alpha^n$, so $y(\mu) \in Y_i^n(\lambda)$—a contradiction.

Let $z = z_k(\mu, o) \notin X_i(\lambda)$. Suppose the set $V_i z X_i(\lambda)$ contains a point

$y(\nu)$; then $\nu \supset_i \mu q(z)$ or $\nu \supset_i q(z)$, and $\nu \supset_i \alpha^n$. Therefore

$$\mathfrak{A}^n\colon \alpha^n \supset_i \mu q(z) \text{ or } \mu q(z) \supset_i \alpha^n \text{ or } \alpha^n \supset_i q(z) \text{ or } q(z) \supset_i \alpha^n.$$

Now $\lambda \neq o$, $\lambda < i$, and $\lambda = \alpha^1$; $\mathfrak{A}^1$ then reduces to $\mu q(z) \supset_i \lambda$; so $\mu \supset_i \lambda$, $q(z) \geqq i$, and consequently $z \in Z_i^1(\lambda)$—a contradiction. If $n > 1$, then $\alpha^n = q(z') \geqq i$ for some $z' \in Z_i^{n-1}(\lambda)$, so $\mathfrak{A}^n$ reduces to: $\mu q(z) \supset_i q(z')$, $\mu \neq o$; or $q(z) = q(z')$. Now $\mu q(z) \supset_i q(z')$, $\mu \neq o$, implies that $\mu \supset_i q(z')$, $q(z) \geqq i$, and hence that $z \in Z_i^n(\lambda)$—a contradiction. And $q(z) = q(z')$ implies that $z = z' \in Z_i^{n-1}(\lambda)$—also a contradiction.

The sets $X_i(\lambda)$ are then isolated subsets of X^{**} for $\lambda \neq o$, $\lambda < i$. Notice that

$$\begin{aligned} x = y(\lambda) &\in X_i(\lambda), && \\ x = y(\lambda') &\notin X_i(\lambda) && \text{if } \lambda' \neq \lambda \text{ and } \lambda' < i, \\ x = z_k(\lambda, o) &\in X_i(\lambda) && \text{if } q(x) \geqq i, \\ x' &\notin X_i(\lambda) && \text{if } x' \in Z \text{ and } q(x') < i. \end{aligned}$$

Now there exists for any two distinct points x, x' in X^{**} an isolated set $X_i(\lambda)$ containing x but not x': if $x = y(\lambda)$, $x' = y(\lambda')$, choose i so that $\lambda\lambda' < i$; if $x = y(\lambda)$ and $x' \in Z$, choose i so that $\lambda q(x') < i$; and if $x = z_k(\lambda, o)$ and $x' \in Z$, choose $i = q(x)$, then $q(x') < i$ and $\lambda < i$, since it may be assumed that $q(x') < q(x)$ and since the mapping p can be selected so that $\mu\nu < p(\mu, \nu)$.

Thus the space X^{**} is totally disconnected. In particular, every two distinct points in X^{**} are $\overline{H}$-separable; hence $y(o)$ is the only $\overline{H}$-inseparable point of X^*.

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ON THE HAUSDORFF DIMENSION OF SOME SETS IN EUCLIDEAN SPACE

PAUL ERDÖS

Let E be a closed set in n-dimensional space, x a point not in E. Denote by $S(x)$ the largest sphere of center x which does not contain any point of E in its interior. Put $\phi(x)=E\cap\bar{S}(x)$. ($\bar{A}$ denotes the closure of A.) Denote by M_k the set of points for which $\phi(x)$ contains k or more linearly independent points (that is, points which do not lie in any $(k-2)$-dimensional hyperplane). M_k is defined for $k\leqq n+1$. In a previous paper I proved that M_2 has n-dimensional measure 0 and conjectured that M_k has Hausdorff dimension not greater than $n+1-k$. In the present note we shall prove this conjecture. In my previous paper I also proved that M_{n+1} is countable, but the proof there given applied only for the case $n=2$; now we are going to give a general proof.

Let R be any set in n-dimensional space. Let $x\in R$. We define the contingent[1] of R at x ($\text{contg}_R\, x$) as follows: The contingent will be a subset of the unit sphere. A point z of the unit sphere belongs to $\text{contg}_R\, x$ if and only if there exists a sequence of points $y_1, y_2, \cdots$ in R converging to x so that the direction of the vector connecting x with y_i tends to the direction of the vector connecting the center of the unit sphere with z. First we state the following lemma.

LEMMA. *Let there be given a set R in n-dimensional space. Assume that for every x, $\text{contg}_R\, x$ does not contain any point of the intersection of the unit sphere with a k-dimensional hyperplane going through its center (the hyperplane can depend on x). Then R is contained in the sum of countably many surfaces of finite $(n-k)$-dimensional measure.*

This lemma is well known.[2]

THEOREM 1. *Let $k<n+1$. Then M_k is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. If $k=n+1$, then M_k is countable.*[3]

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[1] G. Bouligand, *Introduction à la géometrie infinitésimale directe.* Also Saks, *Theory of the integral.*

[2] Saks, ibid. pp. 264–266 and pp. 304–307. Also Roger, C. R. Acad. Sci. Paris vol. 201 (1935) pp. 871–873.

[3] For $n=2$ this theorem is proved by C. Pauc, Revue Scientifique, August, 1939.

Remark. This clearly means that the Hausdorff dimension of M_k $(k \leqq n+1)$ is not greater than $n+1-k$.

Let us first consider the case $k=n+1$. Assume that $x \in M_{n+1}$. Let $z_i \in \phi(x)$, $i=1, 2, \cdots, n+1$, and assume that the z's are linearly independent. Denote by $f(x)$ the maximum value of the volume of the simplices determined by the z's (since $\phi(x)$ is closed the maximum is attained). Define now $N_{n+1}^{(c)}=N$ to consist of all the points $x \in M_{n+1}$ for which $f(x) \geqq c$. It clearly will be sufficient to show that N is countable (for every c). In fact we shall show that N is isolated (in other words no $x \in N$ is a limit point of $N-x$), that is, we shall prove that for every $x \in N$ $\text{contg}_N\, x$ is empty. If this would not hold then N would contain an infinite sequence of points y_j coverging to x so that the direction of the line connecting x with y_j would converge to a fixed direction. Let Z_j be a point of $\phi(x)$ which is closest to y_j, and let A_j be the (unique) hyperplane through Z_j perpendicular to the segment xy_j. It is easy to see that as $j \rightarrow \infty$, A_j converges to a limiting hyperplane A. Moreover it is easily seen that the set $\phi(y_j)$ is ultimately contained in any preassigned neighborhood of A. Thus for large enough j, the volume $f(y_j)$ must be less than c, an evident contradiction; this completes our proof.

Next we prove our theorem in the general case. Let $k \leqq n$ and define M_k' to be the set of all points x for which the maximum number of linearly independent points in $\phi(x)$ is exactly k. It will clearly be sufficient to show that M_k' is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. Let $x \in M_k'$, and let $f(x)$ be the maximum volume of the k-dimensional simplices formed from the points z_i, $i \leqq k+1$, where $z_i \in \phi(x)$. $x \in M_k'^{(c)}=N'$ if $f(x) \geqq c$. Let $x \in N'$, and z_i, $i \leqq k+1$, be the points which determine a simplex of maximal volume. Then a simple geometrical argument (similar to the previous one) shows that $\text{contg}_{N'}\, x$ consists only of the directions through x which are perpendicular to the hyperplane determined by the z_i's, $i \leqq k+1$. Thus our theorem follows from the lemma.

Let E be a closed set, $x \notin E$. Denote by $g(x)$ the distance of x from E. It has been proved[4] that $g(x)$ has a derivative $-\cos \alpha$ in every direction (x, y), where α is the smallest angle formed by the direction (x, y) with the direction (x, z), z in $\phi(x)$. Clearly if $x \in E$ the derivative of $g(x)$ can be 0. We shall show that the derivative of $g(x)$ is 0 for almost all points of E.

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[4] Mises, C. R. Acad. Sci. Paris vol. 205 (1937) pp. 1353–1355. See also Golab, ibid. vol. 206 (1938) pp. 406–408 and Bouligand, ibid. vol. 206 (1938) pp. 552–554.

Let $x \in E$. Denote by $S(x, \epsilon)$ the sphere of center x and radius ϵ. Denote by $G(x, \epsilon)$ the greatest distance of the points of $\bar{S}(x, \epsilon)$ from E. We are going to prove the following theorem.

THEOREM 2. *For almost all points of* E *(that is, for all points of* E *except a set of n-dimensional measure* 0)

$$\lim G(x, \epsilon)/\epsilon = 0.$$

It is well known that almost all points of E are points of Lebesgue density 1. Let x be such a point, and suppose that

$$\lim G(x, \epsilon)/\epsilon \neq 0.$$

This means that there exists an infinite sequence ϵ_i and points z_i, $z_i \in \bar{S}(x, \epsilon_i)$, $\epsilon_i \to 0$, such that the distance of z_i from E is greater than $c\epsilon_i$, where $c > 0$. But this clearly means that x can not have Lebesgue density 1. This contradiction establishes our theorem.

UNIVERSITY OF MICHIGAN

ON MONOTONE RETRACTABILITY INTO SIMPLE ARCS

G. T. WHYBURN

In recent work on the area of surfaces Radó [1][1] has had occasion to use the following properties as applied to locally connected continua A:

(π) *Every simple arc in* A *is a monotone retract of* A;

(II) *Every monotone image of* A *has property* (π).

Radó has noted that (II) implies (π) and that the sphere and 2-cell each have (II). In this paper it will be shown that (1) for locally connected continua in general, property (II) is equivalent to unicoherence, (2) for plane locally connected continua, property (π) is equivalent to unicoherence, and (3) every closed 2-dimensional connected manifold has property (π).

To clarify our meaning, we recall that a continuum is compact, connected and metric. A continuous mapping $f(A) = B$ on a continuum A is *monotone* provided $f^{-1}(y)$ is a continuum for $y \in B$. If

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$X \subset A$ and a monotone mapping $f(A) = X$ exists such that $f(x) = x$ for every $x \in X$, X is a *monotone retract* of A and f is a *monotone retraction* of f into A.

THEOREM 1. *In order that a locally connected continuum M have property* (II) *it is necessary and sufficient that it be unicoherent.*

PROOF. If M is unicoherent, any simple arc ab in M is a monotone interior retract of the cyclic chain $C(a, b)$ in M by pp. 212–213 of [2]; and since $C(a, b)$ is a monotone retract of M, it follows that a monotone retraction of M onto $C(a, b)$ followed by a monotone retraction of $C(a, b)$ onto ab yields a monotone retraction of M onto ab. Thus M has property (π). Since unicoherence is invariant under monotone mappings, it follows that M also has property (II). Thus the condition is sufficient.

To show the necessity, suppose M has property (II), but, contrary to our assertion, M is not unicoherent. We shall show that some monotone image of M fails to have property (π) and thus arrive at a contradiction. It follows by (3.5) and (3.21), pp. 215–216, of [2] that there exist continua A, B, H and K in M such that $A \cdot B = 0$, $H \cdot K = A + B$, and $H + K = M$. For let f be an interior non-alternating retraction of M onto a simple closed curve J as given by (3.2), (3.5) and (3.21), pp. 214–216, of [2]. Let $x_1, x_2 \in \alpha$, $y_1, y_2 \in \beta$ be chosen so that the pairs x_1, y_2 and x_2, y_1 separate each other on J. Then the continua

$$A = f^{-1}(x_1y_1), \qquad B = f^{-1}(x_2y_2),$$
$$H = f^{-1}(y_1x_1x_2y_2), \qquad K = f^{-1}(x_1y_1y_2x_2)$$

satisfy all our requirements.

Now let us decompose M [3] into the sets A, B and single points of $M - (A + B)$. Let $g(M) = M'$ be the associated mapping for this decomposition. Then g is monotone. However, if $a' = g(A)$, $b' = g(B)$, $H' = g(H)$, $K' = g(K)$ and $a'b'$ is an arc in H' from a' to b', there can exist no monotone retraction $\phi(M') = a'b'$. For if so, $\phi(K')$ would be connected; and since it would contain a' and b', it would also contain a point x' interior to $a'b'$. But then $\phi^{-1}(x') \cdot (K' - a' - b') \neq 0 \neq \phi^{-1}(x') \cdot (H' - a' - b')$ and $\phi^{-1}(x') \subset (K' - a' - b') + (H' - a' - b')$, contrary to the fact that $\phi^{-1}(x')$ is connected.

THEOREM 2. *If X is a compact metric space and $f(X) = Y$ is a monotone retraction, for any component R of $X - Y$ we have*

$$f(\overline{R}) = \overline{R} - R.$$

PROOF. Since obviously $f(\overline{R}) \supset f(\overline{R} - R) = \overline{R} - R$ and $R \cdot f(R) = 0$, we have only to show that $f(R)\subset\overline{R}$. To this end let $y\in f(R)$, let $x\in R\cdot f^{-1}(y)$ and let C_x be the component of $f^{-1}(y)-y$ containing x. Since $f^{-1}(y)$ is a continuum, $\overline{C}_x\supset y$. Also, since f is a retraction we have $Y\cdot f^{-1}(y)=y$ and hence $C_x\cdot Y=0$. Thus $C_x\subset X-Y$. Hence $C_x\subset R$ because $R\supset x$. Accordingly $\overline{R}\supset\overline{C}_x\supset y$, as was to be proven.

COROLLARY 1. *If Y is a monotone retract of a compact metric space X, then for any component R of $X-Y$, $\overline{R}-R$ is a continuum.*

COROLLARY 2. *If X is a locally connected continuum and Y is any monotone retract of X, the boundary of each component of $X-Y$ is a continuum.*

THEOREM 3. *In order that a plane locally connected continuum have property (π) it is necessary and sufficient that it be unicoherent.*

The sufficiency results from Theorem 1, since property (II) implies property (π).

To prove the necessity, let M be any plane non-unicoherent locally connected continuum. We shall show that M cannot have property (π). Now E has at least one non-unicoherent true cyclic element E and it will suffice to show that E contains a simple arc α which is not a monotone retract of E. For if M could be retracted onto α by a monotone retraction $g(M)=\alpha$, g would be monotone on E and hence would retract E monotonically onto α.

Now as E is cyclic and non-unicoherent, it contains simple closed curves J and C such that J is the boundary of the unbounded comcomplementary domain U of E and C is the boundary of a bounded complementary domain R of E. We distinguish three cases.

1. *If $J\cdot C=0$*, then since E is cyclic it contains disjoint arcs a_1b_1 and a_2b_2 such that $a_ib_i\cdot J=a_i$ and $a_ib_i\cdot c=b_i$, $i=1, 2$. Then if b_1xb_2 and b_1yb_2 are the two arcs of C from b_1 to b_2, the arc $\alpha=a_1b_1+b_1xb_2+b_2a_2$ cannot be a monotone retract of E. For if Q is the component of $E-\alpha$ containing $b_1yb_2-(b_1+b_2)$, the boundary of Q is not connected because it contains b_1 and b_2 but does not contain x. Hence, by Corollary 2, α is not a monotone retract of E.

2. *If $J\cdot C$ is a single point p*, there is an arc ab in $E-p$ with $ab\cdot J=a$ and $ab\cdot C=b$. Then if bxp and byp are the two arcs of C from b to p, the arc $\alpha=ab+bxp$ cannot be a monotone retract of E, because again the boundary of the component Q of $E-\alpha$ containing $byp-(a+p)$ is not connected.

3. *If $J\cdot C$ contains two distinct points p and q*, and pxq and pyq are the two arcs of C from p to q, then the arc $\alpha=pxq$ cannot be a

monotone retract of E because the boundary of the component Q of $E-\alpha$ containing $pyq-(p+q)$ is not connected.

THEOREM 4. *Any closed 2-dimensional connected manifold has property* (π).

For let M be such a manifold and let α be any simple arc on M. There exists a 2-cell E on M containing α and such that if J is the edge of E, then $J\cdot\alpha=a$ where $\alpha=ab$. There exists a monotone retraction $g(E)=\alpha$ such that $g(J)=a$. For we may suppose E to be the set $|z|\leqq 1$ and α the set $y=0$, $0\leqq x\leqq 1$ in a complex plane; and the mapping $g: w=|z|$, with the w-plane identical with the z-plane, meets our requirements. Then if we define $g(p)=a$ for all $p\in M-E$, clearly g retracts M monotonically onto α.

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ON WEAKLY ORDERED SYSTEMS

M. RICHARDSON

1. **Introduction.** By a *weakly ordered system* we mean a system D of elements $x, y, \cdots$ with a binary relation $\succ$ such that

$$x \succ y \text{ implies } x \neq y \tag{1}$$

and

$$x \succ y \text{ implies } y \succ x \text{ is false.} \tag{2}$$

The statement "$x \succ y$" may be read "x dominates y." Transitivity is not assumed; a transitive weakly ordered system is a partially ordered system. By a *solution* of a weakly ordered system is meant a set V of elements of D such that (a) $x \in V$ and $y \in V$ implies $x \prec y$ is false and (b) $x \in D - V$ implies $y \succ x$ for some $y \in V$. The concept of solution was introduced in J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton, 1944, where it is proved that a weakly ordered system which is strictly acyclic[1] possesses a solution which is unique, and for which a construction is given. This result suggests the problem of finding conditions for the existence and uniqueness of solutions of weakly ordered systems in general. The simplest examples show that if cycles exist neither the existence nor the uniqueness of solutions can be expected in all cases. For example, the system of three elements $a \succ b \succ c \succ a$ has no solution, while the system of four elements $a \succ b \succ c \succ d \succ a$ has the two solutions (a, c) and (b, d). The purpose of this note is to prove the existence of solutions for certain non-acyclic systems. The proof will itself provide a method of construction for the solutions. Zermelo's axiom of choice, the well-ordering theorem, and transfinite induction will be used. The result presented below is a contribution to the general problem suggested above rather than to the theory of games. For the hypothesis of the theorem below precludes transitivity completely; that is, it precludes the existence of three elements a, b, c, such that $a \succ b$, $b \succ c$, and $a \succ c$. This restriction is too severe for the theory of games, just as is the assumption of transitivity. The problem remains open for weakly ordered systems which are not strictly acyclic but also do not satisfy the hypothesis of the theorem below.

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[1] See loc. cit. pp. 590–600 for definitions and proof.

2. **The theorem.** We shall find it convenient to represent the weakly ordered system D by an oriented 1-complex or linear graph whose vertices are in 1-1 correspondence with the elements of D and such that two vertices x and y are joined by an oriented 1-cell xy, oriented from x toward y if and only if the element corresponding to x dominates the element corresponding to y. This oriented graph will also be denoted by D, as well as its set of vertices, and we shall denote indiscriminately by $x, y, \cdots$ either the elements of D or the vertices of the graph with which they are identified. The graph D of a weakly ordered system is not the most general oriented graph; it contains no loops consisting of an oriented 1-cell whose initial and terminal vertices are the same (condition (1) above), and no two vertices are joined by more than one 1-cell (condition (2) above). A graph is called *even* if all its unoriented[2] 1-cycles contain an even number of 1-cells (or vertices). We shall prove the following theorem.

THEOREM. *If the graph D is even, then solutions exist.*

Of course, the hypothesis of this theorem implies that all cycles, in the sense of von Neumann and Morgenstern, loc. cit., are even, but not conversely.

3. **A lemma on graphs.** We shall use the following known lemma whose proof we include for the sake of completeness and to facilitate the construction of examples by the reader.

LEMMA (KÖNIG).[3] *The set of vertices of an even graph D can be decomposed into two disjoint sets P and Q such that every 1-cell of D joins a vertex of P to one of Q.*

PROOF. Select a definite vertex x of any (connected) component of D. Put a vertex y of the same component into P or Q according as the unoriented 1-chains joining x and y have an odd or even number of 1-cells. If a vertex y could be joined to x by two 1-chains, one with an odd and the other with an even number of 1-cells, then there would exist an unoriented 1-cycle with an odd number of 1-cells, contrary to hypothesis. This may be done for each component of D. This proves the lemma.

4. **Proof of the theorem for finite systems.** For simplicity, we give the proof first for finite systems.

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[2] That is, modulo 2.

[3] D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, Math. Ann. vol. 77 (1916) pp. 453–465. Cf. also D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936.

Let P and Q be two classes selected according to the lemma. Let the elements p of P be ordered by means of a subscript index ranging over a lower segment of (finite) ordinal numbers. Let $P_0=0$ and $Q_0=Q$. We now define sets N_k, P_{k+1}, R_k, Q_{k+1} inductively for $k=0, 1, 2, 3, \cdots$. Let N_k be the set of all elements of $P-P_k$ which are not dominated by any element of Q_k. If $N_k=0$, let $V=P_k\cup Q_k$. If $N_k\neq 0$, let p_{α_k} be the first element of N_k, that is, the element with the lowest ordinal subscript. Let $P_{k+1}=P_k\cup[p_{\alpha_k}]$. Let R_k be the set of all elements of Q_k which are dominated by p_{α_k}. Let $Q_{k+1}=Q_k-R_k$.

Since the set P is finite, it is clear that there is a least ordinal λ for which $N_\lambda=0$. We shall prove that

$$V = P_\lambda \cup Q_\lambda \tag{3}$$

is a solution.

Notice the obvious relations

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\lambda \subseteq P, \tag{4}$$

$$Q = Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_\lambda, \tag{5}$$

(6) $Q=(\bigcup_{\beta<\lambda}R_\beta)\cup Q_\lambda$ where the sets $(\bigcup_{\beta<\lambda}R_\beta)$ and Q_λ are disjoint.

We must prove: (a) if $x\in V$, $y\in V$, then $x\prec y$ is impossible; (b) if $x\in D-V$, then there exists an element $y\in V$ such that $x\prec y$.

Proof of (a). Suppose $x\prec y$.

(i) Suppose x and y belonged both to P_λ or both to Q_λ. Since $P_\lambda\subseteq P$ and $Q_\lambda\subseteq Q$, this would contradict the property of P and Q given in the lemma of König.

(ii) If $x\in P_\lambda$ and $y\in Q_\lambda$, then $x=p_{\alpha_\gamma}\in N_\gamma$ for some $\gamma<\lambda$ and hence is not dominated by any element of Q_γ, and, a fortiori, of Q_λ. This is a contradiction.

(iii) If $x\in Q_\lambda$ and $y\in P_\lambda$, then $y=p_{\alpha_\gamma}$ for some $\gamma<\lambda$. Then $x\in R_\gamma$ and cannot be an element of $Q_{\gamma+1}=Q_\gamma-R_\gamma$, nor, a fortiori, of Q_λ. This is a contradiction.

Proof of (b). Suppose no such y existed.

(iv) If $x\in P\cap(D-V)$, then $x\notin P-P_\lambda$ since $N_\lambda=0$. Hence $x\in P_\lambda$. Therefore $x\in V$. This is a contradiction.

(v) If $x\in Q\cap(D-V)$, then, by (6), $x\in R_\gamma$ for some $\gamma<\lambda$. Then there exists a y such that $x\prec y\in P_{\gamma+1}\subseteq P_\lambda\subseteq V$. This is a contradiction.

This completes the proof.

5. **Proof of the theorem for infinite systems.** We indicate the modifications necessary in the infinite case.

We assume that all finite unoriented cycles have an even number of 1-cells. By the component to which a vertex belongs we mean the set of all vertices which can be joined to it by finite unoriented 1-chains. Then the lemma of König holds as before. Now, let the elements p of P be well-ordered by means of a subscript index ranging over a lower segment of ordinal numbers. The construction of the sets N_k, P_{k+1}, R_k, Q_{k+1} will be as before for finite k. But now it may happen that $N_k=0$ for no finite k.

Consider any ordinal number γ. We shall construct the sets N_γ, P_γ, Q_γ, R_γ, and so on, by transfinite induction.

Case I. If γ is a limit ordinal and $N_\beta=0$ for no $\beta<\gamma$ then let $P_\gamma=\bigcup_{\beta<\gamma}P_\beta$ and $Q_\gamma=\bigcap_{\beta<\gamma}Q_\beta$. Let N_γ be the set of those elements of $P-P_\gamma$ not dominated by any element of Q_γ. Let p_{α_γ} be the first element of N_γ, and let $P_{\gamma+1}=P_\gamma\cup[p_{\alpha_\gamma}]$. Let R_γ be the set of those elements of Q_γ dominated by p_{α_γ}, and let $Q_{\gamma+1}=Q_\gamma-R_\gamma$.

Case II. If γ is not a limit ordinal, it has an immediate predecessor $\gamma-1$. If $N_\beta=0$ for no $\beta<\gamma$, let $p_{\alpha_{\gamma-1}}$ be the first element of $N_{\gamma-1}$. Let $P_\gamma=P_{\gamma-1}\cup[p_{\alpha_{\gamma-1}}]$. Let $R_{\gamma-1}$ be the set of those elements of $Q_{\gamma-1}$ dominated by $p_{\alpha_{\gamma-1}}$. Let $Q_\gamma=Q_{\gamma-1}-R_{\gamma-1}$. Let N_γ be the set of those elements of $P-P_\gamma$ not dominated by elements of Q_γ.

Let π be the ordinal number of the set P in the chosen well-ordering. Note that $\beta<\gamma$ implies $\alpha_\beta<\alpha_\gamma$. There must exist a least ordinal λ such that $N_\lambda=0$, $\lambda\leqq\pi$. Let $V=P_\lambda\cup Q_\lambda$. It is readily seen that (4), (5), and (6) hold whether λ is a limit ordinal or not. The proofs of (a) and (b) can now be seen to proceed as in the infinite case.

BROOKLYN COLLEGE

A RESTRICTED CLASS OF CONVEX FUNCTIONS

CECIL L. WOODS

1. **Introduction.** The relationship of the inequalities

$$I(f, x, h, \alpha) \leqq A(f, x, h, \beta) \tag{1.1}$$

and

$$I(f, x, h, \alpha) \geqq A(f, x, h, \beta) \tag{1.2}$$

to general convex and concave functions respectively has been treated by Radó [1].[1] In his and in this paper, $f(x)$ denotes a positive continuous function defined on a given open interval $x_1<x<x_2$. In $I(f, x, h, \alpha)$ and $A(f, x, h, \beta)$, defined by Radó [1, pp. 267, 268] as follows:

$$\begin{aligned} I(f, x, h, \alpha) &= \left[\frac{1}{2h}\int_{-h}^{h} f(x+\xi)^\alpha d\xi\right]^{1/\alpha}, && \text{if } \alpha \neq 0,\\ I(f, x, h, 0) &= \exp\left[\frac{1}{2h}\int_{-h}^{h} \log f(x+\xi) d\xi\right], \end{aligned} \tag{1.3}$$

$$\begin{aligned} A(f, x, h, \beta) &= \left[\frac{f(x-h)^\beta + f(x+h)^\beta}{2}\right]^{1/\beta}, && \text{if } \beta \neq 0,\\ A(f, x, h, 0) &= \exp\left[\frac{\log f(x-h) + \log f(x+h)}{2}\right]\\ &= [f(x-h)f(x+h)]^{1/2}, \end{aligned} \tag{1.4}$$

x and h satisfy the inequalities $x_1<x-h<x+h<x_2$; α and β are real exponents.

To enable us to express his definitions and results concisely, we define four classes of $f(x)$ as follows:

Let K be the class of all $f(x)$ which are convex on $x_1<x<x_2$.

Let $K_{\alpha\beta}$ be the class of all $f(x)$ which satisfy the inequality (1.1).

Let K^* be the class of all $f(x)$ which are concave on $x_1<x<x_2$.

Let $K^*_{\alpha\beta}$ be the class of all $f(x)$ which satisfy the inequality (1.2).

Then the four sets of pairs (α, β) which Radó defined [1, pp. 269, 281] can be defined by the statements:

Let E be the set of all pairs (α, β) for which $K \subset K_{\alpha\beta}$.

Let $\bar{E}$ be the set of all pairs (α, β) for which $K_{\alpha\beta} \subset K$.

Let E^* be the set of all pairs (α, β) for which $K^* \subset K^*_{\alpha\beta}$.

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[1] Numbers in brackets indicate the references at the end of the paper.

Let $\overline{E}^*$ be the set of all pairs (α, β) for which $K^*_{\alpha\beta} \subset K^*$.

We also use two of the functions of α defined by Radó [1, p. 269, footnote]:

$$\tag{1.5} \begin{aligned} \psi_2 &= \psi_2(\alpha) = (\alpha + 2)/3, && -\infty < \alpha < +\infty, \\ \psi_3 &= \psi_3(\alpha) = \begin{cases} 0, & -\infty < \alpha \leqq -1, \\ \dfrac{\alpha \log 2}{\log(\alpha + 1)}, & -1 < \alpha < \infty,\ \alpha \neq 0, \\ \log 2, & \alpha = 0, \end{cases} \end{aligned}$$

to give his theorems [1, pp. 269, 281, 282]:

THEOREM 1.61. *A pair* (α, β) *belongs to* E *if and only if* $\beta \geqq \max(\psi_2, \psi_3)$.

THEOREM 1.62. *A pair* (α, β) *belongs to* $\overline{E}$ *if and only if* $\beta \leqq \psi_2$.

THEOREM 1.63. *A pair* (α, β) *belongs to* E^* *if and only if* $\beta \leqq \min(\psi_2, \psi_3)$.

THEOREM 1.64. *A pair* (α, β) *belongs to* $\overline{E}^*$ *if and only if* $\beta \geqq \psi_2$.

From Theorems 1.61 and 1.63 it is easily seen that E and E^* do not cover the $\alpha\beta$-plane. This leads us to an investigation of the inequalities (1.1) and (1.2) for pairs (α, β) not included in E or E^* and thus to a restriction on the classes K and K^*. For each $M > 0$, we define a class K_M as follows:

Let K_M be the class of all $f(x)$ such that $2^{-1} \log(B/b) \leqq M$, where B and b are respectively the least upper bound and greatest lower bound of $f(x)$ on $x_1 < x < x_2$.

Then we define two sets of pairs (α, β) by the statements:

Let E_M be the set of all pairs (α, β) for which $KK_M \subset K_{\alpha\beta}$.

Let E_M^* be the set of all pairs (α, β) for which $K^*K_M \subset K^*_{\alpha\beta}$.

As we shall show, sets E_M and E_M^* increase as M decreases, and E and E^* are proper subsets of E_M and E_M^* respectively.

2. A word as to method. The inequality (1.1) can obviously be written

$$\tag{2.1} \log \frac{I(f, x, h, \alpha)}{A(f, x, h, \beta)} \leqq 0.$$

We follow Radó in investigating first the special case where $f(x)$ is a linear function $l(x)$. $I(l, x, h, \alpha)$ can be computed for all α, and we choose a variable v defined by

$$v = \frac{1}{2} \log \frac{l(x+h)}{l(x-h)}, \tag{2.2}$$

by which choice the left member of (2.1) in this case is transformed into a function $\phi(v, \alpha, \beta)$ defined for all v, α, and β. We shall explicitly compute $\phi(v, \alpha, \beta)$ and investigate a number of its significant properties. On the basis of this investigation we shall separate the $\alpha\beta$-plane into point sets for each of which $\phi(v, \alpha, \beta)$ has certain invariant properties for all (α, β) in the set. We shall draw curves $\phi = \phi(v, \alpha, \beta)$ in the $v\phi$-plane which exhibit these invariant properties. Finally we shall state and prove two theorems about E_M and E_M^*.

3. I, A, **and** ϕ **expressed as functions of** v. In the case where $f(x)$ is linear, we let A_0 denote $[l(x-h)l(x+h)]^{1/2}$. Performing the integrations of (1.3), factoring out A_0 from the results and from (1.4), and substituting e^{2v} for $l(x+h)/l(x-h)$ according to (2.2), we obtain, for $v \neq 0$,

$$\begin{aligned} I(l, x, h, \alpha) &= A_0 \left[\frac{\sinh (\alpha+1)v}{(\alpha+1)\sinh v}\right]^{1/\alpha}, && \text{if } \alpha \neq 0, -1, \\ I(l, x, h, -1) &= A_0(\sinh v)/v, \\ I(l, x, h, 0) &= A_0 \exp (v \operatorname{ctnh} v - 1), \end{aligned} \tag{3.1}$$

$$\begin{aligned} A(l, x, h, \beta) &= A_0(\cosh \beta v)^{1/\beta}, && \text{if } \beta \neq 0, \\ A(l, x, h, 0) &= A_0. \end{aligned} \tag{3.2}$$

Since $\phi(v, \alpha, \beta)$ is the logarithm of the ratio of $I(l, x, h, \alpha)$ to $A(l, x, h, \beta)$, we see that $\phi(v, \alpha, \beta)$ is independent of A_0. From (3.1) and (3.2) we have, for $v \neq 0$,

$$\begin{aligned} \phi(v, \alpha, \beta) &= \frac{1}{\alpha} \log \frac{\sinh (\alpha+1)v}{(\alpha+1)\sinh v} - \frac{1}{\beta} \log \cosh \beta v, \\ & \qquad\qquad\qquad\qquad \text{if } \alpha \neq 0, -1; \beta \neq 0, \\ \phi(v, 0, \beta) &= v \operatorname{ctnh} v - 1 - (1/\beta) \log \cosh \beta v, && \text{if } \beta \neq 0, \\ \phi(v, -1, \beta) &= \log ((\sinh v)/v) - (1/\beta) \log \cosh \beta v, && \text{if } \beta \neq 0, \\ \phi(v, \alpha, 0) &= \frac{1}{\alpha} \log \frac{\sinh (\alpha+1)v}{(\alpha+1)\sinh v}, && \text{if } \alpha \neq 0, -1, \\ \phi(v, 0, 0) &= v \operatorname{ctnh} v - 1, \\ \phi(v, -1, 0) &= \log ((\sinh v)/v); \end{aligned} \tag{3.3}$$

and

$$\phi(0, \alpha, \beta) = 0.$$

That $\phi(0, \alpha, \beta) = 0$ follows trivially from (2.2), for $v = 0$ implies that $l(x)$ is constant, and each of the means in (1.3) and (1.4) has the same constant value. We observe that $\phi(v, \alpha, \beta)$ is a continuous function of v, α, and β for all finite values of the independent variables. We also see that $\phi(v, \alpha, \beta)$ is an even function of v. Hence we shall in what follows consider $v > 0$.

4. The set in which $\phi(v, \alpha, \beta)$ is identically zero. We have the following result.

THEOREM 4.1. *In the set of just three points which form the complete intersection of E and E^*, namely*: $(-2, 0)$, $(-1/2, 1/2)$, *and* $(1, 1)$, $\phi(v, \alpha, \beta) \equiv 0$.

The proof for the point $(-2, 0)$ lies in the fact that sinh v is an odd function, and for the other two points in the identity: $\sinh 2v = 2 \sinh v \cosh v$. That there are no other points where $\phi(v, \alpha, \beta) \equiv 0$ follows from the facts developed in the next section.

5. The character of $\phi(v, \alpha, \beta)$ for sufficiently small v. For sufficiently small positive v, and, because of the continuity of $\phi(v, \alpha, \beta)$, for all α and β, we have the following expansion of $\phi(v, \alpha, \beta)$ in power series of v:

$$\begin{aligned}(5.1)\quad \phi(v, \alpha, \beta) &= (\alpha + 2 - 3\beta)v^2/6 \\ &\quad - (\alpha^3 + 4\alpha^2 + 6\alpha + 4 - 15\beta^3)v^4/180 + O(v^6).\end{aligned}$$

This is easily verified by integrating the series for ctnh v and tanh v, and applying the results to (3.3).

From (5.1) we obtain the following theorem:

THEOREM 5.2. *There exists an* $\epsilon > 0$ *such that for* $0 < v < \epsilon$,

(1) $\phi(v, \alpha, \beta) > 0$, *if* $\alpha + 2 - 3\beta > 0$, *or if* $\alpha + 2 - 3\beta = 0$, *and* $-\infty < \alpha < -2$; $-1/2 < \alpha < 1$,

(2) $\phi(v, \alpha, \beta) < 0$, *for all other pairs* (α, β) *except those in* EE^*.

The proof consists in the following remarks: for sufficiently small v,

(1) $\operatorname{sgn} \phi(v, \alpha, \beta) = \operatorname{sgn}(\alpha + 2 - 3\beta)$, if $\alpha + 2 - 3\beta \neq 0$,

(2) $\operatorname{sgn} \phi(v, \alpha, (\alpha + 2)/3) = -\operatorname{sgn}(\alpha - 1)(2\alpha + 1)(\alpha + 2)$, if $\alpha \neq -2$, $-1/2$, or 1.

6. The character of $\phi(v, \alpha, \beta)$ as v approaches infinity. We shall show that $\lim_{v \to \infty} \phi(v, \alpha, \beta)$ is not a continuous function of α and β for all values of α and β.

From definitions we obtain the following identities:

$$\log \sinh v = v + \log (1 - e^{-2v}) - \log 2, \tag{6.1}$$

$$\log \cosh v = v + \log (1 + e^{-2v}) - \log 2, \tag{6.2}$$

$$\operatorname{ctnh} v = 1 + 2/(e^{2v} - 1). \tag{6.3}$$

We use (6.1), (6.2), and (6.3) to express $\phi(v, \alpha, \beta)$ for large positive v. We write only those terms which are significant as $v \to \infty$, and indicate by three dots the omission of two kinds of terms: (1) terms involving exponentials which vanish as $v \to \infty$, and (2) constants and terms in α and β, independent of v, in those cases where $\lim_{v\to\infty}\phi(v, \alpha, \beta)$ is positive infinity or negative infinity.

We have, for $\beta > 0$,

$$\begin{aligned} \phi(v, \alpha, \beta) &= (1/\beta) \log 2 - (1/\alpha) \log (\alpha + 1) + \cdots, && \text{if } \alpha > -1, \alpha \neq 0, \\ \phi(v, 0, \beta) &= (1/\beta) \log 2 - 1 + \cdots, && \\ \phi(v, -1, \beta) &= -\log v + \cdots, && \\ \phi(v, \alpha, \beta) &= -2(\alpha + 1)v/\alpha + \cdots, && \text{if } \alpha < -1; \end{aligned} \tag{6.4}$$

for $\beta = 0$,

$$\begin{aligned} \phi(v, \alpha, 0) &= v + \cdots, && \text{if } \alpha > -1, \alpha \neq 0, \\ \phi(v, 0, 0) &= v + \cdots, && \\ \phi(v, -1, 0) &= v - \log v + \cdots, && \\ \phi(v, \alpha, 0) &= -(\alpha + 2)v/\alpha + \cdots, && \text{if } \alpha < -1; \end{aligned} \tag{6.5}$$

and for $\beta < 0$,

$$\begin{aligned} \phi(v, \alpha, \beta) &= 2v + \cdots, && \text{if } \alpha > -1, \alpha \neq 0, \\ \phi(v, 0, \beta) &= 2v + \cdots, && \\ \phi(v, -1, \beta) &= 2v - \log v + \cdots, && \\ \phi(v, \alpha, \beta) &= -2v/\alpha + \cdots, && \text{if } \alpha < -1. \end{aligned} \tag{6.6}$$

From (6.4), (6.5), and (6.6) we conclude:

THEOREM 6.7. *As v approaches infinity*:

(1) $\lim \phi(v, \alpha, \beta) = +\infty$, *for* (1) $\beta < 0$ *and all* α; *and* (2) $\beta = 0$, $\alpha > -2$;

(2) $\lim \phi(v, \alpha, \beta) = -\infty$, *for* (1) $\beta = 0, \alpha < -2$; *and* (2) $\beta > 0, \alpha \leqq -1$;

(3) $\lim \phi(v, \alpha, \beta) = (1/\beta)\log 2 - (1/\alpha)\log(\alpha + 1)$, *for* $\beta > 0$, $\alpha > -1$, $\alpha \neq 0$;

(4) $\lim \phi(v, 0, \beta) = (1/\beta)\log 2 - 1$, *for* $\beta > 0$.

We observe that for $\alpha > -1$, $\beta > 0$, $\lim_{v\to\infty}\phi(v, \alpha, \beta)$ is a continuous function of α and β, and the limit is positive, zero, or negative according as β is respectively less than, equal to, or greater than $\psi_3(\alpha)$, $\alpha > -1$.

7. The number of extrema of $\phi(v, \alpha, \beta)$ for positive v. We shall define

$$\tag{7.1} \psi_4 = \psi_4(\alpha) = \begin{cases} 0, & -\infty < \alpha \leqq -1, \\ \alpha + 1, & -1 < \alpha \leqq 0, \\ 1, & 0 < \alpha < +\infty, \end{cases}$$

and then state and prove the following theorem.

THEOREM 7.2. *For $v > 0$, and $(\alpha, \beta) \neq (-2, 0)$, $(-1/2, 1/2)$ or $(1, 1)$, there is at most one extremum of $\phi(v, \alpha, \beta)$, and an extremum exists if and only if the pair (α, β) satisfies the inequality*: $\min(\psi_2, \psi_4) < \beta < \max(\psi_2, \psi_4)$.

Except for the three (α, β)-points where $\phi(v, \alpha, \beta)$ vanishes identically, $\phi_v(v, \alpha, \beta)$ has at most one zero for positive v. We shall show this, and determine the point set in which a zero exists. Differentiating $\phi(v, \alpha, \beta)$ in (3.3), we have, for $v \neq 0$,

$$\tag{7.3} \begin{aligned} \phi_v(v, \alpha, \beta) &= (1/\alpha)[(\alpha + 1)\,\text{ctnh}\,(\alpha + 1)v - \text{ctnh}\,v] \\ &\quad - \tanh \beta v, \qquad\qquad \alpha \neq 0, -1;\ \beta \neq 0, \\ \phi_v(v, 0, \beta) &= \text{ctnh}\,v - v\,\text{csch}^2\,v - \tanh \beta v, \qquad \beta \neq 0, \\ \phi_v(v, -1, \beta) &= \text{ctnh}\,v - 1/v - \tanh \beta v, \qquad \beta \neq 0, \\ \phi_v(v, \alpha, 0) &= (1/\alpha)[(\alpha + 1)\,\text{ctnh}\,(\alpha + 1)v - \text{ctnh}\,v], \\ &\qquad\qquad \alpha \neq 0, -1, \\ \phi_v(v, 0, 0) &= \text{ctnh}\,v - v\,\text{csch}^2\,v = (2^{-1}\sinh 2v - v)/\sinh^2 v, \\ \phi_v(v, -1, 0) &= \text{ctnh}\,v - 1/v = (v - \tanh v)/(v \tanh v). \end{aligned}$$

We observe that $\phi_v(v, \alpha, \beta)$ is a continuous function of α and β, for all finite v. It is easily verified that $\lim_{v\to 0}\phi_v(v, \alpha, \beta) = 0$, a fact which is more easily seen from (5.1). Expressing $\phi_v(v, \alpha, \beta)$ in terms of hyperbolic sines and cosines only, combining the terms into one fraction, and simplifying the numerators by replacing products of hyperbolic functions by equivalent sums [2, p. 129], we obtain

$$\tag{7.4} \phi_v(v, \alpha, \beta) = \frac{\alpha \sinh (\alpha - \beta + 2)v - (\alpha + 1)\sinh (\alpha - \beta)v - \sinh (\alpha + \beta)v}{2\alpha \sinh (\alpha + 1)v \sinh v \cosh \beta v},$$

$$\alpha \neq 0, -1;\ \beta \neq 0,$$

$$
\begin{aligned}
&\phi_v(v,0,\beta)=\frac{\sinh(2-\beta)v+\sinh\beta v-2v\cosh\beta v}{2\sinh^2 v\cosh\beta v}, && \beta\neq 0,\\
(7.4)\quad &\phi_v(v,-1,\beta)=\frac{2v\cosh(\beta-1)v-\sinh(\beta+1)v+\sinh(\beta-1)v}{2v\sinh v\cosh\beta v}, && \beta\neq 0,\\
&\phi_v(v,\alpha,0)=\frac{\alpha\sinh(\alpha+2)v-(\alpha+2)\sinh\alpha v}{2\alpha\sinh(\alpha+1)v\sinh v}, && \alpha\neq 0,-1.
\end{aligned}
$$

We do not modify $\phi_v(v, 0, 0)$ and $\phi_v(v, -1, 0)$ in this way, for we easily see from the final expressions for them in (7.3) that they are never zero for $v>0$.

We remark that $\phi_v(v, \alpha, \beta)$ has a zero for positive v if and only if the numerator in the right member of an equation of (7.4) is zero for some positive value of v. Letting $N(v, \alpha, \beta)$ denote this numerator, we may write, for $\alpha\neq 0, -1; \beta\neq 0$,

$$
(7.5)\quad
\begin{aligned}
2N(v,\alpha,\beta) &= \alpha e^{(\alpha-\beta+2)v}-\alpha e^{-(\alpha-\beta+2)v}+(\alpha+1)e^{(\beta-\alpha)v}\\
&\quad-(\alpha+1)e^{-(\beta-\alpha)v}+e^{-(\alpha+\beta)v}-e^{(\alpha+\beta)v}.
\end{aligned}
$$

We observe that $N(v, \alpha, \beta)$ is identically zero for all v if (α, β) is in EE^*. For all other (α, β) pairs, the equation $N(v, \alpha, \beta)=0$ has at most five roots counting each root with its proper multiplicity [3, (77), p. 49]. $N(v, \alpha, \beta)$ is obviously an odd function of v, and we shall show that $N(v, \alpha, \beta)=0$ has a triple root for $v=0$, and hence it has at most one positive root. Using the series for hyperbolic sine, we expand $N(v, \alpha, \beta)$ in powers of v, the linear terms vanish, and we have

$$
(7.6)\quad
\begin{aligned}
&N(v,\alpha,\beta)=4\alpha(\alpha+1)(\alpha+2-3\beta)v^3/3!+O(v^5),\\
&\qquad\qquad\qquad\qquad \text{if } \alpha\neq 0,\ -1;\ \beta\neq 0;\ \alpha+2-3\beta\neq 0,\\
&N(v,\alpha,(\alpha+2)/3)=-(64/27)\alpha(\alpha+1)(\alpha+2)(\alpha+1/2)(\alpha-1)v^5/5!\\
&\qquad\qquad +O(v^7), \qquad\qquad \text{if } \alpha\neq 0,\ -1.
\end{aligned}
$$

Inspection of (7.6) shows that $N(v, \alpha, \beta)=0$ has a triple root for $v=0$ if $\alpha\neq 0, -1; \beta\neq 0$; and $\alpha+2-3\beta\neq 0$. The multiplicity is five if $\alpha+2-3\beta=0$. Hence a positive root exists only if $\alpha+2-3\beta\neq 0$. Now for sufficiently small v,

$$
(7.7)\quad
\begin{aligned}
&\operatorname{sgn} N(v,\alpha,\beta)=\operatorname{sgn}\alpha(\alpha+1)(\alpha+2-3\beta),\\
&\alpha\neq 0,\ -1;\ \beta\neq 0;\ \alpha+2-3\beta\neq 0,
\end{aligned}
$$

as one sees from (7.6). As v approaches infinity, $N(v, \alpha, \beta)$ is dominated by the term having the largest exponent in the right member of (7.5). Hence as $v\rightarrow\infty$,

$$\operatorname{sgn} N(v, \alpha, \beta) = \operatorname{sgn} \alpha(\alpha + 1)(\psi_4 - \beta), \qquad \alpha \neq 0, -1. \tag{7.8}$$

Plotting the signs of $N(v, \alpha, \beta)$ for small v and for large v, in the $\alpha\beta$-plane, we find that they are opposite in and only in those point sets specified in Theorem 7.2. If $\beta = 0$, 1 or $\alpha+1$, the number of terms of $N(v, \alpha, \beta)$ in (7.5) having different exponents is four, and hence there are at most three roots. Again $v=0$ is a triple root of $N(v, \alpha, \beta) = 0$, and there are no positive roots.

A similar investigation for $\alpha = -1$ or $\alpha = 0$ shows that a positive root exists for $N(v, \alpha, \beta) = 0$, if $0 < \beta < 1/3$ or $2/3 < \beta < 1$ respectively.

8. The number of zeros of $\phi(v, \alpha, \beta)$ for positive v. We state the following result.

THEOREM 8.1. *A zero of $\phi(v, \alpha, \beta)$ exists for positive v if and only if*

$$\min(\psi_2, \psi_3) \leqq \beta \leqq \max(\psi_2, \psi_3), \tag{8.2}$$

where either both equalities hold or neither holds, and the zero is unique if neither equality holds.

The case where both equalities hold follows from Theorem 4.1, for $\psi_2(\alpha)$ intersects $\psi_3(\alpha)$ in $(-2, 0)$, $(-1/2, 1/2)$, and $(1, 1)$. That a zero exists when neither equality holds follows from the continuity of $\phi(v, \alpha, \beta)$ and Theorems 5.2 and 6.7, since $\operatorname{sgn} \phi(v, \alpha, \beta)$ for small v is opposite to that for large v if and only if both inequalities in (8.2) hold. The necessity of the condition (8.2) and the uniqueness of the zero follow from the facts: $\phi(0, \alpha, \beta) = 0$, and when an extremum exists it is unique.

9. The character of $\phi(v, \alpha, \beta)$ at all points of the $\alpha\beta$-plane. In fig-

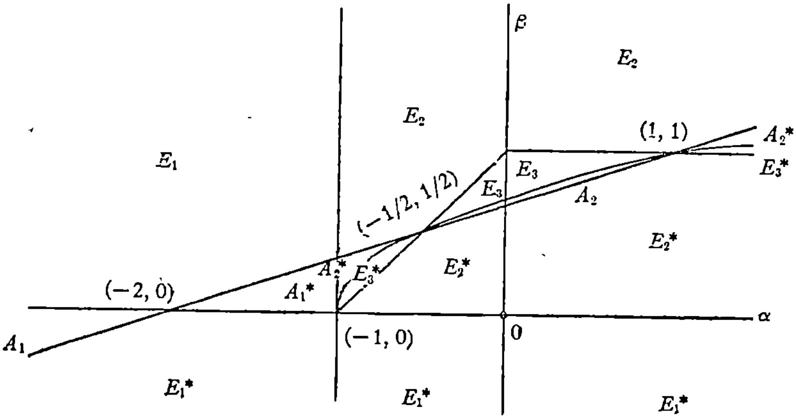

FIG. 1

ure 1 we have drawn the $\alpha\beta$-plane separated into sets and subsets for each of which $\phi(v, \alpha, \beta)$ has the following properties: (1) for sets:

(1) $\operatorname{sgn} \phi(v, \alpha, \beta)$ for sufficiently small v is invariant,

(2) $\operatorname{sgn} \phi(v, \alpha, \beta)$ for sufficiently large v is invariant,

(3) the number of zeros of $\phi(v, \alpha, \beta)$ is invariant,

and in addition to these three, we have (2) for subsets,

(4) $\lim_{v\to\infty}\phi(v, \alpha, \beta)$ is invariantly positive or negative, finite or infinite,

(5) the number of extrema is invariant.

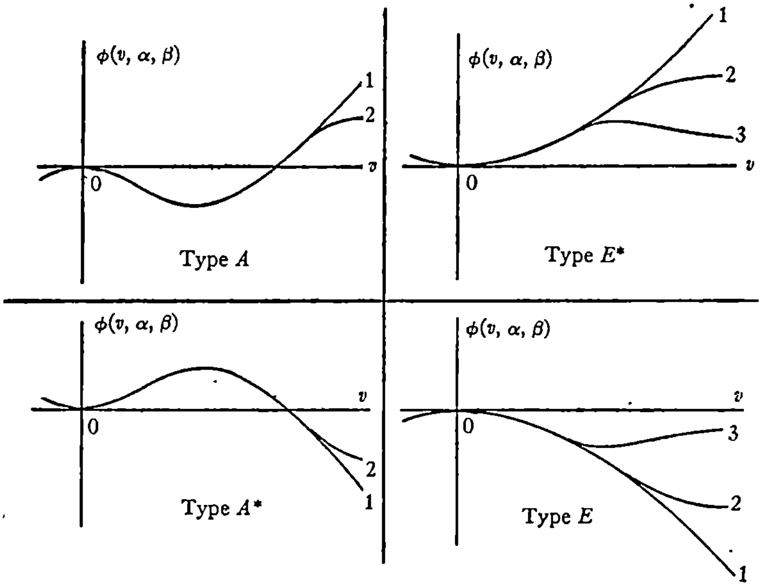

FIG. 2

In figure 2 we have sketched ten curves $\phi = \phi(v, \alpha, \beta)$ in the $v\phi$-plane which exhibit these properties. We shall call these curves the characteristic curves for the sets and subsets of figure 1. The lettering in the two figures is such that the curve marked 1 in type A exhibits the invariant properties of $\phi(v, \alpha, \beta)$ for (α, β) in the set marked A_1, and so on. A brief description of the curves and an exact definition of the point set for each curve follows.

Type A. The characteristic curves for set A (where $\psi_2 < \beta < \psi_3$) have the common properties: a maximum for $v = 0$, a minimum and a zero for positive values of v, and a positive limit as $v \to \infty$. A has two subsets:

$$A_1: -\infty < \alpha < -2, \psi_2 < \beta < \psi_3,$$

and

$$A_2\colon\ -1/2 < \alpha < +1,\ \psi_2 < \beta < \psi_3.$$

In A_1, $\lim_{v\to\infty}\phi(v, \alpha, \beta) = \infty$, but in A_2 the limit is finite.

Type A^*. The characteristic curves for set A^* (where $\psi_3<\beta<\psi_2$) have properties as would a reflection of type A in the v-axis. A^* has two subsets:

$$A_1^*\colon\ -2 < \alpha \leqq -1,\ \psi_3 < \beta < \psi_2,$$

and

$$A_2^*\colon\ -1 < \alpha < -1/2,\ 1 < \alpha < \infty,\ \psi_3 < \beta < \psi_2.$$

Type EE^*. The characteristic curve for the set EE^* is the horizontal axis or v-axis, since for (α, β) in EE^*: $(-2, 0)$, $(-1/2, 1/2)$, $(1, 1)$, $\phi(v, \alpha, \beta) \equiv 0$.

Type E. The characteristic curves for the set $E-EE^*$ have the common properties: a maximum at the origin, and negative values for all v different from zero. There are three subsets:

E_1: $\alpha \leqq -1$, $\beta \geqq \max(\psi_2, \psi_3)$ except for the point $(-2, 0)$ in EE^*,

E_2: $-1<\alpha$, $\beta \geqq \max(\psi_2, \psi_4)$ except for the points $(-1/2, 1/2)$ and $(1, 1)$ in EE^*.

E_3: $-1/2<\alpha<1$, $\psi_3 \leqq \beta < \psi_4$.

In E_1, $\lim_{v\to\infty}\phi(v, \alpha, \beta) = -\infty$. In E_2 and E_3 the limit is finite, and E_3 is distinguished by being the only subset of $E-EE^*$ in which the function has a minimum. When $\beta=\psi_3$ in E_3, $\lim_{v\to\infty}\phi(v, \alpha, \beta)$ is zero.

Type E^*. The characteristic curves for E^*-EE^* have common properties as would a reflection of type E in the v-axis. E^*-EE^* has three subsets:

E_1^*: $-\infty<\alpha<+\infty$, $\beta \leqq \min(\psi_2, 0)$ except for the point $(-2, 0)$,

E_2^*: $-1<\alpha<+\infty$, $0<\beta \leqq \min(\psi_2, \psi_4)$ except for $(-1/2, 1/2)$ and $(1, 1)$.

E_3^*: $-1<\alpha<-1/2$, $\psi_4<\beta \leqq \psi_3$; and $1<\alpha<+\infty$, $\psi_4<\beta \leqq \psi_3$.

10. **The sets E_M and E_M^*.** Radó showed [1, p. 274] that when:

(1) $f(x)$ is a general positive and continuous convex function,

(2) $l(x)$ is the linear function that coincides with $f(x)$ at $x-h$ and $x+h$,

(3) $l(x)$ satisfies the inequality (1.1),

then $f(x)$ satisfies (1.1).

Now for any given $M>0$, the class KK_M is a restricted class of convex functions $f(x)$, containing only those for which $2^{-1}\log(B/b) \leqq M$.

Let us now consider the equation, for any given $M>0$,

$$\phi(M, \alpha, \beta) = 0, \tag{10.1}$$

where ϕ is the function defined in (3.3). We may think of β in (10.1) as an implicit function of α, with M as a parameter, which we shall denote by $\beta(\alpha, M)$. The curve $\beta=\beta(\alpha, M)$ in the $\alpha\beta$-plane lies entirely in the set where (α, β) satisfies the inequalities (8.2) and passes through the points of EE^* for all M. For small M, the curve lies close to the straight line $\beta=\psi_2(\alpha)$; in fact, using (5.1), replacing v by M, setting $\phi(M, \alpha, \beta)=0$, dividing by M^2, we find that $\beta\to(\alpha+2)/3$ as $M\to 0$. For large M, the curve lies close to $\beta=\psi_3(\alpha)$. Approximate values of β for given values of α and M may be easily computed by the use of tables.

We conclude with the theorems:

THEOREM 10.2. *A pair* (α, β) *belongs to* E_M *if and only if* $\beta\geqq\max(\psi_2, \beta(\alpha, M))$.

THEOREM 10.3. *A pair* (α, β) *belongs to* E_M^* *if and only if* $\beta\leqq\min(\psi_2, \beta(\alpha, M))$.

We first prove the sufficiency of the condition in Theorem 10.2. Our hypotheses are: (1) $f(x)$ is in K, (2) $f(x)$ is in K_M, and (3) the pair (α, β) satisfies $\beta\geqq\max(\psi_2, \beta(\alpha, M))$. We wish to prove that: (4) $f(x)$ is in $K_{\alpha\beta}$.

By (3) the pair (α, β) is either in the set E where $\beta\geqq\max(\psi_2, \psi_3)$ or in that part of the set A where $\beta\geqq\beta(\alpha, M)$. If (α, β) is in E it is obviously in E_M and $f(x)$ is in $K_{\alpha\beta}$. If (α, β) is in that part of A where $\beta\geqq\beta(\alpha, M)$ we consider first $\beta=\beta(\alpha, M)$ and then $\beta>\beta(\alpha, M)$.

When $\beta=\beta(\alpha, M)$, $\phi(M, \alpha, \beta)=0$. For the $l(x)$ that coincides with $f(x)$ at $x-h$ and $x+h$, $A(l, x, h, \beta)=A(f, x, h, \beta)$, and by (1), $I(f, x, h, \alpha)\leqq I(l, x, h, \alpha)$. For the same $l(x)$, $v\leqq 2^{-1}\log(B/b)\leqq M$, and therefore $\phi(v, \alpha, \beta)\leqq 0$, for $f(x)$ is in K_M, and the pair (α, β) that we are considering is in A. Hence we have $I(l, x, h, \alpha)\leqq A(l, x, h, \beta)$ and thus $I(f, x, h, \alpha)\leqq A(f, x, h, \beta)$, which proves that $f(x)$ is in $K_{\alpha\beta}$. When $\beta>\beta(\alpha, M)$, $\phi(M, \alpha, \beta)<0$, for $A(f, x, h, \beta)$ is an increasing function of β [4, p. 26], and therefore $\phi(M, \alpha, \beta)$ decreases with increase in β. Thus for all $v\leqq M$, $\phi(v, \alpha, \beta)\leqq 0$, and $f(x)$ is in $K_{\alpha\beta}$.

To prove the necessity of the condition we have as hypotheses: (1), (2) and (4), and wish to prove (3). Suppose, contrary to our conclusion, for all $f(x)$ in KK_M, there exists a $\beta<\max(\psi_2, \beta(\alpha, M))$ such that $f(x)$ is in $K_{\alpha\beta}$. In A, where $\beta(\alpha, M)>\psi_2(\alpha)$, $\beta<\beta(\alpha, M)$ implies that $\phi(M, \alpha, \beta)>0$. In A^*, where $\beta(\alpha, M)<\psi_2(\alpha)$, $\beta<\psi_2(\alpha)$ implies that for sufficiently small v, $\phi(v, \alpha, \beta)>0$. In E^*-EE^*, for all v, $\phi(v, \alpha, \beta)>0$. Thus in any case, we have a value of $v\leqq M$ for which $\phi(v, \alpha, \beta)>0$. Hence for some $v\leqq M$, the inequality (1.1) is not satisfied, in particular, by a linear function in the class KK_M, which gives us a contradiction.

The proof of Theorem 10.3 is similar, but with obvious modifications.

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EMMANUEL MISSIONARY COLLEGE

A SIMPLE SUFFICIENT CONDITION THAT A METHOD OF SUMMABILITY BE STRONGER THAN CONVERGENCE

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1. **Introduction.** A matrix a_{nk} of real or complex constants determines a transformation

$$\sigma_n = \sum_{k=1}^{\infty} a_{nk}s_k \tag{1}$$

and a method A of summability by means of which a given sequence $s_1, s_2, \cdots$ is summable to σ if the series in (1) converge and define numbers $\sigma_1, \sigma_2, \cdots$ such that $\sigma_n \to \sigma$ as $n \to \infty$. If a sequence s_n is summable A, we say that $A\{s_n\}$ exists and that s_n belongs to the summability field of A. If s_n is summable A to σ, we say that $A\{s_n\} = \sigma$. The method A is regular if $A\{s_n\} = \lim s_n$ whenever $\lim s_n$ exists.

Toeplitz [1911] (reference in bibliography at end of this paper) proved that A is regular if and only if the three conditions

$$\sum_{k=1}^{\infty} |a_{nk}| \leqq M, \qquad n = 1, 2, 3, \cdots, \tag{2}$$

$$\lim_{n\to\infty} a_{nk} = 0, \qquad k = 1, 2, 3, \cdots, \tag{3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1 \tag{4}$$

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are satisfied, M being a constant depending on the matrix a_{nk}. Immediately thereafter, Steinhaus [1911] proved that no regular A has a summability field containing all sequences; in fact, to each A correspond sequences, whose elements are zeros and ones, which are nonsummable A. Actually, Toeplitz and Steinhaus considered only row-finite transformations, but their methods give the facts stated.

An attempt to prove that each regular A has a summability field containing some divergent sequences cannot succeed, because there exist regular methods A whose summability fields are identical with the class of convergent sequences. Convergence is the simplest example. A classic theorem of Mercer [1907] and its generalizations, known as Mercerian Theorems, provide further examples. More examples of less special forms have been given by Agnew [1932], Sunouchi [1934] and Radó [1938]. It is the purpose of this paper to prove and discuss the following theorem and a generalization of it.

THEOREM 1. *If A is regular and satisfies the condition*

$$\lim_{n,k\to\infty} a_{nk} = 0 \tag{5}$$

then some divergent sequences of zeros and ones are summable A.

This theorem and the result of Steinhaus given above combine to yield the following theorem.

THEOREM 2. *If A is regular and satisfies* (5), *then some but not all divergent sequences of zeros and ones are summable A.*

It is of course well known and obvious that if the matrix a_{nk} has an inverse α_{nk}, in the sense that (1) holds if and only if

$$s_n = \sum_{k=1}^{\infty} \alpha_{nk}\sigma_k,$$

then at least one divergent sequence s_n is summable A (such that σ_n converges) if and only if the matrix α_{nk} fails to be conservative. The more general case in which the transformation (1) is reversible (in the sense of Mazur and Banach) was treated by J. D. Hill [1942]. The virtue of the criterion (5) lies in the fact that it is a simple criterion involving the matrix a_{nk} itself; it may be used when a_{nk} has no inverse, and when the inverse exists but is so complicated that the test for conservatism is not easily applied.

2. **The condition** (5). Each of the conditions (2), (3), and (4) has an obvious interpretation involving the matrix

$$\begin{matrix} a_{11}a_{12}a_{13} \cdots a_{1k} \cdots \\ a_{21}a_{22}a_{23} \cdots a_{2k} \cdots \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; . \end{matrix} \tag{6}$$

For example, (3) means that the elements of each column form a sequence converging to zero. The meaning of (5) is, roughly, that $|a_{nk}|$ is near zero whenever both n and k are large.

When a matrix satisfies (2), (3), and (4), there are several conditions which imply and are implied by (5). One is the condition (8) displayed below; it says, in other words, that the limit in (3) is uniform over the set $k=1, 2, 3, \cdots$.

3. **A generalization of Theorem 1.** While proving Theorem 1, we can, without introducing complications, prove the following more general theorem.

THEOREM 3. *If the matrix a_{nk} of A is such that*

$$\sum_{k=1}^{\infty} |a_{nk}| < \infty, \qquad n = 1, 2, 3, \cdots, \tag{7}$$

and

$$\lim_{n\to\infty} \max_{k=1,2,\cdots} |a_{nk}| = 0, \tag{8}$$

then there is at least one divergent sequence, whose elements are zeros and ones, which is summable A.

Actually, the proof will show clearly that there are "many" such sequences summable A.

4. **Proof of Theorem 3.** Let a_{nk} be a matrix satisfying (7) and (8). We establish Theorem 3 by exhibiting a divergent sequence s_n, whose elements are all zeros and ones, which is summable A to zero. To simplify typography, we sometimes write $a(n, k)$ and $s(k)$ for a_{nk} and s_k. Let $\alpha(1), \alpha(2), \cdots$ be a sequence, of positive numbers, which converges to 0 so rapidly than $n\alpha(n)\rightarrow 0$; for example, let $\alpha(n)=n^{-2}$. Let $\beta(1), \beta(2), \cdots$ be a sequence, of positive numbers, which converges to 0. The hypothesis (8) implies existence of an increasing sequence $n(1)<n(2)< \cdots$ of positive integers such that, for each $p=1, 2, \cdots$,

$$|a_{n,k}| \leqq \alpha(p), \qquad n \geqq n_p,\ k = 1, 2, \cdots. \tag{9}$$

Such a sequence $n(p)$ being fixed, the hypothesis (7) implies that if

$k(1), k(2), \cdots$ is a sequence of integers which becomes infinite sufficiently rapidly, then, for each $p=1, 2, 3, \cdots$,

$$\sum_{k=k(p)+1}^{\infty} |a_{n,k}| \leqq \beta(p), \qquad n_p \leqq n < n_{p+1}. \tag{10}$$

Let a sequence $k(p)$ be fixed such that (10) holds and $k(p+1) > k(p) + 1$ for each $p=1, 2, \cdots$. Let $s_1, s_2, \cdots$ be the particular sequence of zeros and ones defined by

$$\begin{aligned} s_k &= 1, \qquad & k &= k_1, k_2, k_3, \cdots, \\ s_k &= 0 & &\text{otherwise.} \end{aligned} \tag{11}$$

Then $s_k = 1$ for an infinite set of k's and $s_k = 0$ for an infinite set of k's; hence the sequence is divergent. Moreover the transform $\sigma_1, \sigma_2, \cdots$ of this sequence is such that, when $p=1, 2, 3, \cdots$ and $n_p \leqq n < n_{p+1}$,

$$\begin{aligned} |\sigma_n| &= \left| \sum_{k=1}^{\infty} a(n, k)s(k) \right| = \left| \sum_{j=1}^{\infty} a(n, k_j) \right| \\ &\leqq \sum_{j=1}^{p} |a(n, k_j)| + \sum_{j=p+1}^{\infty} |a(n, k_j)| \\ &\leqq \sum_{j=1}^{p} \alpha(p) + \sum_{k=k(p)+1}^{\infty} |a(n, k)| < p\alpha(p) + \beta(p). \end{aligned}$$

Therefore, since $p\alpha(p) \to 0$ and $\beta(p) \to 0$, $\sigma_n \to 0$. Thus the particular divergent sequence of zeros and ones is summable A to 0 and Theorem 3 is proved.

5. **Relations between two methods of summability.** Let B and C be two matrix methods of summability regular or not, of the form (1). It is the purpose of this section to discuss standard general procedures for determining relations between B and C, and to show how Theorems 1 and 3 may be used. One says that B includes C, or $B \supset C$, if $B\{s_n\} = C\{s_n\}$ whenever $C\{s_n\}$ exists; and that B is stronger than C if $B\{s_n\}$ exists whenever $C\{s_n\}$ exists, while $B\{s_n\}$ exists for at least one sequence for which $C\{s_n\}$ fails to exist. Two methods B and C are consistent if $B\{s_n\} = C\{s_n\}$ whenever both $B\{s_n\}$ and $C\{s_n\}$ exist, and are equivalent if $B\{s_n\} = C\{s_n\}$ whenever at least one of $B\{s_n\}$ and $C\{s_n\}$ exist.

For simplicity, we assume that C is triangular and has an inverse; this means that $c_{nk} = 0$ when $k > n$ and that $c_{nn} \neq 0$ for each $n = 1, 2, \cdots$. Let a_{nk} be the matrix of the transformation A defined by $A = BC^{-1}$. It is standard practice to use the fact that $B \supset C$

if and only if A is regular, that is, if and only if (2), (3), and (4) hold. If, in a particular case, one shows that (2), (3), and (4) hold, one knows that $B \supset C$; the methods must then be consistent, but it remains unknown whether (i) B and C are equivalent or (ii) B is stronger than C. If, in addition to (2), (3), and (4), one shows that (5) holds, then, by Theorem 1, B must be stronger than C.

There are cases in which, by reason of algebraic difficulties or by reason of actual failure of the conditions, one is unable to show that (2), (3), and (4) hold. (Those who work in the field know that the "norm condition" (2) is frequently the troublesome one.) It may nevertheless be possible to show that (7) and (8) hold. In such cases, Theorem 3 implies existence of sequences summable B but nonsummable C.

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CORNELL UNIVERSITY

DERIVATIVES AND FRÉCHET DIFFERENTIALS

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1. **Generalities.** A function $f(x)$, defined on an open set S of a complex Banach space X, with values in a complex Banach space Y, is said to have a Fréchet differential at a point x_0 of S if for $x=x_0$ the following conditions (G), (D), and (P) are satisfied:

(G) The limit $\lim_{\zeta\to 0}[f(x+\zeta h)-f(x)]/\zeta=\delta_x^h f=\delta f(x, h)$ exists for all h in X; (D) this limit is a continuous linear function of h; (P) the Gâteaux differential $\delta f(x, h)$ is a principal part of the increment, that is, $[f(x+h)-f(x)]-\delta f(x, h)=o(\|h\|)$.

We say that $f(x)$ is F-differentiable on S if these conditions hold at every point of S; if the condition (G) is satisfied for every point in S we call the function G-differentiable on S.

The reader will find in [2][1] or [6] a proof to the effect that a function which is G-differentiable on S—or indeed on more general sets—leads to a function $\delta f(x, h)$ which is linear, in the algebraic sense, with respect to h. We may thus replace the condition (D) by the requirement that the Gâteaux differential be continuous with respect to the argument h, which in turn is equivalent to $\delta f(x, h)$ being $O(1)$, $o(1)$ or $O(\|h\|)$ as $\|h\|$ tends to zero.

Our main purpose is to show that (P) is satisfied automatically if (G) and (D) hold on S, giving a new answer to the question: under which conditions is a G-differentiable function F-differentiable?

Previous solutions of this problem have been of two kinds. The first kind operates with topological conditions on the function $f(x)$, like continuity (see [4]), local boundedness (see [2]), or essential continuity (see [6]). The most general characterization theorem of this type seems to be the following: Let $f(x)$ be G-differentiable on the connected open set S, and bounded on a set $V-M$, where V is a nonvoid open subset of S and M is such that the whole space X is not the sum of a countable number of homothetic images $\alpha_n M+a_n$ of M; under these conditions the function $f(x)$ is F-differentiable on S (see [7]).

A solution of the second kind may be abstracted from [2] or [6]: if the higher differentials $\delta^n f(x; h_1, \cdots, h_n)$ are continuous functions of their h-arguments for one value x_0 of x, then $f(x)$ will be F-differentiable on a suitable neighborhood of x_0. The two kinds of characterizations are rather different; the first type refers to the behaviour

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[1] Numbers in brackets refer to the references cited at the end of the paper.

of $f(x)$ on an open set; the second is based on the behaviour on a subset which is only "finitely open" (see the definition (2.1)). The condition (D) belongs to the second class, and we look upon it in this manner:

By virtue of (D) there belongs to x a bounded linear transformation on X to Y, whose value for the argument h is $\delta f(x, h)$. The bounded linear transformations on X to Y, under the standard definition of norms, constitute a (complex) Banach space $[X, Y]$. The above linear transformation is thus the value of a function on S to $[X, Y]$, which we denote by $f'(x)$; the name derivative is justified by the formula $\delta f(x, h) = f'(x)h$.

2. **Derivation of the condition** (P). The functions $f(x)$ we deal with are at first assumed to be G-differentiable on a set D, which is finitely open according to the definition:

(2.1) A subset D of the (complex) Banach space X is finitely open if for x in D, $h_1, \cdots, h_n$ in X, the n-uples $(\zeta_1, \cdots, \zeta_n)$ for which

$$x + \zeta_1 h_1 + \cdots + \zeta_n h_n \in D$$

form an open set of the n-dimensional (complex) number space.

Without making use of the topology or metric of X the G-differential $\delta f(x, h)$ and the higher differentials $\delta^n f(x; h_1, \cdots, h_n)$ may be defined; for instance, $\delta^2 f(x; h, k)$ is $\delta_x{}^k[\delta f(x, h)]$. We shall use the (trivial) observation that the function $f(x)$ is G-differentiable on D if and only if $f(x+\zeta h)$ is a differentiable function of the complex variable ζ.

The topology of the value space is of course being used; and since we want the values of our functions to be in Banach spaces it becomes understandable that we restrict the concept derivative as follows:

(2.2) A function $f'(x)$ on D to the Banach space $[X, Y]$ of all bounded linear transformations on X to Y is called the derivative of $f(x)$ if $\delta f(x, h) = f'(x)h$, for x in D and h in X.

The value of this definition may be gauged by the lemma:

(2.3) *The derivative is G-differentiable on D*;

and the theorem:

(2.4) *If $f(x)$ is G-differentiable on an open set S and possesses a derivative on a nonvoid, finitely open subset D of S it satisfies the condition* (P) *at every point of D.*

We shall arrange the proofs in such a manner that a maximum of information is derived from the behaviour of the function on the finitely open set D alone.

From the theory of the G-differential in [2] or [6] we shall have to use the theorems:

(2.5.1) *The G-differentials* $\delta^n f(x; h_1, \cdots, h_n)$ *exist and they are G-differentiable with respect to* x *on* D, *linear and symmetric with respect to the h-arguments on* X;

(2.5.2) *For* x *fixed in* D *we get* $f(x+h)=\sum_0^\infty \delta^n f(x; h, \cdots, h)/n! = \sum_0^\infty p_n(x, h)$, *where* h *comes from a set* H_x *which is defined by the condition that* $|\zeta| \leqq 1$ *implies* $x+\zeta h \in D$.

From the theory of linear operators we borrow (see [5]):

(2.6) *If the bounded operator* $U(\zeta)$ *depends on the complex number* ζ*—which varies in an open set* Δ*—in such a manner that* $U(\zeta)h$ *is differentiable with respect to* ζ *for any* h *in* X, *then* $U(\zeta)$ *is differentiable with respect to* ζ, *on* Δ.

Proceeding now to the proof of the lemma (2.3) we note that it suffices to show that for k in X the quantity $f'(x+\zeta k)$ is differentiable with respect to ζ. This will follow from A. E. Taylor's theorem (2.6) if we know that $f'(x+\zeta k)h$ or $\delta f(x+\zeta k, h)$ is differentiable with respect to ζ for any h in X; that, however, amounts just to G-differentiability of $\delta f(x, h)$ with respect to x, which is asserted by (2.5.1). The lemma is thus proved and we may apply the theory of the G-differential to the function $f'(x)$. Its higher differentials will exist, and they will be bounded linear transformations. If $U(x)$ is a G-differentiable function on D to $[X, Y]$, we shall have the equality $[\delta U(x, k)]h = \delta_x{}^k\{[U(x)]h\}$, for

$$\left\{\lim_{\zeta\to 0} [U(x+\zeta k) - U(x)]/\zeta\right\} h = \lim_{\zeta\to 0} \{[U(x+\zeta k)]h - [U(x)]h\}/\zeta.$$

With the use of this principle and the symmetry of the differentials in their h-arguments one arrives by way of a mathematical induction at the formula:

$$(2.7) \qquad \delta^n f'(x; h_1, \cdots, h_n)h_{n+1} = \delta^{n+1} f(x; h_1, \cdots, h_{n+1}).$$

The left member of (2.7) is continuous with respect to h_{n+1}; the right member is a symmetric function of the h-arguments, so that the differentials turn out to be partially continuous in these arguments. By a theorem of Mazur and Orlicz (see [3, p. 65] and the references given there; compare also [6, Theorem (3.7)]) they will be continuous jointly in their h-arguments. The functions $p_n(x, h) = \delta^n f(x; h, \cdots, h)/n!$ are therefore continuous in h; in the terminology of [2] and [6] we have shown that the "G-powers" $p_n(x, h)$ are "F-powers" of h.

We show now that in a suitable neighborhood of $h=0$ the power series $\sum_0^\infty p_n(x, h)$ converges uniformly towards a function which has $p_1(x, h)$ as its Fréchet differential. By (2.5.2) the sum of this power series coincides with $f(x+h)$ on the set H_x. The proof is only a slight variation of the arrangement in [6]; we may thus content ourselves with a mere sketch.

The set H_x on which the power series converges is of the second category, since $\bigcup_{n=1}^\infty (nH_x)$ is the whole space X. Since the terms are continuous functions of h, a classical principle shows that they are uniformly bounded on a sphere (compare [1, p. 19]). We may thus assume that for a suitable h_0 and positive numbers ρ, M the inequality $\|h-h_0\| \leqq \rho$ implies, for all n, $\|p_n(h)\| \leqq M$ (we drop the argument x). It is easily seen that due to the homogeneity and G-differentiability of the functions $p_n(h)$ the same uniform bound obtains for $\|h\| \leqq \rho$ (compare Theorem (4.1) of [6]).

For $\|h\| \leqq \sigma < \rho$ we find $\|p_n(h)\| \leqq M(\sigma/\rho)^n$; this ensures uniform convergence of $\sum_0^\infty p_n(h)$ towards a function $g(h)$, for $\|h\| \leqq \sigma < \rho$.

We might stop here with an appeal to the theory of power series (see, for instance, [4, p. 11]); or we may prove, as in [6, Theorem (4.3)], the inequality

$$\|[g(h) - g(0)] - p_1(h)\| \leqq \|h\|^2 M/\rho(\rho - \sigma),$$

which yields for x in D, h in H_x, $\|h\| \leqq \sigma < \rho$,

$$\|[f(x+h) - f(x)] - \delta f(x, h)\| \leqq \|h\|^2 M/\rho(\rho - \sigma), \tag{2.8}$$

where the quantities M and ρ depend on x.

At this point we make use, for the first time, of the premise that $f(x)$ be defined on an open set S which contains D. The number ρ may then be taken so small that for $\|h\| < \rho$ the points $x+h$ are contained in S; we do not ask for more if we want the points $x+\zeta h$ to be in S for $|\zeta| \leqq 1$. By virtue of (2.5.2) the power series $\sum_0^\infty p_n(x, h)$ represents $f(x+h)$ in the sphere $\|h\| < \rho$, so that the inequality (2.8) is valid there. A fortiori, the condition (P) holds at every point of D. We have proved the theorem (2.4); let us add the corollary:

(2.9) *If $f(x)$ possesses a derivative on the open set S it is F-differentiable on S.*

Added in proof, January 20, 1946. Professor A. D. Michal informs me that more than ten years ago, in connection with a first draft of his paper *General tensor analysis* (Bull. Amer. Math. Soc. vol. 43 (1937)), he introduced the notion of a derivative as distinguished from a differential.

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CONTINUED FRACTION EXPANSIONS FOR FUNCTIONS WITH POSITIVE REAL PARTS

H. S. WALL

1. **Introduction.** Let K denote the region of the complex z-plane exterior to the cut along the real axis from -1 to $-\infty$. Let E denote the class of functions $F(z)$ with the following three properties:

$$(1.1)\qquad \begin{aligned} &\text{(a)}\quad F(z) \text{ is analytic over } K;\\ &\text{(b)}\quad F(0) = 1;\\ &\text{(c)}\quad R(F(z)) > 0 \text{ over } K. \end{aligned}$$

The object of this paper is to prove that the class E is coextensive with the class of functions representable in the form

$$(1.2)\qquad \cfrac{(1+z)^{1/2}}{1+\cfrac{g_1z}{1+ir_1(1+z)^{1/2}+\cfrac{(1-g_1)g_2z}{1+ir_2(1+z)^{1/2}+\cfrac{(1-g_2)g_3z}{1+ir_3(1+z)^{1/2}+\cdots}}}},$$

where $0<g_p<1$, $-\infty<r_p<+\infty$, $p=1, 2, 3, \cdots$, or as a terminating continued fraction of this form, in which the last g_p which appears may be equal to unity. The continued fractions converge uniformly over every bounded closed region within K. That branch of $(1+z)^{1/2}$ is to be taken in K which equals 1 for $z=0$.

This result supplements [3].[1] In fact, the continued fraction (1.2) is actually the continued fraction (3.6) of [3]. At that time we did not recognize that the latter can be put in the form (1.2), and we proved convergence only in the neighborhood of the origin. If $r_p=0$, $p=1, 2, 3, \cdots$, the continued fraction (1.2) reduces to a familiar form first considered by E. B. Van Vleck [2], and recently by the present writer [4] in connection with totally monotone sequences. From one point of view, the result is a reformulation of a theorem of Schur [1] on bounded analytic functions.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

2. **Preliminaries.** It will be convenient to introduce, along with the class E, two other classes of analytic functions, U and V. The class U consists of all functions $f(w)$ which are analytic and have moduli not greater than unity for $|w|<1$. The class V consists of all functions $k(w)$ which are analytic and have positive real parts for $|w|<1$, and which have the value 1 for $w=0$. The transformation

$$k(w)=\frac{1+wf(w)}{1-wf(w)} \tag{2.1}$$

maps U one-to-one upon V. We now map the domain $|w|<1$ upon K by means of the transformation

$$z=\frac{4w}{(1-w)^2},\qquad w=\frac{(1+z)^{1/2}-1}{(1+z)^{1/2}+1}. \tag{2.2}$$

The class E then consists of the functions

$$F(z)=k\left(\frac{(1+z)^{1/2}-1}{(1+z)^{1/2}+1}\right),$$

where $k(w)$ is in V. The correspondence set up in this way between E and V is one-to-one.

To each function $f(w)$ of the class U, Schur [1] makes correspond uniquely a finite or infinite sequence of constants α_p such that, in case the sequence is finite and has $n+2$ terms,

$$|\alpha_p|<1,\quad p=0,1,2,\cdots,n,\qquad |\alpha_{n+1}|=1, \tag{2.3}$$

and, in case the sequence is infinite,

$$|\alpha_p|<1,\qquad p=0,1,2,\cdots. \tag{2.4}$$

Conversely, each such sequence determines uniquely a function in the class U. The correspondence is set up in the following way. If $f(w)$ is any function in U, the sequence $\{\alpha_p\}$ is determined recurrently by means of the formulas

$$f_0(w)=f(w),\quad f_{k+1}(w)=\frac{1}{w}\,\frac{\alpha_k-f_k(w)}{1-\bar{\alpha}_k f_k(w)},\qquad \alpha_k=f_k(0),\quad k=0,1,2,\cdots. \tag{2.5}$$

Conversely, if a sequence $\{\alpha_p\}$ satisfying (2.3) or (2.4) is given, we construct the linear transformations

$$s=s_p(w;t)=\frac{\alpha_p-wt}{1-\bar{\alpha}_p wt} \tag{2.6}$$

of the t-plane into the s-plane, the transformations depending upon the parameter w. These have the property that if $|w| < 1$ then $|t| \leqq 1$ implies $|s| < 1$. The product of two or more of the transformations must have the same property. Thus, if

$$S_p(w;\, t) = s_0 s_1 \cdots s_p(w;\, t),$$

then

$$|S_p(w;\, t)| < 1 \quad \text{for} \quad |w| < 1 \quad \text{and} \quad |t| \leqq 1. \tag{2.7}$$

To any finite sequence $\{\alpha_p\}$ satisfying (2.3), there corresponds the function $S_n(w;\, \alpha_{n+1})$ in the class U. To any infinite sequence $\{\alpha_p\}$ satisfying (2.4) there corresponds the function $f(w)$ in U, defined for $|w| < 1$ by

$$f(w) = \lim_{p=\infty} S_p(w;\, t_{p+1}). \tag{2.8}$$

Here $t_1, t_2, t_3, \cdots$ is any sequence of numbers with moduli not greater than unity; the limit exists *uniformly* for $|w| \leqq r$ for every positive constant r less than 1; the value of the limit does not depend upon the particular sequence $\{t_p\}$ which is chosen. This formulation differs from that of Schur only in the introduction of the arbitrary sequence $\{t_p\}$, $|t_p| \leqq 1$, which, moreover, entails no essential modification in the proof.

3. **The main theorem.** Let $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_p$ be numbers with moduli less than unity, and let

$$f_{k+1}(w;\, t) = \frac{1}{w} \frac{\alpha_k - f_k(w;\, t)}{1 - \bar{\alpha}_k f_k(w;\, t)}, \quad f_k(w;\, t) = \frac{\alpha_k - w f_{k+1}(w;\, t)}{1 - \bar{\alpha}_k w f_{k+1}(w;\, t)}, \tag{3.1}$$
$$k = 0, 1, 2, \cdots, p,$$

where, in the notation introduced before, $f_0(w;\, t) = S_p(w;\, t)$, so that $f_{p+1}(w;\, t) = t$. We introduce functions $h_k(w;\, t)$ by means of the equations

$$h_k(w;\, t) = \frac{1 - \beta_k f_k(w;\, t)}{1 + w\beta_k f_k(w;\, t)}, \qquad k = 0, 1, 2, \cdots, p + 1, \tag{3.2}$$

where the β_k are numbers to be determined. If we substitute the values of $f_k(w;\, t)$ and $f_{k+1}(w;\, t)$ obtained from (3.2) into (3.1) we obtain the following formula expressing $h_k = h_k(w;\, t)$ in terms of $h_{k+1} = h_{k+1}(w;\, t)$:

$$h_k = \frac{(\beta_k - \bar{\alpha}_k + \beta_{k+1}\beta_k\alpha_k - \beta_{k+1}) w h_{k+1} + (\beta_{k+1}\beta_k\alpha_k - \beta_{k+1} - w\beta_k + w\bar{\alpha}_k)}{(w\bar{\alpha}_k + w^2\beta_k - \beta_{k+1} - w\beta_{k+1}\beta_k\alpha_k) - (\bar{\alpha}_k + w\beta_k + \beta_{k+1} + w\beta_{k+1}\beta_k\alpha_k) w h_{k+1}}.$$

We now determine the β_m so that the factor multiplying h_{k+1} in the numerator is zero. This will be true if

$$(3.3)\qquad \beta_k = \frac{\beta_{k-1} - \bar{\alpha}_{k-1}}{1 - \beta_{k-1}\alpha_{k-1}}, \qquad k = 1, 2, 3, \cdots, p+1, \beta_0 = 1.$$

We note that $|\beta_k| = 1$, $k = 0, 1, 2, \cdots, p+1$. With these values of the β_m, the above formula may be written:

$$(3.4)\ h_k(w; t) = \frac{|1 - \alpha_k\beta_k|^2}{(1 - \bar{\alpha}_k\bar{\beta}_k) - (1 - \alpha_k\beta_k)w + (1 - |\alpha_k\beta_k|^2)wh_{k+1}(w; t)}.$$

In particular, since $\beta_0 = 1$, $f_0(w; t) = S_p(w; t)$, we have

$$(3.5)\quad \begin{aligned} &\frac{1 - S_p(w; t)}{1 + wS_p(w; t)} \\ &= \frac{|1 - \alpha_0\beta_0|^2}{(1 - \bar{\alpha}_k\bar{\beta}_k) - (1 - \alpha_0\beta_0)w + (1 - |\alpha_0\beta_0|^2)wh_1(w; t)}. \end{aligned}$$

On multiplying both members of (3.5) by $2w/(1-w)$, adding 1 to both sides, and then taking reciprocals, we obtain

$$(3.6)\quad \begin{aligned} &\frac{1 + wS_p(w; t)}{1 - wS_p(w; t)} \\ &= \cfrac{1 + w}{1 - w + \cfrac{2w|1 - \alpha_0\beta_0|^2}{(1 - \bar{\alpha}_k\bar{\beta}_k) - (1 - \alpha_0\beta_0)w + (1 - |\alpha_0\beta_0|^2)wh_1(w; t)}}. \end{aligned}$$

Let

$$(3.7)\qquad \alpha_k\beta_k = 1 - 2u_{k+1}, \qquad k = 0, 1, 2, \cdots, p.$$

Since $|\beta_k| = 1$, $|\alpha_k| < 1$, then

$$(3.8)\qquad |u_k - 1/2| < 1/2, \qquad k = 1, 2, 3, \cdots, p+1.$$

Thus, $R(u_k) > 0$. We now put

$$(3.9)\qquad g_k = |u_k|^2/R(u_k), \qquad r_k = -I(u_k)/R(u_k).$$

Making the substitutions (3.7) and (2.2) in (3.6) and (3.4), we obtain:

$$(3.10)\quad \begin{aligned} &\frac{1 + wS_p(w; t)}{1 - wS_p(w; t)} \\ &= \cfrac{(1 + z)^{1/2}}{1 + \cfrac{g_1 z}{1 + ir_1(1 + z)^{1/2} + (1 - g_1)((1 + z)^{1/2} - 1)h_1(w; t)}}, \end{aligned}$$

and

$$(3.11)\quad h_k(w;t)=\frac{g_{k+1}((1+z)^{1/2}+1)}{1+ir_{k+1}(1+z)^{1/2}+(1-g_{k+1})((1+z)^{1/2}-1)h_{k+1}(w;t)},$$
$$k=1,2,3,\cdots,p.$$

By (3.8) and (3.9), the numbers g_k, r_k satisfy the inequalities

$$(3.12)\qquad 0<g_k<1,\quad -\infty<r_k<+\infty,\qquad k=1,2,3,\cdots,p+1.$$

Conversely, if g_k, r_k are any numbers satisfying (3.12), then numbers u_k satisfying (3.8) are uniquely determined by (3.9), inasmuch as the latter may be written $u_k=g_k\cos\phi_k$, $\arg u_k=-\text{arc}\tan r_k=-\phi_k$, $k=1,2,3,\cdots,p+1$. Then, numbers α_k with moduli less than unity are uniquely determined such that (3.3) and (3.7) hold. In fact,

$$\alpha_0=1-2u_1,\qquad \alpha_k=\frac{u_1u_2\cdots u_k}{\bar{u}_1\bar{u}_2\cdots\bar{u}_k}(1-2u_{k+1}),$$
$$k=1,2,3,\cdots,p.$$

We now suppose that $\{\alpha_k\}$ is any sequence such that (2.4) holds. Then the preceding formulas hold for arbitrarily large values of p. By (3.2) with $k=p+1$ we have, remembering that $f_{p+1}(w;t)=t$:

$$(3.13)\qquad h_{p+1}(w;t)=\frac{1-\beta_{p+1}t}{1+w\beta_{p+1}t}.$$

If we take $t=t_{p+1}=1/\beta_{p+1}$, then $|t_{p+1}|=1$ and $h_{p+1}(w;t_{p+1})\equiv 0$. Then, by (3.10) and (3.11),

$$\frac{1+wS_p(w;t_{p+1})}{1-wS_p(w;t_{p+1})}$$

is the $(p+2)$th approximant of the continued fraction (1.2). Hence, by (2.7) and (2.8), the continued fraction converges uniformly for $|w|\leqq r$ for every constant r less than 1, that is, for z in any bounded closed region within K, to the function

$$k(w)=\frac{1+wf(w)}{1-wf(w)}=F(z),\qquad z=\frac{4w}{(1-w)^2},$$

in the class E. If, on the other hand, (2.3) holds, then we readily verify that

$$\frac{1+wS_n(w;\alpha_{n+1})}{1-wS_n(w;\alpha_{n+1})}=F(z),$$

a function in the class E, is equal to the $(n+2)$th approximant of the continued fraction (1.2) if $1-\alpha_{n+1}\beta_{n+1}=0$, and is equal to the $(n+3)$th approximant, with $g_{n+2}=1$, if $1-\alpha_{n+1}\beta_{n+1}\neq 0$.

Conversely, if we start with a function $F(z)$ in the class E, we see immediately that there is determined uniquely a continued fraction (1.2) or a terminating continued fraction of this form, whose value is $F(z)$.

The proof of the theorem stated in the introduction is now complete. We remark that the condition (b) of (1.1) may be easily removed. If we do this, the result may then be stated as follows.

THEOREM. *If* $c>0$, $0<g_p<1$, $-\infty<r_{p-1}<+\infty$, $p=1, 2, 3, \cdots$, *then the continued fraction*

$$\cfrac{c(1+z)^{1/2}}{1+ir_0(1+z)^{1/2}+\cfrac{g_1z}{1+ir_1(1+z)^{1/2}+\cfrac{(1-g_1)g_2z}{1+ir_2(1+z)^{1/2}+\cfrac{(1-g_2)g_3z}{1+ir_3(1+z)^{1/2}+\cdots}}}} \tag{3.14}$$

converges uniformly over every bounded closed region in the domain K, *and its value is a function* $F(z)$ *which is analytic and has a positive real part throughout* K. *Conversely, if* $F(z)$ *is any function with these properties, then there is a uniquely determined continued fraction of the form* (3.14), *or a terminating continued fraction of this form in which the last* g_p *which appears may be unity, whose value is* $F(z)$.

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ON THE ZEROS OF POLYNOMIALS WITH COMPLEX COEFFICIENTS[1]

EVELYN FRANK

1. **Introduction.** Problems in dynamics very frequently have physically realizable solutions only if the determinantal equation of the system has all its roots in the negative half of the complex plane. It is therefore convenient to have a simple algorithm for testing whether this condition holds without actually computing the roots. Solutions to this problem have been considered by Cauchy [1],[2] Routh [6], and many others. Hurwitz [4] gave a method for polynomials with *real* coefficients of the form

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n. \tag{1.1}$$

According to his rule, all of the roots lie in the half-plane $R(z) < 0$ if and only if all the determinants

$$D_p = \begin{vmatrix} a_1, & a_3, & a_5, & \cdots, & a_{2p-1} \\ 1, & a_2, & a_4, & \cdots, & a_{2p-2} \\ 0, & a_1, & a_3, & \cdots, & a_{2p-3} \\ 0, & 1, & a_2, & \cdots, & a_{2p-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_p \end{vmatrix}, \qquad p = 1, 2, \cdots, n, \quad a_j = 0, \ j > n,$$

are positive.

Recently, Wall [8] formulated and proved this theorem by means of continued fractions. We extend his method to polynomials with *complex* coefficients,

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[1] The author thanks Professor H. S. Wall for suggestions on the writing of this paper.

The referee has kindly called my attention to a recent article by Herbert Bilharz, *Bemerkung zu einem Satze von Hurwitz*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 24 (1944) pp. 77–82 (lithoprinted by Edwards Brothers, Inc., Ann Arbor, Mich., 1945). There is presented in Bilharz' article an algorithm for the computation of determinants of type D_p similar to that given here in §2. Also Theorem 3.2 is essentially the same as the theorem stated and proved by Bilharz (p. 81), and Theorem 4.1 is equivalent to but approached differently from that stated by Bilharz without detailed proof.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$P(z) = z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n, \quad \alpha_k = p_k + iq_k, \tag{1.2}$$

$k=1, 2, \cdots, n$. Therefore, by a rotation and translation, the results can be applied in an *arbitrary* half-plane. We form

$$Q(z) = p_1 z^{n-1} + iq_2 z^{n-2} + p_3 z^{n-3} + iq_4 z^{n-4} + \cdots \tag{1.3}$$

and the J-fraction

$$\frac{Q(z)}{P(z)} = \frac{1}{c_1 z + k_1 + 1} + \frac{1}{c_2 z + k_2} + \frac{1}{c_3 z + k_3} + \cdots + \frac{1}{c_n z + k_n}, \tag{1.4}$$

where the c_p are real and the k_p are pure imaginary or zero. We find that all the zeros of $P(z)$ have negative real parts if and only if the expansion (1.4) exists and the c_p are positive (Theorem 3.1). Moreover, if this expansion exists with k of the c_p negative and $(n-k)$ positive, then k of the zeros of $P(z)$ have positive real parts and $(n-k)$ have negative real parts (Theorem 4.1). We find the proofs of these theorems carry over with no basic changes from those given by Wall [8] for the case of real polynomials, and at one step the proof of Theorem 3.1 is even simpler in the complex case.

We give in §2 some convenient formulas for expanding a rational function into a continued fraction of the form (1.4). This leads to formulations of the preceding theorems by means of determinants analogous to the Hurwitz determinants.

In §5, we give methods for modifying Theorem 4.1 in case the expansion (1.4) fails to exist.

In §6, we obtain formulas similar to those in [8] for finding bounds for the moduli of the zeros of (1.2).

2. Expansion of a rational function into a J-fraction. We consider here the following problem: If

$$\begin{aligned} f_0 &= \alpha_{00} z^n + \alpha_{01} z^{n-1} + \cdots + \alpha_{0n}, \\ f_1 &= \alpha_{11} z^{n-1} + \alpha_{12} z^{n-2} + \cdots + \alpha_{1n} \end{aligned} \tag{2.1}$$

are two polynomials of degree n and $n-1$ respectively, *to determine conditions upon the coefficients* $\alpha_{00}, \cdots, \alpha_{0n}, \alpha_{11}, \cdots, \alpha_{1n}$ *which are necessary and sufficient in order that*

$$\frac{f_1}{f_0} = \frac{1}{r_1 z + s_1} + \frac{1}{r_2 z + s_2} + \cdots + \frac{1}{r_n z + s_n}, \tag{2.2}$$

where the r_p *and* s_p *are constants, the* r_p *different from zero.* This problem is equivalent to the problem of determining polynomials f_p of degree $n-p$, $p=2, 3, \cdots, n-1$, which are connected with f_0 and f_1

by the recurrence relations

$$(2.3)\qquad \begin{array}{lr} f_{p-1} = (r_p z + s_p) f_p + f_{p+1}, & p = 1, 2, \cdots, n, \\ f_{n+1} = 0, \qquad f_n = \alpha_{nn} \neq 0, & r_p \neq 0. \end{array}$$

In other words, the expansion (2.2) exists if and only if the euclidean algorithm for the highest common factor of two polynomials, when applied to f_0 and f_1, gives a system of the form (2.3).

If we examine the long division process involved in the euclidean algorithm, we see that the numbers which contribute to the final result are only those contained in the following table.

$$(2.4)\qquad \begin{array}{llll} \alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \beta_{11}=\dfrac{\alpha_{11}\alpha_{01}-\alpha_{00}\alpha_{12}}{\alpha_{11}} & \beta_{12}=\dfrac{\alpha_{11}\alpha_{02}-\alpha_{00}\alpha_{13}}{\alpha_{11}} & \beta_{13}=\dfrac{\alpha_{11}\alpha_{03}-\alpha_{00}\alpha_{14}}{\alpha_{11}} & \cdots \\ \alpha_{22}=\dfrac{\alpha_{11}\beta_{12}-\beta_{11}\alpha_{12}}{\alpha_{11}} & \alpha_{23}=\dfrac{\alpha_{11}\beta_{13}-\beta_{11}\alpha_{13}}{\alpha_{11}} & \alpha_{24}=\dfrac{\alpha_{11}\beta_{14}-\beta_{11}\alpha_{14}}{\alpha_{11}} & \cdots \\ \beta_{22}=\dfrac{\alpha_{22}\alpha_{12}-\alpha_{11}\alpha_{23}}{\alpha_{22}} & \beta_{23}=\dfrac{\alpha_{22}\alpha_{13}-\alpha_{11}\alpha_{24}}{\alpha_{22}} & \beta_{24}=\dfrac{\alpha_{22}\alpha_{14}-\alpha_{11}\alpha_{25}}{\alpha_{22}} & \cdots \\ \alpha_{33}=\dfrac{\alpha_{22}\beta_{23}-\beta_{22}\alpha_{23}}{\alpha_{22}} & \alpha_{34}=\dfrac{\alpha_{22}\beta_{24}-\beta_{22}\alpha_{24}}{\alpha_{22}} & \alpha_{35}=\dfrac{\alpha_{22}\beta_{25}-\beta_{22}\alpha_{25}}{\alpha_{22}} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

The expansion (2.2) exists if and only if the numbers α_{00}, α_{11}, α_{22}, $\cdots$, α_{nn} are different from zero. When it exists, we have

$$(2.5)\qquad r_p = \frac{\alpha_{p-1,p-1}}{\alpha_{p,p}}, \qquad s_p = \frac{\beta_{p,p}}{\alpha_{p,p}}, \qquad p = 1, 2, \cdots, n.$$

Example. Let $f_0 = z^3 + (2+i)z^2 + (3+i)z + (2i+2)$, $f_1 = 2z^2 + iz + 2$. The table (2.4) in this case is

1	$2+i$	$3+i$	$2i+2$
2	i	2	
$2+i/2$	$2+i$	$2i+2$	
$9/4$	$3i/2$		
$-i/3$	2		
$16/9$			
$3i/2$			

Therefore,

$$r_1 = 1/2, \qquad r_2 = 8/9 \qquad r_3 = 81/64,$$
$$s_1 = 1 + i/4, \qquad s_2 = -\,4i/27, \qquad s_3 = 27i/32,$$

and the expansion (2.2) is

$$\frac{f_1}{f_0} = \frac{1}{z/2 + 1 + i/4 +} \, \frac{1}{8z/9 - 4i/27 +} \, \frac{1}{81z/64 + 27i/32}\,. \tag{2.6}$$

We now formulate the condition for the existence of (2.2) in terms of certain determinants.

THEOREM 2.1. *The quotient* f_1/f_0 *of two polynomials* (2.1) *can be expressed in the form* (2.2) *if and only if*

$$D_p \neq 0, \qquad p = 0, 1, \cdots, n, \tag{2.7}$$

where $D_0 = \alpha_{00}$ *and* $D_1, D_2, \cdots, D_n$ *are the first n principal minors of odd order (blocked off by lines) in the array*

$$\begin{array}{cccccccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \cdots \\
\cline{1-1}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \cdots \\
0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \cdots \\
\cline{1-3}
0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \cdots \\
0 & 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \cdots \\
\cline{1-5}
0 & 0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \tag{2.8}$$

where $\alpha_{0p} = \alpha_{1p} = 0$ *if* $p > n$.

PROOF. We suppose first that the expansion (2.2) exists with $r_p \neq 0$, $p = 1, 2, \cdots, n$, so that the numbers α_{pp}, $p = 0, 1, 2, \cdots$, of (2.4) are not zero. Consider the determinant D_p of order $2p-1$, $2 \leqq p \leqq n$. If we subtract α_{00}/α_{11} times the $(2k-1)$th row from the $2k$th row, for $k = 1, 2, \cdots, p-1$, we find with the aid of (2.4) that

$$D_p = \alpha_{11} \begin{vmatrix}
\beta_{11}, & \beta_{12} & \beta_{13}, & \cdots \\
\alpha_{11}, & \alpha_{12}, & \alpha_{13}, & \cdots \\
0, & \beta_{11}, & \beta_{12}, & \cdots \\
0, & \alpha_{11}, & \alpha_{12}, & \cdots \\
0, & 0, & \beta_{11}, & \cdots \\
\cdot & \cdot & \cdot & \cdot
\end{vmatrix},$$

where the new determinant is of order $2p-2$. On subtracting β_{11}/α_{11} times the $2k$th row from the $(2k-1)$th row, for $k=1, 2, \cdots, p-1$, and making use of (2.4), we readily obtain

$$(2.9) \qquad D_p = (-1)^{p-1}\alpha_{11}^{2}D_{p-1}^{(1)}, \qquad p = 2, 3, \cdots, n,$$

where $D_r^{(k)}$ denotes the determinant D_r with both the subscripts of all its elements increased by k. From (2.9) we then find immediately that

$$(2.10) \qquad D_p = (-1)^{p(p-1)/2}\alpha_{11}^{2}\alpha_{22}^{2}\cdots\alpha_{p-1,p-1}^{2}\alpha_{pp}, \qquad p = 2, 3, \cdots, n.$$

Since $\alpha_{pp}\neq 0$, $p=0, 1, \cdots, n$, it follows from (2.10) that (2.7) holds.

We suppose now, conversely, that (2.7) holds. Then, $\alpha_{00}\neq 0$, $\alpha_{11}\neq 0$, since, by definition, $D_0=\alpha_{00}$, $D_1=\alpha_{11}$. Since $\alpha_{11}\neq 0$, then (2.10) holds for $p=2$, so that $D_2=-\alpha_{11}^2\alpha_{22}\neq 0$, or $\alpha_{22}\neq 0$. This guarantees that (2.10) holds for $p=3$, so that $D_3=-\alpha_{11}^2\alpha_{22}^2\alpha_{33}\neq 0$, or $\alpha_{33}\neq 0$. On continuing this argument, we finally arrive at $\alpha_{nn}\neq 0$, and the proof of Theorem 2.1 is complete.

We observe that if $f_1=c_0z^{n-1}+c_1z^{n-2}+\cdots+c_{n-1}$, $f_0=z^n$, then the condition of Theorem 2.1 reduces to

$$D_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix} \neq 0,$$

$$p = 0, 1, \cdots, n-1 \ (c_p = 0 \text{ for } p > n-1).$$

This leads to the well known condition for a power series

$$\frac{c_0}{z}+\frac{c_1}{z^2}+\frac{c_2}{z^3}+\cdots+\frac{c_{n-1}}{z^n}+\cdots$$

to have a J-fraction expansion. We may obtain this expansion from formulas (2.4), (2.5) if we take $\alpha_{00}=1$, $\alpha_{0p}=0$, $p\geqq 1$; $\alpha_{1p}=c_{p-1}$.

Analogous considerations show that there exists a Stieltjes expansion [7] of the form[3]

$$(2.11) \quad \frac{f_1}{f_0}=\frac{1}{d_1z}+\frac{1}{d_2}+\frac{1}{d_3z}+\cdots+\frac{1}{T}, \quad T=\begin{cases} d_{2n-1}z & \text{if } f_0(z) = 0, \\ d_{2n} & \text{if } f_0(0) \neq 0, \end{cases}$$

---

[3] Expansions for rational functions of the form (2.2) and (2.11) find application in certain problems in electrical network theory (cf. [2, 3]).

where $d_p \neq 0$, $p=1, 2, 3, \cdots$, if and only if, in addition to condition (2.7), it is required that the principal minors of *even* order in the array (2.8) are different from zero up to and including the one of order $2n-2$ or $2n$, according as $f_0(0)=0$ or $f_0(0)\neq 0$, respectively. The coefficients d_p in (2.11) can be computed by forming the table

$$
\begin{array}{lllll}
 & \alpha_0 & \alpha_{01} & \alpha_{02} & \cdots \\
 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\
(2.12) & \alpha_{22}=\dfrac{\alpha_{11}\alpha_{01}-\alpha_{00}\alpha_{12}}{\alpha_{11}} & \alpha_{23}=\dfrac{\alpha_{11}\alpha_{02}-\alpha_{00}\alpha_{13}}{\alpha_{11}} & \alpha_{24}=\dfrac{\alpha_{11}\alpha_{03}-\alpha_{00}\alpha_{14}}{\alpha_{11}} & \cdots \\
 & \alpha_{33}=\dfrac{\alpha_{22}\alpha_{12}-\alpha_{11}\alpha_{23}}{\alpha_{22}} & \alpha_{34}=\dfrac{\alpha_{22}\alpha_{13}-\alpha_{11}\alpha_{24}}{\alpha_{22}} & \alpha_{35}=\dfrac{\alpha_{22}\alpha_{14}-\alpha_{11}\alpha_{25}}{\alpha_{22}} & \cdots \\
 & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Then

$$
(2.13) \qquad d_p = \frac{\alpha_{p-1,p-1}}{\alpha_{pp}}, \qquad p = 1, 2, 3, \cdots .
$$

This may be shown if we apply Theorem 2.1 to the function $zf_1(z^2)/f_0(z^2)$. Since this is an *odd* function, its expansion (2.2) will have $s_p=0$, $p=1, 2, 3, \cdots$. From this, (2.11) can be obtained by simple transformations.

3. Conditions for the zeros of a polynomial to lie in a half-plane. There is no loss in generality if we assume that the given half-plane is $R(z)<0$ since any half-plane can be reduced to this by a rotation and translation.

THEOREM 3.1. *Let $P(z)$ be a polynomial with complex coefficients* (1.2), *and form $Q(z)$* (1.3). *The zeros of $P(z)$ all lie in the half-plane $R(z)<0$ if and only if*

$$
(3.1) \qquad \frac{Q(z)}{P(z)} = \frac{a_0}{z+a_0+b_1} + \frac{a_1}{z+b_2} + \frac{a_2}{z+b_3} + \cdots + \frac{a_{n-1}}{z+b_n},
$$

where the a_p are real and positive and the b_p are pure imaginary or zero.

PROOF. If the expansion (3.1) holds, one may regard the continued fraction as generated by the transformations

$$
t = \frac{a_0}{z+a_0+b_1+w_1}, \quad w_1 = \frac{a_1}{z+b_2+w_2}, \cdots, w_{n-1} = \frac{a_{n-1}}{z+b_n+w_n},
$$

and show exactly as in [8] that $Q(z)/P(z)$ is irreducible, and

$$\left|\frac{Q(z)}{P(z)}-\frac{1}{2}\right| \leqq \frac{1}{2}, \qquad \text{for } R(z) \geqq 0. \tag{3.2}$$

Hence $P(z) \neq 0$ for $R(z) \geqq 0$.

Conversely, let $P(z)$ be a given polynomial whose zeros are all in the half-plane $R(z)<0$. Let $\overline{P}(z)$ denote the polynomial obtained from $P(z)$ by replacing its *coefficients* by their complex conjugates. The polynomial $Q(z)$ of (1.3) is then $[P(z)+\overline{P}(-z)]/2$ or $[P(z)-\overline{P}(-z)]/2$ according as the degree n of $P(z)$ is odd or even, respectively. We note that the set of zeros of $P(z)$ is symmetrical to the set of zeros of $\overline{P}(-z)$, with respect to the imaginary axis. Hence the geometrical argument used in [8] can be applied to show that all the zeros of $Q(z)$ lie on the axis of imaginaries, and that (3.2) holds. Since the zeros of $P(z)$ are in the half-plane $R(z)<0$ while those of $Q(z)$ are on the line $R(z)=0$, it follows that $Q(z)/P(z)$ is irreducible.

By division we now get

$$\frac{Q(z)}{P(z)} = \frac{a_0}{z + a_0 + b_1 + [C(z)/Q(z)]}, \tag{3.3}$$

where a_0 is the negative of the sum of the real parts of the zeros of $P(z)$ and is therefore positive, b_1 is pure imaginary or zero, and $C(z)/Q(z)$ is an irreducible rational fraction in which the denominator is of degree $n-1$ and the degree of the numerator is less than $n-1$. Just as in [8] it follows that $R[C(z)/Q(z)] \geqq 0$ for $R(z) \geqq 0$, and hence that there is a partial fraction expansion of the form

$$\frac{C(z)}{Q(z)} = \sum_{p=1}^{n-1} \frac{L_p}{z + ix_p}, \tag{3.4}$$

where the x_p are real and distinct, and the $L_p>0$. Then

$$\frac{-iC(-iz)}{Q(-iz)} = \sum_{p=1}^{n-1} \frac{L_p}{z - x_p},$$

so that

$$\frac{-iC(-iz)}{Q(-iz)} = \frac{a_1}{z+b_2 i} - \frac{a_2}{z+b_3 i} - \cdots - \frac{a_{n-1}}{z+b_n i},$$

where the a_p are real and positive and the b_p are pure imaginary or zero. On replacing z by iz and dividing both members by $-i$, we get

$$\frac{C(z)}{Q(z)} = \frac{a_1}{z+b_2} + \frac{a_2}{z+b_3} + \cdots + \frac{a_{n-1}}{z+b_n}.$$

On substituting this expression into (3.3), we obtain (3.1), and the proof of the theorem is complete.[4] If we put

$$
\begin{aligned}
(3.5)\qquad & a_0 = \frac{1}{c_1}, \quad a_p = \frac{1}{c_p c_{p+1}}, \quad p = 1, 2, \cdots, n-1, \quad p = 1, 2, \cdots, n, \\
& k_p = b_p c_p,
\end{aligned}
$$

then (3.1) takes the form (1.4). The c_p are evidently positive if and only if the a_p are positive.

The expansion (1.4) may be conveniently obtained by the method of §2. For example, if $P(z) = z^3 + (2+i)z^2 + (3+i)z + (2i+2)$, then $Q(z) = 2z^2 + iz + 2$, and $Q(z)/P(z)$ is the fraction (2.6). Therefore, $c_1 = 1/2$, $c_2 = 8/9$, $c_3 = 81/64$, so that the zeros of $P(z)$ are all in the left half-plane. Here the zeros are actually $-1-i$, $(-1-i\,7^{1/2})/2$, $(-1+i\,7^{1/2})/2$.

From the formulas (2.5) and (2.10), we may formulate the condition for the zeros of $P(z)$ to lie in the half-plane $R(z) < 0$ by means of certain determinants analogous to the Hurwitz determinants [4]. In fact, we conclude at once that the numbers c_p of (1.4) are positive if and only if $(-1)^{p(p-1)/2} D_p > 0$, $p = 0, 1, \cdots, n$, where D_p is the determinant of Theorem 2.1 formed with $f_0 = P(z)$, $f_1 = Q(z)$. By simple transformations of these determinants, one may formulate this result as the following theorem.

THEOREM 3.2. *The polynomial $P(z)$ of Theorem 3.1 has all its zeros in the half-plane $R(z) < 0$ if and only if the determinants*

$$
\begin{aligned}
\Delta_1 &= p_1, \\
\Delta_k &= (-1)^{k(k-1)/2} D_k \\
(3.6)\qquad &= \left|\begin{array}{ccccccccc}
p_1, & p_3, & p_5, & \cdots, & p_{2k-1}, & -q_2, & -q_4, & \cdots, & -q_{2k-2} \\
1, & p_2, & p_4, & \cdots, & p_{2k-2}, & -q_1, & -q_3, & \cdots, & -q_{2k-3} \\
 & & \cdots & & & & \cdots & & \\
0, & & & \cdots, & p_k, & 0, & & \cdots, & -q_{k-1} \\
0, & q_2, & q_4, & \cdots, & q_{2k-2}, & p_1, & p_3, & \cdots, & p_{2k-3} \\
0, & q_1, & q_3, & \cdots, & q_{2k-3}, & 1, & p_2, & \cdots, & p_{2k-4} \\
 & & \cdots & & & & \cdots & & \\
0, & & & \cdots, & q_k, & 0, & & \cdots, & p_{k-1}
\end{array}\right|, \\
& \qquad\qquad k = 2, 3, \cdots, n \ (p_r = q_r = 0 \text{ for } r > n),
\end{aligned}
$$

---

[4] An additional step is needed in the case of real polynomials (cf. §1), namely, to show that the $b_p = 0$.

are all positive.

If the q_p are zero, this reduces to the theorem of Hurwitz [4].

4. **Determination of the number of zeros of $P(z)$ in each of the half-planes $R(z)<0$, $R(z)>0$.** We suppose that $P(z)$ is a polynomial with complex coefficients of the form (1.2). We assume that (1.4) exists and that the $c_p \neq 0$. We then have the following theorem.

THEOREM 4.1. *The polynomial* (1.2) *has k zeros with positive real parts and $(n-k)$ zeros with negative real parts if, in the expansion* (1.4), *k of the coefficients c_p are negative and the remaining $(n-k)$ are positive.*

PROOF. Since the expansion (1.4) exists, $Q(z)/P(z)$ is irreducible. It follows that $P(z)$ cannot have a zero on the imaginary axis. For, if $P(ir)=0$, r real, then (cf. §3) $Q(ir)=[P(ir)\pm\overline{P}(-ir)]/2$ $=[P(ir)\pm\overline{P}(\overline{ir})]/2=0$, which is impossible since $Q(z)/P(z)$ is irreducible. Thus the zeros of $P(z)$ have their real parts different from zero, so that, for $R(z)=0$, we can write $P(z)=re^{i\pi\theta}$, where $r>0$. If we consider $P(z)$ as the product of the vectors from its zeros to the point z, then we see at once that, as z ranges along the axis of imaginaries from $i\cdot\infty$ to $-i\cdot\infty$, then θ decreases by the integral amount $\Delta=N-P$, where N and P are the numbers of zeros of $P(z)$ with negative and positive real parts, respectively. The same evidently holds if, instead of $P(z)$, we consider $i^nP(-iz)=re^{i\pi\theta}$, and let z range along the *real* axis from $-\infty$ to $+\infty$. Now

$$\begin{aligned} i^nP(-iz) &= (z^n - q_1z^{n-1} - p_2z^{n-2} + q_3z^{n-3} + p_4z^{n-4} + \cdots) \\ &\qquad + i(p_1z^{n-1} - q_2z^{n-2} - p_3z^{n-3} + q_4z^{n-4} + \cdots) \\ &= U(z) + iV(z), \end{aligned} \tag{4.1}$$

where

$$\frac{V(z)}{U(z)} = \frac{1}{c_1z + ik_1} - \frac{1}{c_2z + ik_2} - \cdots - \frac{1}{c_nz + ik_n}, \tag{4.2}$$

which is real when z is real. This may be seen as follows:

The pth denominator of (1.4) can be written in the form

$$B_1(z) = c_1z + k_1 + 1,$$

$$B_p(z) = \begin{vmatrix} c_1z + k_1 + 1, & -1, & 0, & 0, \cdots, & 0 \\ 1, & c_2z + k_2, & -1, & 0, \cdots, & 0 \\ 0, & 1, & c_3z + k_3, & -1, \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & c_pz + k_p \end{vmatrix},$$

$p=2, 3, \cdots, n$. Since $c_1c_2\cdots c_nP(z)=B_n(z)$, we then have

$$\begin{aligned} c_1c_2\cdots c_nP(z) & \\ &= \begin{vmatrix} c_1z+k_1, & -1, & 0, & 0,\cdots, & 0 \\ 1, & c_2z+k_2, & -1, & 0,\cdots, & 0 \\ 0, & 1, & c_3z+k_3, & -1,\cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & c_nz+k_n \end{vmatrix} \\ &\quad + \begin{vmatrix} c_2z+k_2, & -1, & 0,\cdots, & 0 \\ 1, & c_3z+k_3, & -1,\cdots, & 0 \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & c_nz+k_n \end{vmatrix} \\ &= H_n(z)+G_n(z) \end{aligned} \tag{4.3}$$

and

$$\frac{G_n(z)}{H_n(z)} = \frac{1}{c_1z+k_1+}\,\frac{1}{c_2z+k_2+}\cdots+\frac{1}{c_nz+k_n}. \tag{4.4}$$

If we replace z by $-iz$ in (4.3) and (4.4) and make some simple transformations [5, p. 194], (4.4) becomes (4.2) and (4.3) becomes (4.1).

From this point on, the proof runs almost exactly the same as in [8]. The number Δ is the net decrease in

$$\theta = \frac{1}{\pi}\arctan\frac{V(z)}{U(z)}$$

as z increases through real values from $-\infty$ to $+\infty$. Using (4.2), we form the sequence $f_0=1, f_1=c_nz+ik_n, \cdots, f_n$, defined by the recurrence formula $f_{p+1}=(c_{n-p}z+ik_{n-p})f_p-f_{p-1}$, $p=1, 2, \cdots, n-1$. These form a Sturm's sequence, and we find that $\Delta=n-2k$, where k is the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$. Therefore, $N-P=n-2k$, $N+P=n$, so that $P=k$, $N=n-k$, as was to be proved.

By means of formulas (2.5) and (3.6), Theorem 4.1 can be formulated in terms of the numbers α_{pp} in the first column of table (2.4), or in terms of the determinants Δ_p of (3.6). In this way the methods of Routh and Hurwitz, respectively, are extended to polynomials with complex coefficients.

The method of Theorem 4.1 applies, save in the exceptional case where some determinant Δ_p vanishes and the expansion (1.4) fails to exist. In the next section, we give simple methods for taking care of this exceptional case.

5. **The case where some $\Delta_p = 0$.** We assume $Q(z)/P(z)$ is irreducible. This is no restriction since the common factor can be removed by the euclidean algorithm. We shall show that the method of §4 may be extended to find the number of roots in each half-plane even if some $\Delta_p = 0$.

THEOREM 5.1. *There exists a number $\delta > 0$ such that for all numbers η in the interval $-\delta < \eta < 0$, the expansion* (1.4) *exists for the quotient $Q(z+\eta)/P(z+\eta)$.*

PROOF. Form the determinants Δ_p of (3.6) for the polynomial

$$\begin{aligned} P(z+\eta) = {} & z^n + ({}_nC_1\cdot\eta + \alpha_1)z^{n-1} \\ & + ({}_nC_2\cdot\eta^2 + \alpha_1\,{}_{n-1}C_1\cdot\eta + \alpha_2)z^{n-2} + \cdots + \alpha_n. \end{aligned} \tag{5.1}$$

Then one may readily verify that the Δ_p are polynomials in η of degree p^2, in which the coefficient of the highest power of η is

$$\begin{vmatrix} {}_nC_1, & {}_nC_3, & {}_nC_5, & \cdots, & {}_nC_{2p-1} \\ 1, & {}_nC_2, & {}_nC_4, & \cdots, & {}_nC_{2p-2} \\ 0, & {}_nC_1, & {}_nC_3, & \cdots, & {}_nC_{2p-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & {}_nC_p \end{vmatrix} \cdot \begin{vmatrix} {}_nC_1, & {}_nC_3, & \cdots, & {}_nC_{2p-3} \\ 1, & {}_nC_2, & \cdots, & {}_nC_{2p-4} \\ 0, & {}_nC_1, & \cdots, & {}_nC_{2p-5} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & {}_nC_{p-1} \end{vmatrix}.$$

These determinants are always positive, for they are the determinants (3.6) formed for the polynomial $(z+1)^n = z^n + {}_nC_1z^{n-1} + {}_nC_2z^{n-2} + \cdots + {}_nC_n$, which has its only zero in the half-plane $R(z) < 0$. Then the Δ_p for (5.1) are polynomials in η which are not identically zero. Hence there must exist a constant $\delta > 0$ such that, for $-\delta < \eta < 0$, none of the Δ_p can vanish, and thus the $c_p \neq 0$ since they are quotients of the Δ_p.

Example. We shall apply this theorem to find the number of zeros in each half-plane for the polynomial $P(z) = z^3 + (2+i)z^2 + (-3/2+i)z + (-5/2 - 5i/2)$. Here $\Delta_2 = 0$ so that the expansion (1.4) cannot be formed. However, if we expand $Q(z+\eta)/P(z+\eta)$ into a J-fraction and *retain at each step only the powers of η which dominate* as η approaches zero, we find that, when η is near zero, c_1, c_2, and c_3 are positive, negative, and positive, respectively. Therefore, $P(z+\eta)$ has two zeros with negative real parts and one zero with positive real part for all $|\eta|$ sufficiently small. Hence $P(z)$ must also have two zeros with negative real parts and one zero with positive real part. Here the zeros are actually $(-1 \pm (11)^{1/2})/2$, $-1-i$.

We now describe two methods for finding the number of roots in each half-plane, which are based on the fact that we may alter the coefficients of $P(z)$ by a very small amount without displacing the zeros of $P(z)$ more than a very small amount. Therefore, since we have assumed that $P(z)$ has no zeros on the imaginary axis, the altered polynomial $P'(z)$ will have the same number of zeros on each side of the imaginary axis as the original polynomial $P(z)$.

In the first method, we actually increase one or more of the coefficients by a small positive amount ϵ, and obtain the J-fraction expansion (1.4) for the altered polynomial. We then count the signs of the c_p as before. The following example illustrates the method.

Example. Let $P(z) = z^5 - 3z^4 - 20z^3 + 60z^2 - z - 78$. We find that the expansion (1.4) does not exist for $Q(z)/P(z)$. We therefore form $P'(z) = z^5 - 3z^4 + (-20+\epsilon)z^3 + 60z^2 - z - 78$, $Q'(z) = Q(z) = -3z^4 + 60z^2 - 78$, and the J-fraction expansion (1.4) for $Q'(z)/P'(z)$. In this expansion we find, for all ϵ sufficiently small, c_1, c_2, c_3 are negative and c_4, c_5 are positive, so that there are three roots of $P'(z)$ and hence of $P(z)$ with positive real parts and two with negative real parts. The zeros of $P(z)$ are approximately -1.0, -4.4, $+4.7$, $+1.86 \pm .5i$.

A second method consists in the formation of the continued fraction expansion for $V(z)/U(z)$ by the euclidean algorithm, that is, we form

$$\frac{V(z)}{U(z)} = \frac{1}{q_1(z)} + \frac{1}{q_2(z)} + \cdots + \frac{1}{q_j(z)}, \qquad j < n, \tag{5.2}$$

where the q_p are certain uniquely determined polynomials. Let us imagine that we have formed a polynomial

$$P'(z) = z^n + \alpha_1' z^{n-1} + \alpha_2' z^{n-2} + \cdots + \alpha_n', \tag{5.3}$$

whose coefficients differ by very small amounts from the coefficients of $P(z)$, and the corresponding expansion

$$\frac{V'(z)}{U'(z)} = \frac{1}{c_1 z + ik_1} - \frac{1}{c_2 z + ik_2} - \cdots - \frac{1}{c_n z + ik_n}, \tag{5.4}$$

and compare this with (5.2). The method can best be explained by examples.

Examples. Consider again $P(z) = z^5 - 3z^4 - 20z^3 + 60z^2 - z - 78$. Here

$$\frac{V(z)}{U(z)} = \frac{1}{-z/3} + \frac{1}{(z^3 + 20z)/9} + \frac{1}{27z/78}. \tag{5.5}$$

We form the Sturm's functions for $P'(z)$, $f_0=1$, $f_1=c_1z, \cdots$, defined by the recurrence relation $f_{p+1}=c_{n-p}z\cdot f_p-f_{p-1}$, $p=1, 2, \cdots$, where the c_p are undetermined. We then compare (5.4) and (5.5), and require that $c_1=27/78$, $c_5=-1/3$, and $f_4=V(z)$. Therefore, $c_2c_3c_4=1/9$, $-c_3c_4-27c_4/78-27c_2/78=60/78$. Since $f_5=U(z)-c_3z(27c_2z^2/78-1)$, we see that if we choose $c_3=-\epsilon$, a small negative number, f_5 and $U(z)$ will differ by a very small amount. Then $c_2=(-29\pm(296+36/\epsilon)^{1/2})/18$, $c_4=-20/9-c_2-26/81c_2$, so that two of the c_2, c_3, c_4 have negative signs and one has a positive sign. These, together with the values of c_1 and c_5, give three negative and two positive c_p for $P'(z)$, so that we find the same result as by the first method.

For the polynomial $P(z)=z^5+(i-5)z^4-10iz^3+(10+50i)z^2-16z+(-16i+80)$ we find by the same method that in the J-fraction expansion of the form (5.4) for $V'(z)/U'(z)$ three of the c_p have negative signs and two have positive. Hence $P'(z)$ and $P(z)$ have three zeros in $R(z)>0$ and two zeros in $R(z)<0$. Here the roots are actually $1+i$, $-1-i$, $2+2i$, $-2-2i$, $5-i$.

Still another method, which, however, may not always be successful, is to replace z by $1/z$ and consider the polynomial $z^nP(1/z)$.

6. **Bounds for the moduli of the zeros of $P(z)$.** We extend the method in [8] to obtain bounds for the moduli. If we set

$$h_1=\frac{1}{(c_1z+k_1+1)(c_2z+k_2)},\quad h_2=\frac{1}{(c_2z+k_2)(c_3z+k_3)},\cdots,$$
$$h_{n-1}=\frac{1}{(c_{n-1}z+k_{n-1})(c_nz+k_n)},$$

then (1.4) becomes

$$\frac{Q(z)}{P(z)}=\frac{(c_1z+k_1+1)^{-1}}{1}\;\frac{h_1}{+1}\;\frac{h_2}{+1}+\cdots\frac{h_{n-1}}{+1}.$$

If $g_1, g_2, \cdots, g_{n-1}$ are numbers such that $0<g_p<1$, $p=1, 2, \cdots, n-1$, then $P(z)\neq0$ if z satisfies the inequalities $|h_1|\leqq g_1$, $|h_2|\leqq(1-g_1)g_2$, $|h_3|\leqq(1-g_2)g_3, \cdots, |h_{n-1}|\leqq(1-g_{n-2})g_{n-1}$. This sequence gives the bound $|z|\geqq c$, where c is the largest of the numbers

$$\frac{1}{2}\left(\left|\frac{k_1+1}{c_1}\right|+\left|\frac{k_2}{c_2}\right|+\left[\left(\left|\frac{k_1+1}{c_1}\right|-\left|\frac{k_2}{c_2}\right|\right)^2+\frac{4}{g_1|c_1c_2|}\right]^{1/2}\right),$$
$$\frac{1}{2}\left(\left|\frac{k_2}{c_2}\right|+\left|\frac{k_3}{c_3}\right|+\left[\left(\left|\frac{k_2}{c_2}\right|-\left|\frac{k_3}{c_3}\right|\right)^2+\frac{4}{g_2(1-g_1)|c_2c_3|}\right]^{1/2}\right),\cdots, \tag{6.1}$$

$$\frac{1}{2}\left(\left|\frac{k_{n-1}}{c_{n-1}}\right|+\left|\frac{k_n}{c_n}\right|+\left[\left(\left|\frac{k_{n-1}}{c_{n-1}}\right|-\left|\frac{k_n}{c_n}\right|\right)^2+\frac{4}{g_{n-1}(1-g_{n-2})\,|c_{n-1}c_n|}\right]^{1/2}\right). \tag{6.1}$$

In the example [illegible] §3, the numbers (6.1) are 3.76, 2.32 for g_1, $g_2=1/2$. The largest, 3.76, gives an upper bound for the moduli of the zeros. The moduli of all the zeros are, in fact, 1.41. By varying the g_p, for instance, $g_1=4/5$, $g_2=3/4$, we obtain a closer upper bound. The values of (6.1) are then 3.38 and 2.87.

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Since this paper was finished, my attention has been called to Bull. Amer. Math. Soc. abstract 50-7-190 by Seymour Sherman, J. DiPaola, and H. F. Frissel, and abstract 51-1-12 by Seymour Sherman.

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A NOTE ON THE DU BOIS-REYMOND EQUATIONS IN THE CALCULUS OF VARIATIONS

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1. Introduction. McShane [2][1] and Tonelli [4] have given conditions on the integrand function which insure that every absolutely continuous solution of the problem of minimizing the integral

$$I[y] = \int_{x_1}^{x_2} f(x, y, y')dx \tag{1.1}$$

satisfies the du Bois-Reymond form of the Euler equations. The principal purpose of the present note is to give an alternate proof of these results of McShane and Tonelli. In addition to being simpler in detail than the previous proofs, the differentiability theorems of §§2 and 3 involve weaker hypotheses than the corresponding theorems of McShane and Tonelli, both in regard to conditions on the integrand function and in regard to the class of arcs considered. Basically, the present proof is intimately related to the proof of the fundamental lemma as given by Bliss [1, pp. 20–21].

2. A general differentiability theorem. Suppose that for $(x, y, r) = (x, y_1, \cdots, y_n, r_1, \cdots, r_n)$ in a region R consisting of all values (x, y, r) satisfying $x_1 \leqq x \leqq x_2$, y in an open region Δ of $(y_1, \cdots, y_n)$-space, and $r = (r_1, \cdots, r_n)$ arbitrary, the integrand function $f(x, y, r)$ satisfies the following conditions:

(H$_1$) *For fixed values of* (y, r), $f(x, y, r)$ *is finite and measurable on* x_1x_2;

(H$_2$) *For fixed values of* x, $f(x, y, r)$ *is of class* C' *in* (y, r).

We shall be interested in proving a differentiability property of an arc $y_i = y_i(x)$ which minimizes $I[y]$ in a class of arcs K. Unquestionably, the case in which K consists of absolutely continuous arcs is the most important. Our argument, however, is not complicated by allowing K to be any prescribed class of arcs which possesses the following properties:

(P$_0$) *If the arc* $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ *belongs to* K, *then for each* x *on* x_1x_2 *the point* $y = [y_i(x)]$ *is in the region* Δ;

(P$_1$) *If* $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ *belongs to* K, *then the functions* $y_i(x)$ *are continuous on* x_1x_2 *and the derivatives* $y_i'(x)$ *exist and are finite a. e.*

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

(almost everywhere) on this interval;

(P_2) *If $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ belongs to K, and the functions $\eta_i(x)$ are Lipschitzian on x_1x_2 and satisfy $\eta_i(x_1) = 0 = \eta_i(x_2)$, then for a suitably small in absolute value the arc $y_i = y_i(x) + a\eta_i(x)$ $(x_1 \leqq x \leqq x_2)$ also belongs to K.*

Actually, in the following proof we use (P_2) only for sets of functions $\eta_i(x)$ for which $n-1$ of these functions are identically zero on x_1x_2.

Along an arc of K the function $f(x, y(x), y'(x))$ is defined a.e. on x_1x_2. In considering (1.1), and similar integrals, it is to be understood that the integrand is the indicated function where this function is well defined, and equal to zero elsewhere on x_1x_2. Correspondingly, a function $F(x)$ which is defined a.e. on x_1x_2 will be said to be measurable on x_1x_2 if the function which is equal to $F(x)$ at the points where it is defined, and to zero elsewhere on x_1x_2, is measurable on this interval. In view of (H_1) and (H_2), and the measurability of a function that is the limit a.e. of a sequence of measurable functions [3, p. 122], it follows that along an arc y of K the function f and its partial derivatives with respect to y_j and r_j $(j = 1, \cdots, n)$ are measurable on x_1x_2.

It is to be noted that we do not suppose that the integral (1.1) exists and is finite for every arc of K. An arc y of K is said to minimize (1.1) in this class if $I[y]$ is finite, and for arbitrary arcs Y of K for which $I[Y]$ is finite we have $I[Y] \geqq I[y]$.

The following lemma will be used in the proof of the subsequent theorems.

LEMMA 2.1. *Let Δ^* be a given bounded and closed subregion of Δ in y-space, and for a given positive integer N denote by R_N the subregion of R consisting of all (x, y, r) satisfying $x_1 \leqq x \leqq x_2$, y in Δ^*, and $|r_i| \leqq N$ $(i = 1, 2, \cdots, n)$. Then there exists a corresponding function $\psi_N(x)$ which is finite and measurable on x_1x_2, and such that for (x, y, r) in R_N we have*

$$(2.1) \qquad \begin{aligned} |f(x, y, r)| \leqq \psi_N(x), \qquad & |f_{y_i}(x, y, r)| \leqq \psi_N(x), \\ |f_{r_i}(x, y, r)| \leqq \psi_N(x) \qquad & (i = 1, 2, \cdots, n). \end{aligned}$$

In view of hypotheses (H_1) and (H_2) it follows that for fixed values of (y, r) the partial derivatives of f with respect to y_i and r_i $(i = 1, \cdots, n)$ are finite and measurable on x_1x_2. Consequently, in R the function

$$(2.2) \quad g(x, y, r) = |f(x, y, r)| + \sum_{i=1}^{n} [|f_{y_i}(x, y, r)| + |f_{r_i}(x, y, r)|]$$

is continuous in (y, r) for fixed values of x on x_1x_2, and measurable on x_1x_2 for fixed values of (y, r). Now R_N is the product of the interval $x_1 \leqq x \leqq x_2$ and the closed and bounded region D_N in (y, r) space consisting of all sets (y, r) having y in Δ^* and $|r_i| \leqq N$ $(i=1, 2, \cdots, n)$. For each x on x_1x_2 define the functional value $\psi_N(x)$ as the maximum of $g(x, y, r)$ on D_N. In view of the continuity of $g(x, y, r)$ as a function of (y, r) it follows that $\psi_N(x)$ is finite on x_1x_2. Moreover, $\psi_N(x)$ is measurable on this interval [3, p. 122], since it is the limit superior of a sequence of measurable functions $g(x, y^{(k)}, r^{(k)})$, where $(y^{(k)}, r^{(k)})$ $(k=1, 2, \cdots)$ is a countable sequence dense in D_N. The function $\psi_N(x)$ thus defined clearly satisfies the conclusion of the lemma.

THEOREM 2.1. *Suppose that* $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ *is a minimizing arc for* (1.1) *in a class* K, *and, in addition to hypotheses* (H_1) *and* (H_2), *the integrand satisfies with this arc the further condition*:

($H_{3;j}$) *There exists a constant* $\delta > 0$ *and a function* $\phi_j(x)$ *integrable on* x_1x_2 *and such that for* $|\bar{y}_j - y_j(x)| < \delta$, $\bar{y}_i = y_i(x)$ $(i \neq j)$, *we have*

$$|f_{y_j}(x, \bar{y}, y'(x))| \leqq \phi_j(x)$$

a.e. on x_1x_2. *Then there exists a corresponding constant* c_j *such that*

$$z_j(x) \equiv f_{r_j}(x, y(x), y'(x)) - \int_{x_1}^{x} f_{y_j}(t, y(t), y'(t))dt$$

is equal to c_j *a.e. on* x_1x_2.

Condition ($H_{3;j}$) implies the existence of the integral appearing in the definition of $z_j(x)$; it also implies that the integral

$$\int_{x_1}^{x_2} f(x, Y(x), y'(x))dx$$

exists for all arcs $Y_i = Y_i(x)$ of K satisfying $|Y_j(x) - y_j(x)| < \delta$, $Y_i(x) = y_i(x)$ $(i \neq j)$ on x_1x_2, since for such arcs,

$$(2.3) \qquad |f(x, Y(x), y'(x))| \leqq |f(x, y(x), y'(x))| + \delta\phi_j(x).$$

Corresponding to a positive integer N, let E_N denote the set of all points x on x_1x_2 at which the derivatives y_i' exist and $|y_i'| \leqq N-1$ $(i=1, \cdots, n)$. We may suppose that N is sufficiently large to insure that $m(E_N)$, the measure of E_N, is positive, since E_N is a subset of E_{N+1} $(N=1, 2, \cdots)$ and $\lim_{N\to\infty} m(E_N) = x_2 - x_1$. Furthermore, let the region Δ^* of Lemma 2.1 be such that it contains in its interior all the points $y = [y_i(x)]$ $(x_1 \leqq x \leqq x_2)$ belonging to the supposed minimizing arc, and denote by $E_{N;M}$ the subset of E_N where the function $\psi_N(x)$

of that lemma satisfies the condition $\psi_N(x) \leqq M$ $(M=1, 2, \cdots)$. Since $E_{N;M}$ is a subset of $E_{N;M+1}$ and $\lim_{M\to\infty} m(E_{N;M}) = m(E_N) > 0$, we may also assume that M is so chosen that $m(E_{N;M}) > 0$.

Now on $E_{N;M}$ the function $z_j(x)$ defined in $(\mathrm{H}_{3;j})$ is bounded and measurable, hence integrable. Consequently, there is a unique constant c_j such that

$$(2.4) \qquad 0 = \int_{E_{N;M}} [z_j(x) - c_j]dx = \int_{E_{N;M}} z_j(x)dx - c_j m(E_{N;M}).$$

Now define $\eta_i(x) \equiv 0$ for $i \neq j$, and

$$(2.5) \qquad \eta_j(x) = \int_{x_1}^{x} [z_j(t) - c_j] I_{N;M} dt,$$

where $I_{N;M}$ is the characteristic function of the set $E_{N;M}$. The functions $\eta_i(x)$ are clearly Lipschitzian on x_1x_2 and satisfy $\eta_i(x_1) = 0 = \eta_i(x_2)$; moreover, $\eta_i' = 0$ a.e. on $CE_{N;M}$, the complement of $E_{N;M}$ on x_1x_2. In view of the Lipschitzian character of the functions $\eta_i(x)$ and property (P_2), there exists a positive constant a_0 such that for $|a| < a_0$ the arc $y_i = y_i(x) + a\eta_i(x)$ belongs to K and lies in the region Δ^* of Lemma 2.1. We shall also suppose that a_0 is chosen so small that $a_0|\eta_j(x)| < \delta$ on x_1x_2 and $a_0|\eta_j'(x)| \leqq 1$ a.e. on this interval. In particular, we have that $|y_j'(x) + a\eta_j'(x)| \leqq N$ a.e. on E_N, and hence a.e. on the subset $E_{N;M}$. For such values of a the integral $I[y+a\eta]$ exists and is finite, and

$$(2.6) \qquad \begin{aligned} I[y + a\eta] = &\int_{CE_{N;M}} f(x, y + a\eta, y')dx \\ &+ \int_{E_{N,M}} f(x, y + a\eta, y' + a\eta')dx. \end{aligned}$$

The existence and finiteness of the first integral of (2.6) follows from (2.3) for $Y(x) = y(x) + a\eta(x)$. The existence and finiteness of the second integral of (2.6) follows from the measurability of the integrand, together with the fact that a.e. on $E_{N;M}$ the set $(x, y(x)+a\eta(x), y'(x) + c\eta'(x))$ is in the region R_N of Lemma 2.1, and consequently $|f(x, y(x)+a\eta(x), y'(x)+a\eta'(x))| \leqq \psi_N(x) \leqq M$ on this set.

Now, in view of $(\mathrm{H}_{3;j})$, for $|a| < a_0$ the partial derivative of $f(x, y(x)+a\eta(x), y'(x))$ with respect to a exists and does not exceed in absolute value the integrable function $|\eta_j(x)|\phi_j(x)$, which, in turn, is not greater than $(\delta/a_0)\phi_j(x)$. Moreover, a.e. on $E_{N;M}$ the partial derivative of $f(x, y(x)+a\eta(x), y'(x)+a\eta'(x))$ with respect to a exists and does not exceed in absolute value $(|\eta_j(x)|+|\eta_j'(x)|)\psi_N(x) \leqq [(\delta+1)/a_0]M$. It then follows [3, p. 216] that the derivative of (2.6) with respect to a exists for $|a|<a_0$, and is given by the usual differentiation under the integral sign. For $a=0$ this derivative is

$$\begin{aligned}(2.7)\qquad &\int_{CE_{N;M}} f_{y_j}\eta_j dx+\int_{E_{N;M}} [f_{y_j}\eta_j+f_{r_j}\eta_j']dx \\ &\qquad =\int_{x_1}^{x_2}[f_{y_j}\eta_j+f_{r_j}\eta_j']dx=\int_{x_1}^{x_2} z_j\eta_j' dx,\end{aligned}$$

where the arguments of the partial derivatives of f occurring in (2.7) are $(x, y(x), y'(x))$. The last relation follows by the usual integration by parts, and use of the conditions $\eta_j(x_1)=0=\eta_j(x_2)$. Since the minimizing property of y requires that this derivative be zero, and as the integral of $\eta_j'(x)$ over x_1x_2 is equal to zero, it follows, in particular, that

$$\begin{aligned}(2.8)\qquad 0&=\int_{x}^{x_2}[z_j(x)-c_j]\eta_j'(x)dx=\int_{x_1}^{x_2}[z_j(x)-c_j]^2 I_{N;M}dx \\ &=\int_{E_{N;M}}[z_j(x)-c_j]^2dx.\end{aligned}$$

From (2.8) we have that $z_j(x)=c_j$ a.e. on $E_{N;M}$. Now the constant c_j is seemingly dependent on N and M. Since, however, $E_{N;M}$ is a subset of $E_{N;M+1}$ $(M=1, 2, \cdots)$ and $\lim_{M\to\infty} m(E_{N;M})=m(E_N)$, it is seen that a.e. on E_N we have $z_j(x)=c_j$, where c_j is the constant determined by any set $E_{N;M}$ for which $m(E_{N;M})>0$. Similarly, since E_N is a subset of E_{N+1} $(N=1, 2, \cdots)$ and $\lim_{N\to\infty} m(E_N)=x_2-x_1$, it follows that $z_j(x)=c_j$ a.e. on x_1x_2.

3. **A second differentiability theorem.** We shall now prove another differentiability property of a minimizing arc for (1.1) under the further hypotheses:

(H) *$f(x, y, r)$ is of class C' in all its arguments on R;*

(P_2') *If $y_i=y_i(x)$ $(x_1\leqq x\leqq x_2)$ is an arc of K, and the function $\zeta(x)$ is such that $\zeta(x_1)=x_1$, $\zeta(x_2)=x_2$, while there is a constant $k\geqq 1$ such that $(x'-x'')/k\leqq \zeta(x')-\zeta(x'')\leqq k(x'-x'')$ for arbitrary values x', x'' satisfying $x_1\leqq x''<x'\leqq x_2$, then $y_i=y_i(\zeta(x))$ is also an arc of K.*

Condition (P_2') is satisfied if K is either the class of arcs of bounded variation, the class of absolutely continuous arcs, or the class of Lipchitzian arcs.

THEOREM 3.1. *Suppose that $y_i=y_i(x)$ $(x_1\leqq x\leqq x_2)$ is a minimizing* *arc for* (1.1) *in a class* K *satisfying* (P_0), (P_1) *and* (P_2'), *and, in addition to hypothesis* (H), *the integrand satisfies with this arc the further condition:*

$(\mathrm{H}_{3,0})$ *There exists a constant* δ *and a function* $\phi_0(x)$ *integrable on* x_1x_2 *and such that if* x *and* $\bar{x}$ *are on this interval and satisfy* $|\bar{x}-x|<\delta$, *then a.e. on* x_1x_2,

$$|f_x(\bar{x}, y(x), y'(x))| \leqq \phi_0(x).$$

Then there exists a constant c_0 *such that*

$$z(x) \equiv f(x, y(x), y'(x)) - \sum_{i=1}^{n} y_i'(x)f_{r_i}(x, y(x), y'(x))$$
$$- \int_{x_1}^{x} f_x(t, y(t), y'(t))dt$$

is equal to c_0 *a.e. on* x_1x_2.

Condition $(\mathrm{H}_{3;0})$ implies, in particular, that the partial derivative $f_x(x, y(x), y'(x))$ is integrable on x_1x_2. Corresponding to a positive integer N, again let E_N denote the subset of x_1x_2 on which the derivatives $y_i'(x)$ exist and satisfy $|y_i'(x)| \leqq N-1$ $(i=1, \cdots, n)$. Hypothesis (H) implies that along the minimizing arc the integrand f and its partial derivatives with respect to r_i $(i=1, \cdots, n)$ are bounded functions of x on E_N. As in the proof of Theorem 2.1, we may suppose that N is so large that $m(E_N)>0$. Now on E_N the function $z(x)$ is bounded and measurable, and hence integrable. Let c_0 be the constant determined by the condition

$$0 = \int_{E_N} [z(x) - c_0]dx = \int_{E_N} z(x)dx - c_0 m(E_N),$$

and set

$$\eta(x) = \int_{x_1}^{x} [z(t) - c_0]I_N dt,$$

where I_N is the characteristic function of the set E_N. Then $\eta(x)$ is Lipschitzian on x_1x_2, $\eta(x_1)=0=\eta(x_2)$, and for a_0 sufficiently small the function $T(t; a)=t+a\eta(t)$ is such that $(t'-t'')/2 \leqq T(t'; a)-T(t''; a) \leqq 2(t'-t'')$ for all values of a such that $|a|<a_0$, and for arbitrary t', t'' satisfying $x_1 \leqq t''<t' \leqq x_2$; moreover, $T(x_1; a)=x_1$, $T(x_2; a)=x_2$. Hence for such values of a the equation

$$x = t + a\eta(t) \equiv T(t; a)$$

has a unique solution $t=\zeta(x; a)$, which is Lipschitzian on x_1x_2 and such that $(x'-x'')/2 \leqq \zeta(x'; a) - \zeta(x''; a) \leqq 2(x'-x'')$ for arbitrary values x', x'' satisfying $x_1 \leqq x'' < x' \leqq x_2$. We have also that $\zeta(x_1, a) = x_1$, $\zeta(x_2, a) = x_2$, while $\zeta(x; 0) \equiv x$ on x_1x_2. It then follows from (P_2') that $y_i(x; a) = y_i(\zeta(x; a))$ $(x_1 \leqq x \leqq x_2)$ defines an arc of K such that $y_i(x; 0) = y_i(x)$.

From known theorems on the change of variable in a simple integral [3, p. 215] we then have that the integral (1.1) calculated along the arc $y_i = y_i(x; a)$ is finite and equal to

$$\int_{x_1}^{x_2} f(t + a\eta(t), y(t), y'(t)/[1 + a\eta'(t)])[1 + a\eta'(t)]dt$$

$$= \int_{E_N} f(t + a\eta(t), y(t), y'(t)/[1 + a\eta'(t)])[1 + a\eta'(t)]dt \tag{3.1}$$

$$+ \int_{CE_N} f(t + a\eta(t), y(t), y'(t))dt.$$

Since $|y_i'| \leqq N-1$ on E_N, it follows from (H) that a.e. on E_N the partial derivative with respect to a of the integrand of the first integral of (3.1) exists for $|a| < a_0$; moreover, this partial derivative is uniformly bounded for x on this subset of E_N and $|a| < a_0$. Hypotheses (H) and $(H_{3;0})$ imply that for sufficiently small values of a the partial derivative with respect to a of the integrand of the second integral exists and does not exceed in absolute value the integrable function $|\eta(t)|\phi_0(t)$. It then follows from the previously applied theorem on differentiability of integrals that the derivative of (3.1) exists for $a = 0$, and is given by

$$\int_{x_1}^{x_2} \left[\left(f - \sum_{i=1}^{n} y_i' f_{r_i}\right)\eta' + f_x\eta\right] dx, \tag{3.2}$$

where the arguments of f and its partial derivative occurring in (3.2) are $(x, y(x), y'(x))$.

In view of the minimizing property of the arc $y_i = y_i(x)$, the integrability of f_x along this arc, and the conditions $\eta(x_1) = 0 = \eta(x_2)$, we conclude that

$$0 = \int_{x_1}^{x_2} z(x)\eta'(x)dx = \int_{x_1}^{x_2} [z(x) - c_0]\eta'(x)dx$$

$$= \int_{x_1}^{x_2} [z(x) - c_0]^2 I_N dx = \int_{E_N} [z(x) - c_0]^2 dx.$$

That is, $z(x) = c_0$ a.e. on E_N. As E_N is a subset of E_{N+1} $(N = 1, 2, \cdots)$

and $\lim_{N\to\infty} m(E_N) = x_2 - x_1$, it then follows that $z(x) = c_0$ a.e. on x_1x_2.

4. **Further remarks.** The general differentiability theorems of §§2 and 3 involve, respectively, hypotheses $(H_{3;j})$ and $(H_{3,0})$ for the considered minimizing arc. Now suppose that the integrand function $f(x, y, r)$ satisfies (H_1) and (H_2), and, in addition to (P_0), (P_1) and (P_2), the class K contains only arcs $y_i = y_i(x)$ such that the derivative functions $y_i'(x)$ are integrable on x_1x_2. Then every arc of K for which $I[y]$ is finite satisfies $(H_{3;j})$ whenever the integrand function satisfies the following condition:

$(H^*_{3;j})$ *There exist positive constants* M_1, M_2, δ *such that if* (x, y, r) *and* $(x, \bar{y}, r)$ *are points of* R *with* $|\bar{y}_j - y_j| < \delta$, $\bar{y}_i = y_i$ $(i \neq j)$ *then*

$$|f_{y_j}(x, \bar{y}, r)| \leqq M_1 |f(x, y, r)| + M_2\left[1 + \sum_i r_i^2\right]^{1/2}. \tag{4.1}$$

Correspondingly, if $f(x, y, r)$ satisfies (H), and, in addition to (P_0), (P_1) and (P_2'), the class K contains only arcs $y_i = y_i(x)$ such that the functions $y_i'(x)$ are integrable on x_1x_2, then every arc of K for which $I[y]$ is finite satisfies $(H_{3;0})$ under the following additional hypothesis:

$(H^*_{3;0})$ *There exist positive constants* M_1, M_2, δ *such that if* (x, y, r) *and* $(\bar{x}, y, r)$ *are points of* R *with* $|\bar{x} - x| < \delta$, *then*

$$|f_x(\bar{x}, y, r)| \leqq M_1 |f(x, y, r)| + M_2\left[1 + \sum_i r_i^2\right]^{1/2}. \tag{4.2}$$

The above hypotheses $(H^*_{3;j})$ and $(H^*_{3,0})$ have been used by McShane [2]. In connection with the above remarks it is to be noted that if the term $M_2[1 + \sum_i r_i^2]^{1/2}$ in (4.1) and (4.2) is replaced by M_2, then the condition that the functions $y_i'(x)$ be integrable on x_1x_2 is not needed to insure that every arc of K for which $I[y]$ is finite will satisfy the corresponding condition $(H_{3;j})$ or $(H_{3;0})$.

Finally, it is to be remarked that as a very special case of Theorem 2.1 one has the following generalization of the fundamental lemma of the calculus of variations:

If $M(x)$ *is finite and measurable on* x_1x_2, *and the integral*

$$\int_{x_1}^{x_2} M(x)\eta'(x)dx$$

is zero for every Lipschitzian function $\eta(x)$ *satisfying* $\eta(x_1) = 0 = \eta(x_2)$, *and for which the above integral exists and is finite, then there exists a constant* c *such that* $M(x) = c$ *a.e. on* x_1x_2.

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A NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

M. EVANS MUNROE

Pettis[1] raised the question whether or not separability of the range space implies almost everywhere weak differentiability of Pettis integrals. Phillips[2] has given an example which answers this question in the negative. His construction is based on a sequence of orthogonal vectors in Hilbert space. We present here a different example of the same type of function. Our basic construction is that of a function defined to the space C. Using that function as a basis, we are able to give a specific construction of such a function defined to each member of a large class of Banach spaces.

1. Metric density properties of a non-dense perfect set. Let $B\subset[0, 1]$ be a non-dense perfect set of measure one-half, and let $\overline{B}$ be its complement. $\overline{B}$ may be constructed by taking the sum of a set of open intervals classified as follows:

1 interval of length 1/4,
2 intervals each of length 1/16,
4 intervals each of length 1/64,
. . . .
2^{n-1} intervals each of length $1/2^{2n}$,
. . . .

We shall refer to the intervals of length $1/2^{2n}$ as *intervals of $\overline{B}$ of order n.* We shall assume that each interval of $\overline{B}$ of order n is the center portion of the space either between two intervals of $\overline{B}$ of lower order or between one such interval of $\overline{B}$ and an end point of the unit interval. These spaces we shall refer to as *gaps of order n*, and we shall denote such a gap by the symbol G_n. If $\overline{B}$ is constructed as noted above, then for each n, any two sets each of the form $G_n\cdot\overline{B}$ are congruent; hence we shall use G_n to denote a gap of order n, and we shall not find it necessary to specify which one.

The following three lemmas are now obvious.

1.1. LEMMA. $|\overline{B}\cdot G_n| = 1/2^{2n-1}$.

1.2. LEMMA. $|G_n| = 1/2^n + 1/2^{2n-1}$.

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[1] See [3, p. 303]. Numbers in brackets refer to the references cited at the end of the paper.

[2] See [4, p. 144].

1.3. Definition. If I is any subinterval of $[0, 1]$, we define

$$\rho(I) = |\overline{B}\cdot I|/|I|.$$

1.4. Lemma. $\rho(G_n)=1/(1+2^{n-1})$.

The following important lemma demonstrates a lower bound on the density function ρ. This lower bound may tend to zero as $|I|$ tends to zero, but it is independent of the location of I.

1.5. Lemma. *If*

$$|I| \geqq |G_n| = 1/2^n + 1/2^{2n-1},$$

then

$$\rho(I) > 1/(2 + 2^{n-1}).$$

First suppose $|I| = |G_n|$. For all such I the G_n have minimum values for ρ; for if $I=G_n$ and then is moved a little to the right or left, some points of B are excluded and only points of $\overline{B}$ are included—thus obviously increasing $\rho(I)$—until the interval of $\overline{B}$ is covered. However, this interval of $\overline{B}$ is of order at most $n-1$ and hence has measure at least $1/2^{2n-2}$; thus so long as I contains this interval of $\overline{B}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^n + 1/2^{2n-1}} = \frac{2}{1+2^{n-1}} = 2\rho(G_n).$$

If I is moved on beyond this interval of $\overline{B}$ the above argument applies again by considering the movement in the reverse direction. Comparison with Lemma 1.4 now shows that Lemma 1.5 is established in case $|I| = |G_n|$.

Consider now the effect of increasing $|I|$. If $|I| = |G_n|$ and I contains no interval of $\overline{B}$ of order less than or equal to $n-1$, then one end point of I (let us assume it is the right-hand one) lies in the closure of such an interval of $\overline{B}$. Thus any small extension of I to the right will (until the interval of $\overline{B}$ is covered) add to I only points of $\overline{B}$, thus obviously increasing $\rho(I)$. If $I=G_n$ at the start, a small extension to the left will have the same effect. Otherwise an extension to the left may be regarded as a translation to the left and a subsequent extension to the right, and these cases have already been discussed. In case I contains an interval of $\overline{B}$ of order not greater than $n-1$, we have $|\overline{B}\cdot I| > 1/2^{2n-2}$; thus if $|I| \leqq |G_{n-1}| = 1/2^{n-1}+1/2^{2n-3}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^{n-1} + 1/2^{2n-3}} = \frac{1}{2+2^{n-1}}.$$

If $|I| > |G_{n-1}|$, the above argument may be repeated with $n-1$ substituted for n, thereby obtaining an even larger lower bound for $\rho(I)$. This proves Lemma 1.5 for all cases.

1.6. LEMMA. *For all* $I \subset [0, 1]$,

$$\rho(I) > |\overline{B}\cdot I|^{1/2}/2^{5/2}.$$

We shall prove Lemma 1.6 by showing that, for each n, the required inequality holds for

$$1/2^{2n+1} < |\overline{B}\cdot I| \leqq 1/2^{2n-1}.$$

This will cover all possibilities. For $|\overline{B}\cdot I|$ in this range, we consider first the case $|I| < |G_n| = 1/2^n + 1/2^{2n-1}$. In this case

$$\begin{aligned} \frac{1}{\rho(I)} &= \frac{|I|}{|\overline{B}\cdot I|} < \frac{|I|}{1/2^{2n+1}} < \frac{1/2^n + 1/2^{2n-1}}{1/2^{2n+1}} = 4 + 2^{n+1} \\ &= 4 + 2^{3/2}[1/2^{2n-1}]^{-1/2} \leqq 4 + 2^{3/2}|\overline{B}\cdot I|^{-1/2}. \end{aligned}$$

Considering now the case $|I| \geqq |G_n|$ (and assuming $|\overline{B}\cdot I|$ still in the same range) we have, using Lemma 1.5 and the above inequalities,

$$1/\rho(I) < 2 + 2^{n-1} < 4 + 2^{n+1} \leqq 4 + 2^{3/2}|\overline{B}\cdot I|^{-1/2}.$$

Now for all $I \subset [0, 1]$, $|\overline{B}\cdot I| \leqq 1/2$; hence $|\overline{B}\cdot I|^{-1/2} \geqq 2^{1/2}$; hence $4 \leqq 2^{3/2}|\overline{B}\cdot I|^{-1/2}$. Combining this with the above results, we have

$$1/\rho(I) < 4 + 2^{3/2}|\overline{B}\cdot I|^{-1/2} \leqq 2^{5/2}|\overline{B}\cdot I|^{-1/2},$$

whence

$$\rho(I) > |\overline{B}\cdot I|^{1/2}/2^{5/2}.$$

2. An approximately continuous function whose integral is non-differentiable. For each $t \in B$ we define the function $f_t(x)$ for $x \in [0, 1]$ as follows

$$f_t(x) = \begin{cases} 0 & \text{for } x \leqq t \text{ or } \quad x \in B, \\ |\overline{B}\cdot[t, x]|^{-3/4} & \text{for } x > t \text{ and } x \in \overline{B}. \end{cases}$$

2.1. THEOREM. *For each* $t \in B$, $f_t(x)$ *is an integrable function of* x, *and for* $x_2 \geqq x_1 \geqq t$,

$$\int_{x_1}^{x_2} f_t(x)dx = 4(|\overline{B}\cdot[t, x_2]|^{1/4} - |\overline{B}\cdot[t, x_1]|^{1/4}).$$

Let $z = |\overline{B}\cdot[t, x]|$. Since the intervals of $\overline{B}$ are dense in $[0, 1]$, this defines z as a strictly monotone function of x; hence x is a single-valued function of z, and we may write $f_t[x(z)]$. Now for $x > t$ and $x \in \overline{B}$, $dx = dz$; thus the function $z(x)$ is measure preserving over $\overline{B}$

$\cdot [t, x]$ and hence maps $B \cdot [t, x]$ into a set of measure zero. Therefore, for almost all z, $f_t[x(z)] = z^{-3/4}$; and $dx = dz$ except where $f_t(x) = 0$; so

$$\int_{x_1}^{x_2} f_t(x)dx = \int_{z_1}^{z_2} z^{-3/4} dz = 4(z_2^{1/4} - z_1^{1/4}).$$

2.2. THEOREM. *For $t \in B$, the function*

$$F_t(x) = \int_t^x f_t(u)du$$

is not differentiable with respect to x at $x = t$.

Again letting $z = |\overline{B} \cdot [t, x]|$, and using Theorem 2.1 and Lemma 1.6, we have

$$\frac{F_t(x)}{x - t} = \frac{4z^{1/4}}{x - t} = 4z^{-3/4}\left(\frac{z}{x - t}\right) = 4z^{-3/4}\rho([t, x])$$
$$> 4z^{-3/4}(z^{1/2}/2^{5/2}) = (4z)^{-1/4}.$$

Thus

$$\limsup_{x \to t} \frac{F_t(x)}{x - t} \geq \lim_{z \to 0} (4z)^{-1/4} = \infty.$$

In the next section we shall make further use of the functions $f_t(x)$ and their properties as shown in Theorems 2.1 and 2.2. We might note here, however, that $f_t(x)$ is approximately continuous at t provided $\overline{B}$ has metric density zero at t. This is true for almost all t in B; hence for such t, $f_t(x)$ furnishes a specific example of an approximately continuous function whose integral is not differentiable.

3. A Pettis integral in the space C which is not almost everywhere weakly differentiable. We shall here define a function $\phi(x)$ from $[0, 1]$ to the space C. Our notation will be as follows: For each $x \in [0, 1]$, $\phi(x)$ stands for a continuous function on $[0, 1]$; we denote this continuous function by $\phi_x(t)$. We shall define the functions $\phi_x(t)$ by defining a function $\phi(x, t)$ over the unit square and setting $\phi_x(t) = \phi(x, t)$. We first define $\phi(x, t)$ over a portion of the unit square as follows:

$$\phi(x, t) = \begin{cases} 0 & \text{for} \quad x \in B, \\ f_t(x) & \text{for} \quad t \in B. \end{cases}$$

Since $f_t(x) = 0$ for $x \in B$, these statements are consistent.

3.1. LEMMA. *For a fixed x, $\phi(x, t)$ is continuous in t over B.*

This statement follows immediately from the fact that if one end point of I is fixed, $|\overline{B}\cdot I|^{-3/4}$ is a continuous function of the other end point over any set such that $|I|$ is bounded away from zero. This latter restriction causes no difficulties here. If $x\in B$, $\phi(x, t)=0$; if $x\in\overline{B}$, dist $(x, B)>0$.

We now continue the definition of $\phi(x, t)$. For each $x\in\overline{B}$, let $\phi(x, t)$ be continued linearly over each interval of the set $t\in\overline{B}$. This completes the definition of $\phi(x, t)$ over the entire unit square, and it is clear that for each x, $\phi(x, t)$ is continuous in t over $[0, 1]$.

3.2. THEOREM. *$\phi(x)$ is integrable in the sense of Pettis. For each measurable set $E\subset[0, 1]$, its integral over E is the function*

$$\Phi_E(t) = \int_E \phi(u, t)du.$$

We show this by considering the functions $\phi^{(n)}(x)$ whose values are the continuous functions $\phi_x^{(n)}(t)=\phi^{(n)}(x, t)$ where

$$\phi^{(n)}(x, t) = \begin{cases} \phi(x, t) & \text{for } \phi(x, t) \leqq n, \\ n & \text{for } \phi(x, t) > n. \end{cases}$$

It is easily seen that for each $t\in[0, 1]$, each $\phi^{(n)}(x, t)$ is bounded and continuous in x over $\overline{B}$. Thus[3] each $\phi^{(n)}(x)$ is weakly continuous over $\overline{B}$. Since $\phi^{(n)}(x)=\theta$ for $x\in B$, it is clear that each $\phi^{(n)}(x)$ is weakly measurable. Since C is a separable space, it follows[4] that each $\phi^{(n)}(x)$ is measurable. Now each $\phi^{(n)}(x)$ is bounded, hence Bochner integrable, hence Pettis integrable, therefore integrable with respect to each of the linear functionals $\gamma_t[\phi^{(n)}(x)]=\phi_x^{(n)}(t)$; thus

$$\Phi^{(n)}(E) = \int_E \phi^{(n)}(u)du$$

is the continuous[5] function

$$\Phi_E^{(n)}(t) = \int_E \phi^{(n)}(u, t)du.$$

Clearly for each x and each t,

$$\lim_{n\to\infty} \phi_x^{(n)}(t) = \phi_x(t);$$

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[3] See [1, p. 224, Theorem 8].

[4] See [3, Theorem 1.1].

[5] An independent proof of continuity of $\Phi_E^{(n)}(t)$ is unnecessary. Since $\phi^{(n)}(x)$ is Pettis integrable, it is integrable to an element of C; and the set $\{\gamma_t\}$ of linear functionals defines this element uniquely.

furthermore this approximation is monotone in n. Thus $\|\phi^{(n)}(x)-\phi(x)\|$ is bounded in n for each x; hence[6] $\phi^{(n)}(x)\rightarrow\phi(x)$ weakly for each x. It now follows[7] that $\phi(x)$ is Pettis integrable provided the sequence $\{\Phi^{(n)}(E)\}$ converges with respect to the norm in C; that is, provided $\{\Phi_E^{(n)}(t)\}$ converges uniformly in t. We shall complete the proof of Theorem 3.2 by showing that for each measurable $E\subset[0, 1]$

$$\Phi_E(t) = \int_E \phi(u, t)du$$

exists for each t and that this function is the uniform limit of the sequence $\{\Phi_E^{(n)}(t)\}$.

To show that $\Phi_E(t)$ exists is trivial. For $t\in B$, this follows from Theorem 2.1. For each x, $\phi(x, t)$ is extended linearly over each interval of $t\in\overline{B}$; hence for $t\in\overline{B}$, $\phi(x, t)\leqq\phi(x, t_1)+\phi(x, t_2)$ where t_1 and t_2 are each in B. This completes the proof of integrability.

Now with each $t\in[0, 1]$ we associate two numbers t_1 and t_2 as follows: t_1 is the greatest number such that $t_1\in B$ and $t_1\leqq t$; t_2 is the smallest number such that $t_2\in B$ and $t_2\geqq t$. Geometrically this means that if $t\in B$, $t_1=t=t_2$, while if $t\in\overline{B}$, t_1 and t_2 are the left and right points respectively of the interval of $\overline{B}$ in which t is located.

Now for $t_1\leqq x<t_2$, $\phi(x, t)\leqq\phi(x, t_1)$ while for $x\geqq t_2$, $\phi(x, t)\leqq\phi(x, t_2)$. Thus for any given $t\in[0, 1]$, it is possible to have $\phi(x, t)>n$ only for those values of x for which either

$$t_1 \leqq x < t_2 \quad\text{and}\quad |\overline{B}\cdot[t_1, x]| < n^{-4/3}$$

or

$$x \geqq t_2 \quad\text{and}\quad |\overline{B}\cdot[t_2, x]| < n^{-4/3}.$$

Outside these two intervals $\phi(x, t)-\phi^{(n)}(x, t)=0$; hence if we denote these intervals by I_1 and I_2, we have

$$\begin{aligned}\int_E [\phi(x, t) - \phi^{(n)}(x, t)]dx &\leqq \int_{I_1}\phi(x, t)dx + \int_{I_2}\phi(x, t)dx \\ &\leqq \int_{I_1}\phi(x, t_1)dx + \int_{I_2}\phi(x, t_2)dx \\ &< 2\int_0^{n^{-4/3}} z^{-3/4}dz = 8n^{-1/3}.\end{aligned}$$

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[6] See [1, p. 224, Theorem 8].

[7] See [3, Theorem 4.1].

Thus, clearly, $\lim_{n\to\infty}\Phi_E^{(n)}(t)=\Phi_E(t)$ uniformly in t. This completes the proof of Theorem 3.2.

3.3. THEOREM. *If* $x_0 \in B$, $\Phi(x)=\int_0^x\phi(u)du$ *is not weakly differentiable at* x_0.

By Theorem 2.2, it fails to be differentiable at x_0 with respect to the linear functional $\gamma_{x_0}[\Phi(x)]=\Phi_x(x_0)$.

4. **Extension to other spaces of continuous functions.** The function $\phi(x)$ of §3 may be used as the basis for the construction of a large set of examples as follows:

4.1. THEOREM. *If* Ω *is a compact metric space containing non-denumerably many points and if* $C(\Omega)$ *is the Banach space of all continuous functionals on* Ω, *then there is a function* $\psi(x)$ *from the unit interval to* $C(\Omega)$ *such that* $\psi(x)$ *is Pettis integrable but* $\Psi(E)=\int_E\psi(x)dx$ *fails to be weakly differentiable on a set of positive measure.*

Since Ω is non-denumerable, it contains a perfect set. This perfect set is a complete metric space which is dense in itself and hence contains a homeomorph Π of the Cantor set B.[8]

Let $B=h(\Pi)$ be the homeomorphic mapping of Π into B. Then $h(\omega)$ is a continuous function defined over Π, assuming values between 0 and 1, and assuming for some $\omega\in\Pi$ each value in the set B. Let $H(\omega)$ be a continuous extension[9] of $h(\omega)$ over the whole of Ω with $0\leqq H(\omega)\leqq 1$.

Now for each $t\in[0, 1]$ we define

$$K(t) = \underset{\omega}{\mathcal{E}}\{H(\omega) = t\}.$$

It should be noted that although for some t, $K(t)$ may be vacuous, for each $t\in B$, $K(t)$ contains at least one point.

Referring back to the functions $\phi_x(t)$ of §3, we now define

$$\psi_x(\omega) = \phi_x(t) \quad \text{for} \quad \omega \in K(t).$$

It follows from the continuity in t of each function $\phi_x(t)$ and from the continuity of $H(\omega)$ that for each $x\in[0, 1]$, $\psi_x(\omega)$ is continuous over Ω. For each $x\in[0, 1]$ we now let $\psi(x)$ be the element $\psi_x(\omega)$ of $C(\Omega)$.

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[8] See [2, p. 228]. The author is indebted to the referee for the suggestion that non-denumerability of Ω is sufficient to insure the existence of Π.

[9] See [2, p. 211]. In connection with our remark in the introduction that we have a specific construction applicable to the more general spaces, it should be noted that this extension theorem is not merely an existence proof. A definite formula for the extension is given.

That $\psi(x)$ has the required properties may be seen as follows: To show integrability, we note that for each $\omega \in \Omega$, $\psi_x(\omega)$ is identical (as a function of x) with $\phi_x(t)$ for some $t \in [0, 1]$. Then noting that Banach's criterion for weak convergence[10] applies to the space $C(\Omega)$, we apply the proof of Theorem 3.2. To show non-differentiability, we note that for each $t \in B$ there is an $\omega \in \Omega$ such that $\psi_x(\omega) = \phi_x(t)$ for all $x \in [0, 1]$. The proof of Theorem 3.3 then applies.

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[10] This is used in the proof of Theorem 3.2. See footnotes 3 and 6.

NOTE ON A NOTE OF H. F. TUAN

I. S. COHEN

The following theorem is proved.

THEOREM. *If Z is a nilpotent matrix with elements in a field K, then the replicas of Z are those and only those matrices which are of the form $f(Z)$, where $f(x)$ is an additive*[1] *polynomial in $K[x]$.*

The concept of a replica was introduced by Chevalley,[2] who proved this theorem when K is of characteristic zero. The theorem was proved in general by H. F. Tuan[3] by elementary methods. The object of this note is to give a simplification of Tuan's proof; in particular, computations involving the specific form of Z are avoided.

If $h(x)$ is additive, then according as K is of characteristic 0 or p, $h(x)$ will have one of the two forms

$$(1) \qquad tx, \qquad \sum_{j=1}^{m} t_j x^{p^j} \qquad (t, t_j \in K).$$

For if $h(x)$ had any other terms, then $h(x)+h(y)=h(x+y)$ would contain product terms $x^\alpha y^\beta$, $\alpha>0$, $\beta>0$. Conversely, polynomials of the form (1) are clearly additive. If $h(x)=\sum_{k=0}^{s} c_k x^k (c_k \in K)$, then we define

$$h^{[i]}(x) = \sum_{k=i}^{s} C_{k,i} c_k x^{k-i},$$

where the $C_{k,i}$ are binomial coefficients. Evidently

$$h^{(i)}(x) = i! h^{[i]}(x), \qquad h(x+y) = \sum_{i=0}^{s} h^{[i]}(x) y^i.$$

It follows from this that $h(x)$ is additive if and only if $c_0=0$ and $h^{[i]}(x)=c_i$ for $i>0$.

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[1] A polynomial $f(x)$ is additive if $f(x+y)=f(x)+f(y)$. The statement of the theorem in terms of the additivity of $f(x)$ rather than in terms of the explicit form (1), as well as the use of the derived polynomials $f^{[i]}(x)$ to replace explicit computation with binomial coefficients, was suggested by Professor Jacobson.

[2] Claude Chevalley, *A new kind of relationship between matrices*, Amer. J. Math. vol. 65 (1943) pp. 521–531. We make use of the definitions and notations of this paper.

[3] Hsio-Fu Tuan, *A note on the replicas of nilpotent matrices*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 305–312, in particular Theorems (A) and (D).

If $f(x)$ is additive then it is easily seen that $f(Z)_{r,s}=f(Z_{r,s})$; hence (Z) is a replica of Z if $f(x)$ is additive. The converse follows from:[4]

LEMMA 1. *Let Z be a nilpotent matrix over a field K and let Z' be a matrix such that $Z'=f(Z)$, $Z'_{0,2}=g(Z_{0,2})$, where $f(x)$ and $g(x)$ are polynomials without constant terms. Then f may be assumed to be additive.*

This lemma does indeed imply the theorem since Chevalley[2] has proved (p. 529) that a replica Z' of Z satisfies the hypothesis of the lemma.

We now recall some definitions. If A and B are $n\times n$ matrices over K, then $A\times B$ is the $n^2\times n^2$ matrix formed by the $n\times n$ array of matrices $a_{ij}B$, where $A=(a_{ij})$. The following statements are evident:

$$\begin{aligned} (A\times B)(C\times D) &= AC\times BD,\\ (A+A_1)\times B &= A\times B+A_1\times B,\\ cA\times B &= c(A\times B)\quad\text{if}\quad c\in K,\\ A\times B=0\quad&\text{implies}\quad A=0\quad\text{or}\quad B=0. \end{aligned}\tag{2}$$

Finally, $Z_{0,2}$ is defined as $Z\times E+E\times Z$; E is the $n\times n$ unit matrix, where n is the dimension of Z.

Since Z is nilpotent and since it may be assumed that $Z\neq 0$ (for Lemma 1 is trivial if $Z=0$), there is an integer m such that

$$Z^m\neq 0,\qquad Z^{m+1}=0,\qquad 1\leqq m\leqq n-1.\tag{3}$$

LEMMA 2. *If $A_0, A_1, \cdots, A_m$ are $n\times n$ matrices such that*

$$A_0\times E+A_1\times Z+\cdots+A_m\times Z^m=0,$$

then $A_0=A_1=\cdots=A_m=0$.

PROOF. Multiplying by $E\times Z^m$, we obtain $A_0\times Z^m=0$, $A_0=0$ by (2) and (3). Multiplying successively by $E\times Z^{m-1}$, $E\times Z^{m-2}$, $\cdots$, we obtain $A_1=A_2=\cdots=0$.

PROOF OF LEMMA 1. In view of (3) the polynomial $f(x)$ may be assumed to be of degree at most m. We show that it must then be additive. Now

$$(Z_{0,2})^k=(Z\times E+E\times Z)^k=\sum_{i=0}^{k}C_{k,i}Z^{k-i}\times Z^i.$$

It follows that $(Z_{0,2})^{2m+1}=0$, so that $g(x)$ may be assumed of degree at most $2m$. We observe incidentally that Lemma 2 implies that

---

[4] Loc. cit., Theorems (B) and (C).

$(Z_{0,2})^m \neq 0$; if K is of characteristic 0, then also $(Z_{0,2})^{2m} \neq 0$, for $(Z_{0,2})^{2m} = C_{2m,m} Z^m \times Z^m$.

Now we have

$$
\begin{aligned}
Z'_{0,2} &= g(Z_{0,2}) = g(Z \times E + E \times Z) = \sum_{i=0}^{2m} g^{[i]}(Z \times E)(E \times Z)^i \\
&= g(Z) \times E + g^{[1]}(Z) \times Z + g^{[2]}(Z) \times Z^2 + \cdots \\
&\quad + g^{[m]}(Z) \times Z^m.
\end{aligned} \tag{4}
$$

On the other hand, placing $f(x) = \sum_{i=1}^{m} a_i x^i$, we have

$$
\begin{aligned}
Z'_{0,2} &= Z' \times E + E \times Z' = f(Z) \times E + E \times f(Z) \\
&= f(Z) \times E + a_1 E \times Z + a_2 E \times Z^2 + \cdots + a_m E \times Z^m.
\end{aligned} \tag{5}
$$

A comparison of (4) and (5) gives, by Lemma 2,

$$
\begin{aligned}
g(Z) &= f(Z), & g^{[i]}(Z) &= a_i E, & i &= 1, \cdots, m; \\
g(x) &\equiv f(x)(x^{m+1}), & g^{[i]}(x) &\equiv a_i(x^{m+1}), & i &= 1, \cdots, m.
\end{aligned}
$$

From the first congruence, $g^{[i]}(x) \equiv f^{[i]}(x)(x^{m+1-i})$, and from the second, $f^{[i]}(x) \equiv a_i(x^{m+1-i})$. Since $f^{[i]}(x)$ is of degree at most $m-i$, $f^{[i]}(x) = a_i$. By a previous remark it follows that $f(x)$ is additive, and this completes the proof.

HARVARD UNIVERSITY

NOTE ON THE MERSENNE NUMBERS M_{157} AND M_{167}

H. S. UHLER

The information here offered supplements two short papers written respectively by Charles B. Barker[1] and the present author.[2] The delay in presenting this material was due primarily to the miscarriage of a letter dated February 2, 1945, to Barker from the author which was not delivered to the addressee nor returned to the sender's address as given on the envelope.

With reference to the smaller Mersenne number Barker wrote (January 23, 1945): "I also completed the proof for M_{157}—still ignorant of your result—and I found precisely the same value for the last residue that you determined." This agreement should establish beyond reasonable doubt the composite character of M_{157}. The significance of this verification is somewhat enhanced by the facts that Barker performed direct long division while my potential divisions were effected by multiplication by the reciprocal of the modulus M_{157}, and that the first computation was done on a ten-bank electrically driven machine and the second on a similar eight-bank machine.

In the case of M_{167} the value of the 166th residue as finally obtained by me on December 2, 1944 was 59077 89471 97183 05021 04043 18653 76339 69475 17591 49076. Unfortunately the two investigators used different Lucasian sequences for r_{166}. Barker employed 3, 7, 47, 2207, $\cdots$ and the writer used 4, 14, 194, 37634, $\cdots$. This fully explains the lack of agreement between the datum given by Barker[1] and the residue shown above. Nevertheless since both values are nonzero they agree qualitatively as to the composite character of M_{167}.

Incidentally Barker has just informed me that he used other checks than the one given in the Bulletin, such as the auxiliary moduli 10^5+1 and 10^7+1. Finally the character of the Mersenne number M_{229} is now under investigation by the author on the basis of the sequence 4, 14, 194, $\cdots$.

YALE UNIVERSITY

Received by the editors October 25, 1945.

[1] Charles B. Barker, *Proof that the Mersenne number M_{167} is composite*, Bull. Amer. Math. Soc. vol. 51 (1945) p. 389.

[2] H. S. Uhler, *First proof that the Mersenne number M_{157} is composite*, Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) pp. 314–316.

ON THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

PAUL ERDÖS

The cyclotomic polynomial $F_n(x)$ is defined as the polynomial whose roots are the primitive nth roots of unity. It is well known that

$$F_n(x) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)}.$$

For $n<105$ all coefficients of $F_n(x)$ are ± 1 or 0. For $n=105$, the coefficient 2 occurs for the first time. Denote by A_n the greatest coefficient of $F_n(x)$ (in absolute value). Schur proved that $\lim\sup A_n=\infty$. Emma Lehmer[1] proved that $A_n>cn^{1/3}$ for infinitely many n. In fact she proved that infinitely many such n's are of the form pqr with p, q, and r prime. In the present note we are going to prove that $A_n>n^k$ for every k and infinitely many n. This is implied by the still sharper theorem:

THEOREM 1.[2] *For infinitely many n*

$$A_n > \exp\left[c_1(\log n)^{4/3}\right].$$

Specifically we may take $n=2\cdot 3\cdot 5\cdots p_k$ for sufficiently large k.

Since

$$\max_{|x|=1} |F_n(x)| \leqq A_n[\phi(n)+1],$$

Theorem 1 follows at once from the following theorem.

THEOREM 2. *For infinitely many n*

$$\max_{|x|=1} |F_n(x)| > \exp\left[c_2(\log n)^{4/3}\right].$$

For the proof of Theorem 2 we require several lemmas.

LEMMA 1. *Let $f(x)$ be a polynomial of highest coefficient 1 of degree m with all its roots on the unit circle. Suppose that in the unit circle $f(x)$ assumes its maximum at x_0 ($|x_0|=1$), and let y_0 be the root of $f(x)$ closest to x_0. Then the arc between x_0 and y_0 is not less than π/m; and if it equals π/m, $f(x)=x^m-1$.*

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[1] Bull. Amer. Math. Soc. vol. 42 (1936) p. 389. Reference to the older literature can be found in this paper.

[2] Throughout the paper c_i denotes a positive constant.

This is a theorem of M. Riesz.[3]
Set $n=2\cdot 3\cdot 5\cdots p_k$.

LEMMA 2. $p_k\sim\log n$.

LEMMA 3. $\phi(n)\sim e^{-\gamma}n/\log\log n$, *where* γ *is Euler's constant.*

Lemma 2 is a well known consequence of the prime number theorem, and Lemma 3 follows from Lemma 2 and a theorem of Mertens.[4]

LEMMA 4. *Suppose* $p_k^a\leqq u\leqq p_k^{4/3}$ *where* $1<a\leqq 4/3$, *and let* N *be the number of integers not greater than* u *which are prime to* n. *Then for sufficiently large* k,

$$N>(1+c_3)u\phi(n)/n.$$

PROOF. The integers in question are primes greater than p_k. By the prime number theorem

$$N\sim u/\log u-p_k/\log p_k\sim u/\log u.$$

Now $1/\log u\geqq 3/(4\log p_k)$; and, by Lemmas 2 and 3, $\log p_k\sim\log\log n\sim e^{-\gamma}n/\phi(n)$. Lemma 4 now follows from $e^{-\gamma}<3/4$.

LEMMA 5. *Suppose that for an infinite number of integers* m *we are given a polynomial* $g_m(x)$ *of highest coefficient* 1 *of degree* m, *with all its roots on the unit circle and symmetric with respect to the real axis, and with* $|g_m(1)|=1$, *Let* t_m *be a function of* m *such that* $t_m/m<\pi$ *and* $t_m\rightarrow\infty$ *as* $m\rightarrow\infty$. *Suppose constants* c_4, ϵ $(0<\epsilon<1\quad 0<c_4<1)$ *given such that for any* u *with* $t_m^{1-\epsilon}\leqq u\leqq t_m$ *the number of roots of* $g_m(x)=g_m(e^{i\theta})$ *with* $|\theta|\leqq u/m$ *is greater than* $(1+c_4)u/\pi$, *that is greater than* $(1+c_4)$ *times the number of roots of* $x^m=1$ *in the same interval. Then for sufficiently large* m

$$\max_{|x|=1}|g(x)|>\exp(c_5t_m).^5$$

PROOF. Denote by A, B, C the following arcs:

$$A:\ |\theta|\leqq t_m^{1-\epsilon}/m,$$
$$B:\ |\theta|\leqq t_m/m,$$
$$C:\ |\theta|\leqq (t_m+\pi)/m.$$

We define new polynomials $h_m(x)=x^m+\cdots$ as follows. Outside B,

---

[3] Jber. Deutschen Math. Verein. vol. 23 (1914) pp. 354–368.

[4] See, for example, Hardy and Wright, *Introduction to the theory of numbers*, p. 349.

[5] An analogous but weaker theorem has been stated in a previous paper (Ann. of Math. vol. 44 (1943) p. 337.

h_m and g_m have the same roots. In A, h_m has no roots. On $B-A$ we place consecutive roots spaced by the angle $2\pi/m$. Finally the remaining roots of h_m are placed at the end points of B, half at each.

Let $\theta_1, \theta_2, \cdots$ and $\phi_1, \phi_2, \cdots$ denote the arguments of the roots of g_m and h_m in B above the real axis; we number them in increasing order of magnitude. Our construction implies

$$\phi_r \geq \min (t_m^{1-\epsilon}/m + 2\pi r/m,\ t_m/m) \tag{1}$$

while the hypothesis of Lemma 5 translates into

$$\theta_r \leq \max (t_m^{1-\epsilon}/m,\ 2\pi r/(1 + c_4)m). \tag{2}$$

From (1) and (2) we deduce $\phi_r \geq \theta_r$, that is, the process has pushed roots of g_m away from 1. If $e^{i\theta}$, $e^{i\alpha}$ are points above the real axis respectively inside and outside B, then

$$\partial \,|\, (e^{i\alpha} - e^{i\theta})(e^{i\alpha} - e^{-i\theta}) \,|\, /\partial\theta = 8 \sin \theta(\cos \alpha - \cos \theta) < 0$$

so that the process reduces g_m outside B, that is,

$$|\, h_m(x) \,| \leq |\, g_m(x) \,| \tag{3}$$

outside B.

We shall next prove

$$|\, h_m(1) \,| > \exp (c_6 t_m). \tag{4}$$

Take m large enough so that $t_m^{\epsilon} \geq 2$ and confine r to the interval

$$\begin{aligned} (1 + c_4)t_m^{1-\epsilon}/2\pi &\leq r \\ &\leq (1 + c_4)t_m/4\pi. \end{aligned} \tag{5}$$

Then (2) reduces to

$$\theta_r \leq 2\pi r/(1 + c_4)m. \tag{2$'$}$$

Since from (5) and $c_4 < 1$ we have $2\pi r \leq t_m$, (1) similarly becomes

$$\phi_r \geq 2\pi r/m. \tag{1$'$}$$

Combining (1′) and (2′) we find $\phi_r/\theta_r - 1 \geq c_4$ whence

$$|\, 1 - \exp (i\phi_r) \,| \geq c_7(1 - \exp (i\theta_r) \,|.$$

From this it follows that $|h_m(1)| \geq c_7^R |g_m(1)|$, where R is the number of values of r permitted in (5). Since for large m, $R > c_8 t_m$, we have $c_7^R > \exp(c_6 t_m)$, proving (4).

Let X denote the number of roots of h_m at the end points of B.

It follows from our hypothesis that $X > c_4 t_m/\pi$. We define a further polynomial $k_m(x) = x^m + \cdots$ by placing roots at the points with arguments $\pm\pi/m$, $\pm 3\pi/m$, $\pm 5\pi/m$, $\cdots$ on the arc A. If the number of these points is Y, then $Y < c_9 t_m^{1-\epsilon}$. We place $(X-Y)/2$ roots of k_m at each end point of B and otherwise the roots of h_m and k_m coincide.

In moving the Y roots to pass from h_m to k_m the greatest migration along the arc is from t_m/m to π/m. Hence

$$|k_m(1)| \geqq (c_{10}/t_m)^Y |h_m(1)|. \tag{6}$$

Outside the arc C the movement of roots tends to increase h_m; the worst place is right at the end points of C and there we have the similar estimate

$$|k_m(x)| \leqq (c_{11}t_m)^Y |h_m(x)| \tag{7}$$

outside C. Now k_m has roots all through B spaced $2\pi/m$ apart, and $k_m \neq x^m - 1$. By Lemma 1, k_m must assume its maximum at a point x_0 outside C. Then, applying (3), (7), (6), and (4) in succession, we obtain

$$\begin{aligned} |g_m(x_0)| &> (c_{11}t_m)^{-Y}(c_{10}/t_m)^Y \exp(c_6 t_m) \\ &= (c_{12}/t_m)^{2Y} \exp(c_6 t_m) \\ &> \exp(c_5 t_m), \end{aligned}$$

which completes the proof of Lemma 5.

Proof of Theorem 2. Take $n = 2\cdot 3\cdot 5 \cdots p_k$. It is well known that $|F_n(1)| = 1$. In view of Lemma 4, we may apply Lemma 5 with m, $g_m(x)$, t_m, ϵ replaced by $\phi(n)$, $F_n(x)$, $p_k^{4/3}$ and $1/6$ respectively. The conclusion is precisely Theorem 2.

Theorem 2 is probably not the best result. It should not be difficult to extend the method to show that

$$A_n > \exp(\log n)^k$$

for every k and infinitely many n. A very much stronger result may be true, namely

$$A_n > \exp(c_{13}n/\log\log n) \tag{8}$$

for infinitely many n. If true, this would be essentially the best possible result, because for a certain c_{14} and all n,

$$A_n < \exp(c_{14}n/\log\log n).$$

(The proof is omitted.)

The possibility that (7) may be true is indicated in the following theorem.

THEOREM 3. *Let n be the product of k distinct primes $p_1, p_2, \cdots, p_k$ and denote by $f(x)$ the number of integers not greater than x which are relatively prime to n. Let*

$$P=(1-1/p_1)\cdots(1-1/p_k),$$
$$g(x)=f(x)-Px.$$

Then there exists an x_0, $1\leqq x_0<n$, such that

$$|g(x_0)|>c_{15}2^{k/2}(\log k)^{-1/2}. \tag{9}$$

The connection between Theorem 3 and (8) is as follows. The function $g(x)$ measures how much the roots of $F_n(x)$ are displaced from the uniform distribution. Lemma 5 then suggests that it might be possible to prove

$$\max_{|x|=1}|F_n(x)|>\exp\left[c_{16}2^{k/2}(\log k)^{-1/2}\right]. \tag{10}$$

If in particular we take $n=2\cdot 3\cdot 5\cdots p_k$, then

$$p_k\sim\log n\sim k\log k,$$

and (10) is a result similar to (8).

PROOF OF THEOREM 3. The usual sieve process gives

$$f(x)=[x]-\sum_{p|n}\left[\frac{x}{p}\right]+\sum_{pq|n}\left[\frac{x}{pq}\right]-\cdots=\sum_{r|n}\mu(r)[x/r].$$

Define $(x/r)=x/r-[x/r]$, so that $g(x)=\sum_{r|n}\mu(r)(x/r)$. Then

$$\sum_{x=1}^{n}[g(x)]^2=\sum_{r,s|n}\mu(r)\mu(s)\sum_{x=1}^{n}(x/r)(x/s).$$

Let $r=ud$, $s=vd$, $(u,v)=1$. Then the final sum becomes

$$\begin{aligned}\sum_{x=1}^{n}(x/r)(x/s)&=nd(rs)^{-2}\sum_{a=0}^{d-1}[a+a+d+\cdots+a+(u-1)d]\\&\quad\cdot[a+a+d+\cdots+a+(v-1)d]\\&=n(3rs-3r-3s+d^2+2)/12rs.\end{aligned}$$

In carrying out the second summation, the first three terms vanish. Hence

$$\begin{aligned}12\sum_{x=1}^{n}[g(x)]^2&=n\sum_{r,s|n}(d^2+2)\mu(r)\mu(s)/rs\\&=n(2^kP+2P^2).\end{aligned}$$

Now $P > c_{17}/\log k$, as follows a fortiori from Lemma 3. Hence

$$\sum_{x=1}^{n} [g(x)]^2 > c_{18} n 2^k/\log k,$$

from which the existence of an x_0 satisfying (9) follows at once.

I am indebted to Dr. Irving Kaplansky who shortened some of the proofs and extensively revised the first draft of the manuscript.

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ON SOME ASYMPTOTIC FORMULAS IN THE THEORY OF PARTITIONS

PAUL ERDÖS

Let $p(n)$ denote the number of unrestricted partitions of n. $p_k(n)$ denotes the number of partitions of n into precisely k summands, or what is the same into partitions whose largest summand is k. Auluck, Chowla and Gupta[1] announced the following conjecture:

For n fixed let $p_{k_0}(n)$ be the greatest $p_k(n)$; that is, $p_{k_0}(n) \geqq p_k(n)$. Then

$$k_0 \sim c^{-1}n^{1/2}\log n, \qquad c = \pi(2/3)^{1/2}. \tag{1}$$

They prove that

$$n^{1/2} < k_0 < (1+\delta)c^{-1}n^{1/2}\log n$$

for every $\delta > 0$ if n is sufficiently large.

In the present note we shall prove (1). In fact we shall prove that

$$k_0 = c^{-1}n^{1/2}\log n + an^{1/2} + o(n^{1/2}) \quad \text{where} \quad c/2 = e^{-ca/2}. \tag{2}$$

They also conjectured that for $k_1 < k_2 \leqq k_0$, $p_{k_1}(n) \leqq p_{k_2}(n)$ and for $k_0 < k_1 < k_2$, $p_{k_1}(n) < p_{k_2}(n)$. They verify this conjecture for $n \leqq 32$. Recently Todd[2] published a table of all the $p_k(n)$ for $n \leqq 100$, and it is easy to verify the conjecture for $n \leqq 100$. I am unable to prove or disprove this conjecture. They also remark that $p_{k_0}(n)$ differs from $c^{-1}n^{1/2}\log n$ by less than 1 for $n \leqq 32$; (2) shows that for large n the difference tends to infinity.

Lehner and I[3] proved that if we denote

$$P_k(n) = \sum_{r \leqq k} p_r(n)$$

then for $k = c^{-1}n^{1/2}\log n + \lambda n^{1/2}$ we have the asymptotic formula

$$P_k(n)/p(n) = (1+o(1))\exp\left(-(2/c)e^{-cx/2}\right). \tag{3}$$

In proving (2) we shall use (3) a great deal, we shall also use the well known asymptotic formula

$$p(n) = (1+o(1))(1/4\cdot 3^{1/2}n)\exp(cn^{1/2}). \tag{4}$$

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[1] J. Indian Math. Soc. vol. 6 (1942) pp. 105–112.

[2] Proc. London Math. Soc. vol. 48 (1944) pp. 229–242.

[3] Duke Math. J. vol. 8 (1941) pp. 335–345.

Let $f(n)$ tend to infinity arbitrarily slowly; we easily obtain from (3) that for $k_1 = [c^{-1}n^{1/2}\log n + f(n)n^{1/2}]$, $k_2 = [c^{-1}n^{1/2}\log n - f(n)n^{1/2}]$,

$$(1/p(n))(P_{k_1}(n) - P_{k_2}(n)) \to 1 \quad \text{as} \quad n \to \infty. \tag{5}$$

We immediately obtain from (4) and (5) that for some $k_2 < k_3 < k_1$

$$p_{k_3}(n) > c_1 p(n)/n^{1/2} > (c_2/n^{3/2}) \exp (cn^{1/2}). \tag{6}$$

$c_1, c_2, \cdots$ denote absolute constants. Thus

$$p_{k_0}(n) \geqq p_{k_3}(n) > (c_2/n^{3/2}) \exp (cn^{1/2}). \tag{7}$$

Now we show that for sufficiently large c_3

$$k_0 < c^{-1}n^{1/2}\log n + c_3 n^{1/2}. \tag{8}$$

Let $k_4 \geqq c^{-1}n^{1/2}\log n + c_3 n^{1/2}$. It clearly follows from the definition of $p_k(n)$ and $P_k(n)$ that $p_{k_4}(n) = P_{k_4}(n-k_4) < p(n-k_4)$. Thus from (4)

$$\begin{aligned} p_{k_4}(n) &< (c_4/n) \exp (c(n - k_4)^{1/2}) < (c_4/n) \exp c(n^{1/2} - k_4/2n^{1/2}) \\ &< (c_4/n) \exp (c(n^{1/2} - \log n/2 - c_3/2)) \\ &< (c_2/n^{3/2}) \exp (cn^{1/2}) < p_{k_0}(n) \end{aligned}$$

for sufficiently large c_3, and this proves (8).

Next we prove that for sufficiently large c_5

$$k_0 > c^{-1}n^{1/2}\log n - c_5 n^{1/2}. \tag{9}$$

Suppose (9) does not hold. We obtain from (7) that for some $k_0 < c^{-1}n^{1/2}\log n - c_5 n^{1/2}$

$$p_{k_0}(n) > (c_2/n^{3/2}) \exp (cn^{1/2}). \tag{10}$$

We shall show that (10) leads to a contradiction. First we show that

$$p_k(n) \leqq p_{k+i}(n + j) \qquad \text{for } j \geqq i. \tag{11}$$

We have

$$p_k(n) \leqq p_{k+i}(n + i) \leqq p_{k+i}(n + j). \tag{12}$$

The first inequality of (12) we obtain by mapping the partition $a_1 + \cdots + k$ of $p_k(n)$ into $a_1 + \cdots + (k+i)$ which belongs to $p_{k+i}(n+i)$, the second part we obtain by adding $j-i$ 1's to every partition of $p_{k+i}(n+i)$; this proves (11).

Put $[n^{1/2}] = b$; we have from (10) and (11) for $0 \leqq i \leqq b$

$$\begin{aligned} p_{k_0+i}(n + b) &\geqq p_{k_0}(n) > (c_2/n^{3/2}) \exp (cn^{1/2}) \\ &> (c_6/n^{3/2}) \exp (c(n + b)^{1/2}). \end{aligned}$$

Thus

$$\sum_{i=0}^{b} p_{k_0+i}(n+b) > (c_6/n) \exp (c(n+b)^{1/2}). \tag{13}$$

Now we obtain from (5) that for every ϵ and sufficiently large c_5 and n

$$\sum_{k>k_0+b} p_k(n+b) > (1-\epsilon)p(n+b). \tag{14}$$

The proof of (14) follows immediately from the fact that $k_0+b < c^{-1}n^{1/2} \log n-(c_5-1)n^{1/2}$, thus (5) can be applied. From (13) and (14) we have

$$\begin{aligned} p(n+b) &> \sum_{i=0}^{b} p_{k_0+i}(n+b) + \sum_{k>k_0+b} p_k(n+b) \\ &> (1-\epsilon)p(n+b) + (c_{6/n}) \exp (c(n+b)^{1/2}). \end{aligned}$$

Thus

$$\epsilon p(n+b) > (c_{6/n}) \exp (c(n+b)^{1/2}),$$

which contradicts (4); this proves (9).

We now know from (8) and (9) that k_0 has to satisfy

$$c^{-1}n^{1/2} \log n - c_5 n^{1/2} < k_0 < c^{-1}n^{1/2} \log n + c_3 n^{1/2}.$$

Put

$$k_0 = c^{-1}n^{1/2} \log n + xn^{1/2}.$$

We obtain from (3) and (4) that

$$\begin{aligned} p_{k_0}(n) &= P_{k_0}(n-k_0) \\ &= (1+o(1))p(n)n^{-1/2} \exp (-cx/2-(2/c) \exp (-cx/2)).^{4} \end{aligned} \tag{15}$$

The right side is maximal if $c/2=\exp(-cx/2)$, which completes the proof of (2).

We immediately obtain from (2) and (15) that

$$\lim p_{k_0}(n)n^{1/2}/p(n) = \exp (-ca/2-(2/c) \exp (-ax/2)).$$

It would be easy to sharpen the error term $o(n^{1/2})$ in (2) by getting an error term in (3), but it seems very hard to get a sufficiently good inequality to prove the conjecture of Auluck, Chowla and Gupta.

Denote by $Q(n)$ the number of partitions of n into unequal parts. $Q_k(n)$ denotes the number of partitions of n into precisely k unequal parts. Define k_0 by

$$Q_{k_0}(n) \geqq Q_k(n).$$

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[4] This formula is due to Auluck, Chowla and Gupta (ibid).

It has been conjectured that for $k_1 < k_2 \leqq k_0$, $Q_{k_1}(n) < Q_{k_2}(n)$ and for $k_0 < k_1 < k_2$, $Q_{k_1}(n) \geqq Q_{k_2}(n)$. This conjecture we can not decide. But by using Theorem 3.3 of our paper with Lehner we can show that

$$k_0 = 2 \log 2n^{1/2}/\pi(1/3)^{1/2} + dn^{1/4} + o(n^{1/4})$$

for a certain constant d. Also

$$\lim n^{1/4}Q_{k_0}(n)/Q(n) \to e, \quad \text{for a certain constant } e.$$

We do not discuss the proofs. They are similar but slightly more complicated than the proof of (2).

It would be interesting to get an asymptotic formula for $p_k(n)$ and $Q_k(n)$. Perhaps the first step would be to get an asymptotic formula for $\log p_k(n)$. It is easy to see that for $k = o(n^{1/2})$

$$\log p_k(n) = o(n^{1/2})$$

and if $k/n^{1/2} \to \infty$

$$\log p_k(n)/\log p(n) \to 1.$$

The proofs can be obtained easily by simple Tauberian theorems.

University of Michigan

THOMAS FRANKLIN HOLGATE

1859-1945

The death of Thomas F. Holgate on April 10, 1945, at the age of eigthy-six years, marks the passing of another of the figures prominent in the early history of the Society. He was one of the organizers of the Chicago Section, and was its first secretary, serving in that capacity for eight years. Although the press of administrative work thereafter lessened his active participation, he missed but few meetings of the Society in Chicago up to the time of his death.

His early life was spent in Canada, where he was born, not far from Kingston, on April 8, 1859. From the University of Toronto he received a bachelor's degree in 1884 and a master's degree in 1889. Meantime he had been teaching, first in rural schools, then in a collegiate institute. At the end of his work at Toronto he had definitely decided to become a mathematician in the field of pure geometry, in spite of a comparatively late start. The next year he enrolled as a graduate student at Clark University, and obtained his doctor's degree there in 1893.

At Clark he had formed a close friendship with Henry S. White. The latter had come to Northwestern University in 1892, replacing, in the course of that year, E. H. Moore, who had accepted the headship of the department of mathematics in the newly founded University of Chicago. When this vacancy developed, White urged Holgate to come to Evanston, and in 1893 the latter began his career at Northwestern, which was to last for fifty-two years, as professor of mathematics, dean of the college, acting president of the university, and finally as dean emeritus.

Dean Holgate was fond of quoting John Bunyan's line, "I was once a promising professor," thus referring to the nine years from 1893 to 1902 before he started on the administrative work of his next seventeen years. To this time belong his published papers on various subjects in pure geometry, including ruled surfaces of the fourth order, and loci determined in part by imaginary elements. He had been inspired by Reye's *Geometrie der Lage*, a translation of which he published in 1898. To the same period belongs his text *Elementary geometry, plane and solid*, following the Euclidean form of presentation. At a later time, in 1911, he wrote the monograph on *Modern pure geometry* in J. W. A. Young's *Monographs on modern mathematics*. For many years thereafter he was developing and recasting his ideas on this subject. The final result was his text on *Projective pure geometry*, published in 1930.

In 1902 he became dean of the College of Liberal Arts. From that time on his opportunity to do mathematical work was more and more limited. In 1904 he was a secretary of the geometry section of the mathematical congress at St. Louis, and in 1908 he acted as a secretary of the International Congress of Mathematicians at Rome.

A new responsibility was added to his duties as dean when, in 1904, the president of the university, E. J. James, unexpectedly resigned to take a similar post at the University of Illinois. For the two years 1904–1906, Dean Holgate, as acting president, carried the university through a critical period when substantial funds were obtained without which the development of the institution would have been seriously checked. In 1905 Professor White resigned his headship of the department of mathematics to go to Vassar. These cumulating responsibilities led Dean Holgate, with great regret, to give up the secretaryship of the Chicago Section, a post thereafter filled so capably by H. E. Slaught. He retained, however, his titular headship of the department of mathematics for nearly twenty years. The writer, as acting head during that period, found the relation an ideal one, terminated only at Dean Holgate's request that he be allowed to drop the title.

In the trying years from 1916 to 1919 he was again called to be acting president of the university. His sound judgment and administrative ability brought Northwestern through what seemed then to be a crisis, but proved to be the beginning of a period of unprecedented expansion and addition to resources. The strain was great, and at the age of sixty, in 1919, he retired from both the presidency and deanship with the title of dean emeritus. He did not cease teaching, however, until he reached the age of seventy-five, and his active interest in the department of mathematics continued until his death. In his last years he prepared a history of the department, with some account of the well known mathematicians who had studied or taught there before 1910 (E. H. Moore, J. W. Young, W. H. Bussey, L. L. Dines, C. S. Slichter, among others).

His later years of teaching included the year 1921–1922 at the University of Nanking. He had always been deeply interested in foreign students, and returned to Northwestern to become more than ever their guide and friend, and even financial helper.

If he had never undertaken administrative work, Dean Holgate would still have been notable as a teacher. He was a master of presentation, with a fine sense of proportion, and a feeling that what he taught was significant and important.

He is best known, however, to thousands of Northestern alumni, as "the Dean," who somehow, in their student years, managed to be the friend and adviser of every one of them, and never forgot their names or faces. "Nothing human was foreign to him"; his life was made up of friendships, where he always gave more than he took. We of the faculty remember him best thus, but we cannot forget, either, his wise advice and practical sense, and judgment of men. In spite of his kindliness he had a firmness of character that once drew from a colleague the remark, "There he stands, like a rock."

No account of Dean Holgate could omit his many activities outside the University. He was particularly interested in the Methodist Church, serving five times as an active and prominent member of its quadrennial General Conference. He became deeply interested in negro colleges, advising and visiting them frequently, and helping in their financial affairs. He was a member of the Church Board of Education for over twenty years, and for a time its treasurer. During the years 1923–1925 he was president of the Chicago Church Federation. In 1917–1918 he was president of the North Central Association of Colleges and Secondary Schools. This is by no means a complete list of his activities. He was never one to rust away, even in his old age. He used to say, with a touch of self-deprecating humor, that as he grew older he became busier and busier—on less and less.

During his life he received many honors, among them L.L.D. degrees from the University of Illinois (1905), Queen's University (1919), and Northwestern University (1937). A far greater tribute was the respect and affection of students and colleagues, and of his wide circle of friends outside the university.

D. R. CURTISS

CLASSES OF SEQUENCES OF POSITIVE NUMBERS

E. H. MOORE[1]

The results presented in the following pages engaged the attention of Professor E. H. Moore at various times, but were never written up for publication. Notes dated December 1909 indicate that the impetus came in connection with utilization of the "no last absolutely convergent series" theorem of Du Bois-Reymond (see *General analysis*, p. 48) and the paper of Landau in Nachr. Ges. Wiss. Göttingen (1907) pp. 25–27) in which Landau proves that if $\{a_n\}$ is a sequence such that $\sum |a_n b_n|$ converges for all sequences such that $\sum |b_n|^p$, with $p>1$, converges, then $\sum |a_n|^{p/(p-1)}$ converges. This latter theorem proved interesting not only because it is a sort of converse to the theorem: If $\sum |a_n|^p$ converges and if $\sum |b_n|^{p/(p-1)}$ converges then $\sum |a_n b_n|$ converges, a consequence of Hoelder's inequality, but also because the theorem which Landau actually proved was the contrapositive equivalent theorem, namely, if $\sum |a_n|^{p/(p-1)}$ diverges, then there exists a sequence such that $\sum |b_n|^p$ converges, but $\sum |a_n b_n|$ diverges. By noting the trivial identity $a_n = (a_n b_n)(1/b_n)$, the last theorem has the skeleton form: If the sequence $\{a_n\}$ belongs to a class $\mathfrak{M}_1$ (namely, the divergent $p/(p-1)$ power) then there exist two sequences $\{b_n\}$ and $\{c_n\}$ such that $a_n = b_n c_n$, and $\{b_n\}$ belongs to $\mathfrak{M}_2$ (convergent p the power) and $\{c_n\}$ to $\mathfrak{M}_3$ (divergent). Since logical questions were always of great interest to him, this instance in which the contrapositive of a given theorem has an independently interesting statement led him to speculations concerning the true nature of contrapositive proof. Among other things he stressed the idea that many so-called contrapositive proofs could be formulated to advantage as direct proofs.

From his notes, it appears that the subject matter of the first half of this paper engaged his continued attention from December 1910 to January 1911. But apparently he made no attempt to write up the results for publication until about 1918. However, only a few sections of the manuscript were completed. On several occasions he worked out a "multiplication table" (see end of Part II) and added to it from time to time. We have not always been able to connect the references which some of these tables contain with his notes, and consequently reconstruct his proofs. Instead we have used the final re-

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[1] Prepared for publication by H. H. Goldstine and T. H. Hildebrandt.

sults and taken the liberty of arranging the proofs in what appears to us as a possible logical order.

The first part of the paper, which serves as an introduction, gives the basic notation and certain general theorems and their interrelations (most of which are taken from the unfinished manuscript). In the second part of the paper, we give a body of theorems sufficient to cover all of the cases of the "multiplication table" which concerns itself with a system of classes of sequences of positive numbers, and is in a sense closed in this system. The third part of the paper is concerned with what Professor Moore called a General Multiplication Theorem, which he stated in a letter to T. H. Hildebrandt dated January 4, 1926, but gave no hint as to its proof. Since no proof appears in his notes, we have worked out a proof of the theorem in question.

Whether Professor Moore would have approved of the publication of these results, since in many ways they seem trivial, we do not know. We have found some of these things interesting to play with and it is not impossible that they may suggest further results which are worth while. The connection with linear functional operations on sequences, the closure of the multiplication table, and the fact mentioned in footnote 5 that the Hoelder inequality basic to the Pringsheim-Landau Theorem, and the proof of the Abel-Dini Theorem basic to the Landau Theorem, both can be derived from the same inequality: $x^e \leqq 1+e(x-1)$ for $0<e<1$, $x>0$, are some of the points which made these developments attractive to us.

H. H. GOLDSTINE
T. H. HILDEBRANDT

I. INTRODUCTION

1.1. **The class V.** We consider throughout sequences of positive numbers: $\alpha=\{a_n\}$, with $a_n>0$. Such sequences may be regarded as positive-valued functions on the range $\mathfrak{P}^{\text{III}}$ of positive integers, or geometrically as points in the space of infinite dimensions, limited however to points all of whose coordinates are positive. We designate the space of all such sequences by V and call its elements either sequences, functions or points. Since we shall be interested only in elements of V, it will be understood that the word *sequence refers only to a sequence consisting of positive terms.*

We shall be interested in certain subspaces or subclasses of V, the points of each class being all of the points of V which possess a certain property. For convenience we use the same notation for the class and the characteristic property, that is, the class P is the class of all sequences having the property P. We shall use the standard notation relative to sum and product of classes, that is, $P \cup Q$, the *sum* of P and Q, or the totality of all sequences which have either the property P or the property Q, $P \cap Q$, the *product* of P and Q, or all sequences having both the property P and Q. $-P$ shall be the *complement* of P relative to V, that is, the class of sequences not having the property P.

In an obvious way we say that a class P contains a class Q if every element of Q is contained in P. With respect to properties, Q implies P if the class P contains Q. Under the circumstances we can speak of a monotonically increasing or decreasing set of classes.

1.2. **Fundamental subclasses of V.** We consider the following basic classes (properties):

F: the class of all bounded sequences, that is, of all points in which the least upper bound of the coordinates is finite;

L_0: the class of all sequences having zero as limit;

L_∞: the class of all sequences having infinity as limit;

C: the class of all sequences with convergent sum;

D: the class of all sequences with divergent sum.

Obviously $C = -D$.

Any sequence and real number p defines the sequence $\alpha^p = \{a_n^p\}$. For every property P and number p, the property P_p of a point α is defined as the property P of α^p. Thus C_p is the class of all sequences such that $\sum a_n^p$ is convergent; D_p is defined in a similar way.

We note that if $0<e<f$, then the class C_f properly contains the class C_e, C_{-f} properly contains C_{-e}, D_f is contained in D_e and D_{-f} is contained in D_{-e}. Obviously if $e>0$, then F and F_e, L_0 and $(L_0)_e$, L_∞ and $(L_\infty)_e$ are each identical. But $(L_0)_{-e} = (L_0)_{-1}$ is L_∞ and $(L_\infty)_{-e} = (L_\infty)_{-1}$ is L_0. F_{-e} is the same as F_{-1}, which is the set of sequences such that $\limsup_n (1/a_n) < \infty$, that is, the class of sequences bounded from zero. Then $-F_{-1}$ is the class such that $\liminf_n a_n = 0$.

1.3. **Derived classes.** If for $e>0$, the system of classes P_e is monotonic increasing in e, then we define

$$P_e' = \bigcup_{0<f<e} P_f \quad \text{and} \quad P_e'' = \bigcap_{f>e} P_f.$$

Thus $C_e' = \bigcup_{0<f<e} C_f$ is the class of sequences α such that there exists an $f<e$ such that α^f is in C. C_e'' is the class of sequences such that α^f is in C for all $f>e$.

If for $e>0$ the system of classes P_e is monotonic decreasing in e, then the inequalities in the definitions are reversed, that is, we define

$$P_e' = \bigcup_{f>e} P_f \quad \text{and} \quad P_e'' = \bigcap_{0<f<e} P_f.$$

This applies then to the definitions of D_e' and D_e'' as well as D'_{-e} and D''_{-e} with $e>0$. In general P' is a sum of classes, P'' is a product of classes.

The application of these notions to the limiting cases $e=0$, ∞, is obvious, but one can define only the following:

C_0'' : for every e the property C_e; C''_{-0}: for every e the property C_{-e};
C'_∞: for some e the property C_e; $C'_{-\infty}$: for some e the property C_{-e};
D_0' : for some e the property D_e; D'_{-0}: for some e the property D_{-e};
D''_∞: for every e the property D_e; $D''_{-\infty}$: for every e the property D_{-e}.

An application of the same process to the classes C' and D' yields nothing new in classes, for we have the following results for $e>0$:

$$1.31 \qquad \begin{array}{llll} \text{(a)} & \bigcup\limits_{0<f<e} C_f' = C_e'; & \text{(b)} & \bigcap\limits_{f>e} C_f' = C_e''; \\ \text{(c)} & \bigcup\limits_{0<f<e} C_f'' = C_e'; & \text{(d)} & \bigcap\limits_{f>e} C_f'' = C_e''; \end{array}$$

and similar results for classes of divergent series and negative exponents. Of these (a) and (d) are obvious. For (b), suppose α belongs to C_e''. Then α is of C_f for all $f>e$. Consequently α belongs to C_g' for all $g>f>e$, that is, for all $g>e$. Conversely, suppose that α belongs to $\bigcap_{f>e}C_f'$. Then α is of C_f' for all $f>e$, that is, there exists a $g<f$, such that α is of C_g. As a consequence α is of C_h for all $h>g>e$, that is, α is of C_e''. Similar considerations show the validity of (c). In a way the classes P' are of the nature of greatest of limits of a set of classes, and P'' of the least of limits, as these results seem to indicate.

1.4. **Fundamental systems of subclasses of** V. We shall be interested in the following two systems of subclasses of V:

SYSTEM I

$$0,\ C_0'',\ C_e',\ C_e,\ C_e'',\ C_f',\ C_f,\ C_f'',\ C'_\infty,\ L_0,\ F\cap -F_{-1},\ F \text{ or } -F_{-1},$$
$$F\cup -F_{-1},\ -L_\infty,\ D''_{-\infty},\ D'_{-f},\ D_{-f},\ D''_{-f},\ D'_{-e},\ D_{-e},\ D''_{-e},\ D'_{-0},\ V.$$

SYSTEM II

$$V,\ D_0',\ D_e'',\ D_e,\ D_e',\ D_f'',\ D_f,\ D_f',\ D''_\infty,\ -L_0,\ -F\cup F_{-1},\ -F \text{ or } F_{-1},$$
$$-F\cap F_{-1},\ L_\infty,\ C'_{-\infty},\ C''_{-f},\ C_{-f},\ C'_{-f},\ C''_{-e},\ C_{-e},\ C'_{-e},\ C''_{-0},\ 0.$$

In the first system, the classes are of increasing content, assuming $0<e<f$, each class being a proper subclass of every succeeding class (excepting that there is a certain ambiguity in the middle of each, the remark preceding holding in each system whichever reading is

adopted). In the second system the classes are complements relative to V of the corresponding classes of the first system, so that the second system is of decreasing content.

The proper monotoneity is in general obvious, excepting for cases of the type: $C'_e < C_e < C''_e$. For this it is sufficient to consider the case where $e=1$. We note that the sequence $a_n = 1/n$ is an element of C_e for $e>1$, but not for $e=1$, that is, is of C''_1, but not of C_1. On the other hand if $a_n = 1/n \log^2 n$, then α is an element of C_1 but not of C_e for $e<1$, that is, of C_1 but not of C'_1. Since $P<Q$ implies $P_{-1}<Q_{-1}$, we get at once $C'_{-e}<C_{-e}<C''_{-e}$. The corresponding results for divergent series follow by taking complements.

1.5. **Multiplications of sequences, and classes of sequence multiplications.** If $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ are any two sequences of V, then $\alpha \times \beta$ is the sequence $\{a_n b_n\}$ of V.

In a similar manner if P is the class of sequences α having the property P and Q the class of sequences β having the property Q, then we denote by $P \times Q$ *the class of sequences* $\alpha \times \beta$ *with* α *in* P *and* β *in* Q, that is, γ is in $P \times Q$ if and only if there exists an α of P and a β of Q such that $\gamma = \alpha \times \beta$. The Pringsheim-Landau Theorem gives a characterization of the classes $P \times Q$, where $P = C_{1/e}$, $Q = C_{1/f}$, and $P \times Q = C_{1/(e+f)}$, $e>0, f>0$.

By analogy, we define the class P/Q *as the class of all elements* γ *such that for all* β *of* Q, *the product sequence* $\gamma \times \beta$ *is of* P. The Landau Theorem states a result for the class P/Q for the case where $P = C_{1/(e+f)}$, and Q is $C_{1/f}$, namely, that P/Q is the class $C_{1/e}$, $e>0$, $f>0$; or as a matter of fact, where P is C_1, Q is C_p and P/Q is $C_{p'}$ with $p>1$ and $(1/p)+(1/p')=1$.

The following relations are self-evident:

1.51 $$P \times Q = Q \times P;\ (P \times Q)_{-1} = P_{-1} \times Q_{-1};\ (P/Q)_{-1} = P_{-1}/Q_{-1}.$$

There also exist relations between the product and quotient processes embodied in the following theorem:

1.52 $$P/Q = -(Q \times -P_{-1})_{-1},$$

from which we deduce by an obvious substitution the corollary:

1.53 $$P \times Q = -(-P_{-1}/Q)_{-1}.$$

For the proof suppose that $P/Q = R$, that is, γ in R is equivalent to a γ such that, for every β of Q, it is true that $\gamma \times \beta$ is in P. Then γ in $-R$ is equivalent to a γ such that there exists a β of Q for which $\gamma \times \beta$ is of $-P$. Let $\gamma \times \beta = \alpha$. Then $\beta \times \alpha^{-1} = \gamma^{-1}$. Then γ of $-R_{-1}$ is equivalent to a γ such that there exists a β of Q and an α of $-P_{-1}$

such that $\gamma=\beta\times\alpha$, symbolically:

$$-R_{-1}=Q\times -P_{-1}\quad\text{and so}\quad R=-(Q\times -P_{-1})_{-1}.$$

The following additional equivalences are now obvious:

1.54
$$\begin{aligned}P/Q &= -(Q\times -P_{-1})_{-1} = -(Q_{-1}\times -P)\\ &= -(-P\times Q_{-1}) = -Q_{-1}/-P_{-1}.\end{aligned}$$

1.55
$$P\times Q = -(-P_{-1}/Q)_{-1} = -(-Q_{-1}/P)_{-1}.$$

A theorem of the form $P\times Q=R$ or $P/Q=R$ can be divided into parts, namely, $P\times Q\supset R$, $P\times Q\subset R$, and $P/Q\supset R$ and $P/Q\subset R$ respectively, the theorem being considered as a relation between the classes. A theorem of the form $P\times Q\supset R$ is called a *factorization* theorem since it asserts that if γ is of R then there exists α of P and β of Q such that $\alpha\times\beta=\gamma$. A theorem of the form $P\times Q\subset R$ will be called a *multiplication* theorem, since it asserts that if α is of P and β is of Q than $\alpha\times\beta$ is of R. A theorem of the form $P/Q\subset R$ will be called a *division* or *Landau* theorem, since it asserts that if α is such that for every β of Q it is true that $\alpha\times\beta$ is of P, then α is of R. The theorem $P/Q\supset R$ is really a multiplication theorem since we have:

1.56. *$P/Q\supset R$ is equivalent to $R\times Q\subset P$.*

For the statement $P/Q\supset R$ is equivalent to the statement that if α is of R then, for every β of Q, it is true that $\alpha\times\beta$ is of P, which is logically equivalent to: if α is of R and β is of Q then $\alpha\times\beta$ is of P, that is, a multiplication theorem. We have as an immediate corollary:

1.57. *$P/Q\supset R$ is equivalent to $P/R\supset Q$.*

Since P/Q is the same as $-Q_{-1}/-P_{-1}$, we have also the following:

1.58. *$P/Q\supset R$ is equivalent to $-Q_{-1}/-P_{-1}\supset R$ is equivalent to $-R_{-1}/-P_{-1}\supset Q$ is equivalent to $-Q_{-1}/R\supset -P_{-1}$ is equivalent to $-R_{-1}/Q\supset -P_{-1}$.*

The theorem and corollary also give:

1.59. *$P\times Q\subset R$ is equivalent to $P\times -R_{-1}\subset -Q_{-1}$ is equivalent to $-R_{-1}\times Q\subset -P_{-1}$.*

We note however that the theorem and corollary are not in general true if the implications are reversed, that is, we have:

1.510. *$P/Q\subset R$ does not always imply $Q\times R\supset P$.*

1.511. *$Q\times R\supset P$ does not always imply $P/Q\subset R$.*

1.512. *$P/Q\subset R$ does not always imply $P/R\subset Q$.*

Instances of these are the following, which depend on theorems to be proved later:

ad 1.510. Take $P=L_0$, $Q=C_1$, $R=F$. Then by 2.43

$$P/Q=L_0/C_1=-(C_1\times -L_\infty)_{-1}=F.$$

On the other hand by 2.41

$$Q \times R = C_1 \times F = C_1,$$

and C_1 does not contain $L_0 = P$.

ad 1.511. Take $Q = C_1$, $R = L_0$, and $P = C_1$. Then by 2.44

$$Q \times R = C_1 \times L_0 = C_1,$$

but by 2.21

$$P/Q = C_1/C_1 = -(C_1 \times -C_{-1})_{-1} = -(C_1 \times D_{-1})_{-1} = F$$

and F is not contained in $R = L_0$.

ad 1.512 Take $P = L_0$, $Q = C$ and $R = F$. Then as in 1.510, $P/Q \subset R$, but by 2.41

$$P/R = L_0/F = -(F \times -L_\infty)_{-1} = L_0$$

and L_0 is not contained in C.

Other instances of 1.510 and 1.512 are obtained if $P = V$, $Q = 0$ and $R = V$.

Even so, every division or Landau theorem is equivalent to a factorization theorem, and conversely, but the equivalence is not as simple as one might expect. We have:

1.513. *The statement* $P/Q \subset R$ *is equivalent to the statement* $Q \times -P_{-1} \supset -R_{-1}$.

This follows immediately from 1.52

In view of these theorems, we shall in the sequel focus primary attention on the product class $P \times Q$, of two classes P and Q. We emphasize however that while a theorem of the form $P/Q \supset R$ has as its equivalent the form $Q \times R \subset P$, a theorem of the form $P/Q \subset R$ has the more complicated equivalent $Q \times -P_{-1} \supset -R_{-1}$.

1.6. **Set addition and set product.** We consider next the effect of set addition and set product on the theorems of multiplication and factorization type. We have the following theorems.

1.61. THEOREM. *If* P_p *and* R_p *are any systems of classes such that for every* p: $P_p \times Q \supset R_p$, *then* $(\cup P_p) \times Q \supset \cup R_p$. *If for every* p: $P_p \times Q \subset R_p$, *then* $(\cup P_p) \times Q \subset \cup R_p$. *Consequently if* $P_p \times Q = R_p$ *then* $(\cup P_p) \times Q = \cup R_p$.

1.62. THEOREM. *If for every* p: $P_p \times Q \subset R_p$, *then* $(\cap P_p) \times Q \subset \cap R_p$.

These theorems follow immediately from the definitions. However, conclusions concerning $(\cap P_p) \times Q \supset \cap R_p$ as a consequence of $P_p \times Q \supset R_p$ for every p are not so immediate. Suppose for instance that $P_1 \times Q \supset R_1$ and $P_2 \times Q \supset R_2$, and assume γ is of $R_1 \cap R_2$. Then there exist α_1 of P_1 and β_1 of Q_1 and α_2 of P_2 and β_2 of Q_2 such that $\gamma = \alpha_1 \times \beta_1 = \alpha_2 \times \beta_2$. In order that γ be of $(P_1 \cap P_2) \times Q$, it would be necessary to establish the existence of a single β of Q and an α of $P_1 \cap P_2$ such that $\gamma = \alpha \times \beta$. A sufficient condition for this can be obtained as follows. If Q has the dominance property[2] Δ_1 (if β_1 and β_2 are of Q then there exists a β of Q such that $\beta \geqq \beta_1$ and $\beta \geqq \beta_2$ (pointwise)) and if $\alpha = \gamma/\beta \leqq \gamma/\beta_1 = \alpha_1$ and $\alpha = \gamma/\beta \leqq \gamma/\beta_2 = \alpha_2$ were in both P_1 and P_2, then γ would have the property desired. This would certainly occur if the classes P_1 and P_2 had the dominance property Δ_0': if α_1 is of P and $\alpha \leqq \alpha_1$ then α is of P. Since the same method of proof would apply to a finite number of classes we can state the theorem:

1.63. THEOREM. *If Q has the dominance property Δ_1 and $P_1, \cdots, P_n$ have each the dominance property Δ_0', and if $P_i \times Q \supset R_i$ for each i, then $(\bigcap P_i) \times Q \supset \bigcap R_i$.*

The case when sequences of classes $\{P_m\}$ and $\{R_m\}$ are involved would obviously require some alteration in these conditions. We replace the property Δ_0' by the following modification:

A class P of sequences has the *dominance property* Δ_0'' if any sequence α which is ultimately dominated by some sequence α_0 of P belongs to P.

We replace the dominance property Δ_1 by one involving a sequence of elements, and define:

A class Q of sequences has the *dominance property* K if for any sequence of elements $\beta_1, \cdots, \beta_m \cdots$ of Q there exists an element β_0 of Q so that each β_m is ultimately dominated by β_0, that is, if $\beta_m = \{b_{mn}\}$ and $\beta_0 = \{b_{0n}\}$, then $\limsup_n b_{mn}/b_{0n} \leqq 1$ for all m.

An examination of the proof of the simple case above shows that the same method will give the theorem:

1.64. THEOREM. *If each of the classes P_m has the property Δ_0'', and Q has the property K, and if $P_m \times Q \supset R_m$, then we have $(\bigcap P_m) \times Q \supset \bigcap R_m$ and consequently if $P_m \times Q = R_m$ then $(\bigcap P_m) \times Q = \bigcap R_m$.*

All of the classes in System I have the property Δ_0''. On the other hand the dominance property Δ_1 is not found in the classes $-F_{-1}$, $F \cup -F_{-1}$, $F \cap -F_{-1}$, $-L_\infty$, D_{-0}. These do not have the additive property either. For instance, for D_{-1}, if $a_n = n^2$ for n even and n for n

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[2] For dominance properties see E. H. Moore, *Introduction to a form of general analysis*, Amer. Math. Soc. Colloquium Publications, vol. 2, New York, 1910, pp. 39–41. Since we have used the letter D to signify divergence, we shall use the corresponding Greek letter Δ for dominance properties.

odd, while $b_n = n$ for n even and n^2 for n odd, then α and β belong to D_{-1} but $\alpha + \beta$ does not,[3] and α and β do not have a common dominant in D_{-1}.

The dominance property K is obviously implied by the dominance property[4] K_0, which is defined as follows: For any sequence of sequences β_m of Q there exists a sequence β_0 such that $\lim_n b_{mn}/b_{0n} = 0$ for every m. The classes V, C_e and L_0 have the property K_0 and consequently K, as demonstrated in *Introduction to a form of general analysis*, pp. 44 ff., but since D_{-e} does not have the property Δ_1, even in the ultimate sense, D_{-e} does not have the property K.

For purposes of reference, we mention the following theorem whose proof is obvious:

1.65. THEOREM. *If* $P_1 \subset P_2$, *then* $P_1 \times Q \subset P_2 \times Q$. *Consequently if* $P_1 \times Q = P_2 \times Q = R$, *and if* $P_1 \subset P \subset P_2$, *then* $P \times Q = R$.

II. THE CLASSES $P \times Q$ IN SYSTEM I

In this part we shall prove theorems sufficient to characterize the class $P \times Q$ where P and Q are any two classes in the System I of 1.2. In view of Theorems 1.52, 1.53 and 1.54 this will give not only a complete set of multiplication and factorization theorems in this system, but also division or Landau theorems. We note that corresponding theorems for the classes of System II are immediate consequences of those in System I, since when P is in I then P_{-1} is in II, and $P \times Q = R$ implies $P_{-1} \times Q_{-1} = R_{-1}$.

2.1. We note in the first place:

2.11. THEOREM I. $-F_{-1} \times -F_{-1} = V$.

Since the multiplication of any two sequences is a sequence, it is only necessary to prove that if $\gamma = \{c_n\}$ is any sequence, then there exist two sequences $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ such that $\liminf a_n = 0$, and $\liminf b_n = 0$ and $c_n = a_n b_n$. For this purpose we set $a_n = 1/n$, $b_n = c_n n$ for n even, and $a_n = c_n n$, $b_n = 1/n$ for n odd.

An immediate consequence of this theorem is:

2.12. $P \times Q = V$ *for all* P *and* Q *of the system* I, *such that* $-F_{-1} \subseteq P \subseteq V$, $-F_{-1} \subseteq Q \subseteq V$.

For under these circumstances we have

$$V = -F_{-1} \times -F_{-1} \subset P \times Q \subset V.$$

As an instance of this we might note $D_{-e} \times D_{-f} = V$.

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[3] As a matter of fact $D_{-1} + D_{-1} = V$.

[4] Cf. Moore, loc. cit. p. 44.

2.2. Hoelder-Landau theorems.

2.21. THEOREM II. *We have* (a) $C_e \times C_f = C_{ef/(e+f)}$ *if* $e>0$, $f>0$; (b) $C_e \times D_{-f} = D_{-ef/(e-f)}$ *if* $e>f>0$; (c) $C_e \times D_{-f} = -F_{-1}$ *if* $f \geqq e > 0$.

The first of these: (a) is mentioned in the introduction. From the inequality:[5] $(e+f)(ab)^{1/(e+f)} \leqq ea^{1/e} + fb^{1/f}$ it follows that if α is of $C_{1/e}$ and β is of $C_{1/f}$ then $\alpha \times \beta = \gamma$ is of $C_{1/(e+f)}$. The substitution $e' = 1/e$, $f' = 1/f$, $g = 1/(e+f) = e'f'/(e'+f')$ gives the result $C_{e'} \times C_{f'} \subset C_g$. Conversely suppose γ is of C_g with $g = ef/(e+f)$. Set $\alpha = \gamma^{g/e}$ and $\beta = \gamma^{g/f}$. Then α is of C_e and β is of C_f, so that $C_e \times C_f \supset C_g$.

For the second part (b) of this theorem, we use from 1.59 the equivalence of $P \times Q \subset R$ to $P \times -R_{-1} \subset -Q_{-1}$, so that part (a) yields $C_e \times D_{-g} \subset D_{-f}$. Setting $g = ef/(e+f) = f'$ and $f = g'$ gives $g' = ef'/(e-f')$, so that we must assume $e > f'$, that is, we have, dropping primes, $C_e \times D_{-f} \subset D_{-ef/(e-f)}$. The converse of this statement, that is, $C_e \times D_{-f} \supset D_{-ef/(e-f)}$, is the contrapositive of the Landau Theorem. Assume that γ is of D_{-g}, where $g = ef/(e-f)$, and set $\sigma = \{s_n\} = \{\sum_1^n c_m^{-g}\}$. Then by the Abel-Dini Theorem,[6] $\gamma^{-g} \times \sigma^{-v}$ is of C_1 for $v>1$ and of D_1 for $v \leqq 1$. Consequently if we let $\alpha = \gamma^{-g/e} \times \sigma^{-1/f}$ and $\alpha \times \beta = \gamma$, that is, $\beta = \gamma^{g/f} \times \sigma^{1/f}$, then $\alpha^e = \gamma^{-g} \times \sigma^{-e/f}$ which since $e>f$ is of C_1, so that α is of C_e, while $\beta^{-f} = \gamma^{-g} \times \sigma^{-1}$ is of D_1, that is, β is of D_{-f}.

Note the similarity in form between the first two parts of this theorem.

For the last part (c) of this theorem, we show first that $C_e \times D_{-f} \subset -F_{-1}$, using a contrapositive method of proof. Suppose $\alpha = \{a_n\}$ is of C_e and $\beta = \{b_n\}$ is of D_{-f} and if possible lim inf $a_n b_n > 0$. Then there exists an $e_0 > 0$, such that for all n, $a_n b_n > e_0$. Consequently

$$\sum_n a_n^f \geqq e_0^f \left(\sum b_n^{-f}\right) = \infty,$$

that is, α is of D_f contrary to hypothesis. On the other hand we show that $C_e \times D_{-f} \supset -F_{-1}$ for any positive e and f. Let $\gamma = \{c_n\}$ be such that lim inf $c_n = 0$, and select a sequence $\alpha_0 = \{a_{0n}\}$ from C_e. For each k determine $n_k > n_{k-1}$ so that $c_{n_k} < a_{0k}$, and define the sequence $\alpha = \{a_n\}$ as follows: $a_n = a_{0n}$ if $n \neq n_k$, $a_n = c_n$ for $n = n_k$. Then obviously:

$$\sum (a_n)^e < 2 \sum a_{0n}^e < \infty,$$

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[5] This inequality can be deduced from the well known inequality: $x^e \leqq 1 + e(x-1)$, $0 \leqq e \leqq 1$, $x > 0$. Replace x by A/B, with $A > 0$, and $B > 0$. Then $A^e B^{1-e} \leqq eA + (1-e)B$. Let $e = f/(f+g)$, $f>0$, $g>0$. Then $1 - e = g/(f+g)$. Also set $A^f = a$, and $B^g = b$. Then the inequality $(f+g)(ab)^{1/(f+g)} \leqq fa^{1/f} + gb^{1/g}$ results. We might note that $x^e \geqq 1 + e(x-1)$ for $e > 1$, and $e < 0$, and that each of the three cases can be deduced from each other.

[6] See for instance Knopp, *Infinite series*, London, 1928, §39, p. 290. Note that the C_1 property follows from the initial inequality of the preceding footnote.

that is, α is of C_e. Obviously $b_{n_k}=1$ for all k. Hence $\beta=\{b_n\}$ is of D_{-f} for all f, or of $D'_{-\infty}$.

An examination of the proof just given shows that it applies to a much wider group of classes. In particular we have:

2.22. $P\times Q\supset -F_{-1}$, *for any class P contained in L_0 and any class Q containing* $-L_\infty$.

From the fundamental Theorem 2.21 we obtain theorems concerning $P\times Q=R$ where P and Q are sums and products of classes of the C or D_{-1} series. The procedure is illustrated in the proof of:

2.23. THEOREM III. *If* $1/g=1/e+1/f$ *or* $g=ef/(e+f)$, *then*

$$C_e' \times C_f = C_e' \times C_f' = C_e' \times C_f'' = C_g'$$

while $C_e\times C_f''=C_e''\times C_f''=C_g''$.

From $C_e\times C_f=C_g$, we obtain as a consequence of the sum theorem 1.61

$$\Big(\bigcup_{0<e'<e} C_{e'}\Big)\times C_f = \bigcup_{0<g'<g} C_{g'} \quad \text{or} \quad C_e' \times C_f = C_g',$$

since $g=ef/(e+f)$ is continuous and monotonic in e for fixed f. By repeating the operation we get since $\bigcup_{g'<g}C'_{g'}=C_g'$: $C_e'\times C_f'=C_g'$.

Since C_e has the property K for all $e>0$, as well as the property Δ_0'', we can also apply the product theorems 1.62 and 1.64 to $C_e\times C_f=C_g$ and obtain $C_e\times C_f''=C_g''$, it being assumed that f' runs over a monotonic decreasing sequence approaching f, and the corresponding g' obviously approaches g in the same way. If we apply the sum theorem 1.61 to this result we obtain by 1.31c

$$C_e' \times C_f'' = \bigcup_{h<g} C_h'' = C_g'.$$

Finally to prove that $C_e''\times C_f''=C_g''$, we note that by 1.62 we have

$$\bigcap_{e'>e} C_e' \times C_f'' \subset \bigcap_{g'>g} C_{g'}'', \quad \text{that is,} \quad C_e''\times C_f''\subset C_g''.$$

On the other hand since $C_e''\supset C_e$, we have

$$C_g'' = C_e \times C_f'' \subset C_e'' \times C_f'',$$

from which two statements the desired conclusion follows.

We might note that if in the proof of $C_e'\times C_f''=C_g'$ we had started with the identity $C_e'\times C_f=C_g'$ and been able to apply the product theorem then the equality $C_e'\times C_f''=C_g''$ would have resulted, in other words the product theorem 1.64 is not applicable. Since the class C_f has the property Δ_0'', it follows from this that the class C_e' does not have the property K.

For the case where $e=\infty$, we need consider only C'_∞. We obtain at once for $f>0$: $C'_\infty \times C_f = C'_\infty$. From this follows at once: $C'_\infty \times C_f'' = C'_\infty \times C_f = C'_\infty \times C_f' = C'_\infty \times C'_\infty = C'_\infty$.

For the case where $e=0$, we need consider only the class C_0''. Obviously Theorem 1.64 also applies here if $f \neq 0$, that is, we have

$$C_0'' \times C_f = C_0'' \qquad \text{for all} \quad 0 < f < \infty.$$

Consequently for the same range of f we have also

$$C_0'' \times C_f' = C_0'' \times C_f'' = C_0'' \times C'_\infty = C_0''.$$

It is further obvious from 1.62 that $C_0'' \times C_0'' \subset C_0''$. To prove the inclusion in the other direction, we take an arbitrary sequence γ of C_0'' and let $\alpha = \gamma^{1/2}$ and $\beta = \gamma^{1/2}$, and then obviously α and β are of C_0''.

We incorporate all of these results in the form of a theorem.

2.24. *The classes of System* I *between* C_0'' *and* C'_∞ *form a closed system under multiplication* ($\times$), *the various multiplications can be read from the following table, in which* $g = ef/(e+f)$:

	C_0''	C_f'	C_f	C_f''	C'_∞
C_0''	C_0''	C_0''	C_0''	C_0''	C_0''
C_e'	C_0''	C_g'	C_g'	C_g'	C'_∞
C_e	C_0''	C_g'	C_g	C_g''	C'_∞
C_e''	C_0''	C_g'	C_g''	C_g''	C'_∞
C'_∞	C_0''	C'_∞	C'_∞	C'_∞	C'_∞

We turn next to the extension of 2.21(b) to the classes C', C'', D' and D''. We have:

2.25. THEOREM IV. *If* $g = ef/(e-f)$ *and* $e > f > 0$, *then*

$$C_e' \times D_{-f} = C_e \times D'_{-f} = C_e' \times D'_{-f} = C_e' \times D''_{-f} = C_e'' \times D'_{-f} = D'_{-g}$$

and $C_e \times D''_{-f} = C_e'' \times D_{-f} = C_e'' \times D''_{-f} = D''_{-g}$.

Starting from the equation $C_e \times D_{-f} = D_{-g}$, the methods of proof used for 2.23 or Theorem III apply here also, with the exception of the characterization of the classes $C_e'' \times D'_{-f}$ and $C_e'' \times D_{-f}$. For the class $C_e''\times D'_{-f}$, we note that the product theorem 1.62 applied to $C_e\times D_{-f}\subset D_{-g}$ yields $C_e''\times D_{-f}\subset D''_{-g}$. Applying the sum theorem 1.61 and also 1.31c gives $C_e''\times D'_{-f}\subset D'_{-g}$. On the other hand $C_e''\times D'_{-f}\supset C_e\times D'_{-f}=D'_{-g}$. So that $C_e''\times D'_{-f}=D'_{-g}$.

For the class $C_e''\times D_{-f}$, we need only show that it contains D''_{-g}, as the inverse inclusion is contained in the preceding paragraph. It is not permissible here to use Theorem 1.64, because D_{-f} does not have the dominance property K. Instead we proceed directly. Suppose that γ is of D''_{-g}, that is, there exists a monotonic increasing sequence g_n approaching g, such that γ is of D_{-g_n} for every n. Let e_n be determined from the equation $g_n=e_nf/(e_n-f)$, so that e_n approaches e monotonically decreasing. Since $D_{-g_n}=C_{e_n}\times D_{-f}$ there exist $\alpha_n=\{a_{nm}\}$ of C_{e_n} and $\beta_n=\{b_{nm}\}$ of D_{-f} so that $\gamma=\alpha_n\times\beta_n$ for every n. Let the sequence d_n, with $0<d_n<1$, be such that $\sum d_n$ converges. Determine the sequence $m_n>m_{n-1}$ so that

$$\sum_{m=m_n}^{\infty} a_{nm}^{e_n} < d_n, \qquad \sum_{m_n\leqq m<m_{n+1}} b_{nm}^{-f} > 1.$$

Define $a_m=a_{nm}$ and $b_m=b_{nm}$ for $m_n\leqq m<m_{n+1}$, with $a_m=a_{1m}$ and $b_m=b_{1m}$ for $m\leqq m_1$. Then obviously $\gamma=\alpha\times\beta$. α belongs to C_e for every $e'>e$ since $a_m<1$ for $m\geqq m_1$, so that if $e'>e_n$, $m>m_1$, then $a_m^{e'}<a_m^{e_n}$ and consequently $\sum_{m_n}^{\infty}a_m^{e'}<\sum d_n$. Obviously β belongs to D_{-f}.

The equation $C_e''\times D'_{-f}=D'_{-g}$ could also have been deduced by the sum theorem 1.61 from $C_e''\times D_{-f}=D''_{-g}$. Also the case $e=\infty$ which concerns C'_∞ follows immediately so that we have for $f>0$

$$C'_\infty\times D_{-f}=C'_\infty\times D'_{-f}=C'_\infty\times D''_{-f}=D'_{-f}$$

and $C'_\infty\times D'_{-0}=D'_{-0}$.

2.26. THEOREM V. *If* $0<e<f$, *then* $C_e^{(1)}\times D_{-f}^{(2)}=-F_{-1}$, *where* $C_e^{(1)}$ *may be either* C_e', C_e *or* C_e'' *and* $D_{-f}^{(2)}$ *may be either* D'_{-f}, D_{-f} *or* D''_{-f}.

This is an immediate consequence of the inclusion theorem 1.65.

For the case $e=f$, Theorem IV separates into two parts, namely

$$\begin{aligned} C_e\times D_{-e}&=C_e'\times D_{-e}=C_e\times D'_{-e}=C_e'\times D'_{-e}\\ &=C_e'\times D''_{-e}=C_e''\times D'_{-e}=-F_{-1}. \end{aligned} \tag{2.27}$$

This includes also the case where $e=0$ or ∞, namely

$$C'_\infty\times D''_{-\infty}=C_0''\times D'_{-0}=-F_{-1}. \tag{2.28}$$

On the other hand

2.29 $$C_e'' \times D_{-e} = C_e \times D_{-e}'' = C_e'' \times D_{-e}'' = D_{-\infty}''.$$

For 2.27 the first class is covered in 2.21(c), the next three classes each are contained in $C_e \times D_{-e}$ and by 2.22 contain $-F_{-1}$. For the last two the method of procedure is

$$C_e' \times D_{-e}'' = \left(\bigcup_{f<e} C_f \right) \times D_{-e}'' = \bigcup_{f<e} (C_f \times D_{-e}'') = -F_{-1},$$

$$C_e'' \times D_{-e}' = C_e'' \times \left(\bigcup_{f>e} D_{-f} \right) = \bigcup_{f>e} (C_{-e}'' \times D_{-f}) = -F_{-1}.$$

The first of these two is also valid for $e = \infty$ and the second for $e = 0$, giving 2.28.

For 2.29 the methods used in the proof of the corresponding results in 2.25 are applicable.

To finish this group of cases, we need to consider yet the possibility of having $e = 0$, that is, the class C_0''; $f = 0$, that is, the class D_{-0}'; and the case $f = \infty$, that is, $D_{-\infty}''$.

For C_0'' we have:

2.31. *$C_0'' \times P = -F_{-1}$ for every class P between and including $-F_{-1}$ and D_{-0}'.*

In 2.28 it was shown that $C_0'' \times D_{-0}' = -F_{-1}$. So it remains to prove that $C_0'' \times -F_{-1} = -F_{-1}$. It is obvious that since $C_0'' \subset F$, we have $C_0'' \times -F_{-1} \subset -F_{-1}$. On the other hand suppose $\gamma = \{c_n\}$ is of $-F_{-1}$ and let $\alpha_0 = \{a_{0n}\}$ be any element of C_0'', that is, $\sum_n a_{0n}^e < \infty$ for $e > 0$. Select a sequence n_k so that $c_{n_k} < a_{0k}/k$. Then define $a_n = a_{0n}$ for $n \neq n_k$ and $a_n = a_{0k}$ for $n = n_k$. Then obviously $\sum_n a_n^e < 2\sum_n a_{0n}^e$, for $e > 0$, that is, α is of C_0''. On the other hand since $b_{n_k} = c_{n_k}/a_{n_k} = c_{n_k}/a_{0k} < 1/k$, β is of $-F_{-1}$.

For D_{-0}' we have:

2.32. *$D_{-0}' \times P = D_{-0}'$ for all classes P between C_0'' and F, including F.*

Obviously $D_{-0}' \times F = D_{-0}'$. It is then sufficient to prove that $D_{-0}' \times C_e = D_{-0}'$ for all $e > 0$, which follows immediately from 2.25, namely, $C_e \times D_{-f} = D_{-g}$, by applying the sum theorem 1.61.

For $D_{-\infty}''$ we have:

2.33. *$D_{-\infty}'' \times P = -F_{-1}$ for all classes P between and including C_0'' and C_∞'.*

The equality $D_{-\infty}'' \times C_0'' = -F_{-1}$ is included in 2.31. Since for every $e > 0$, C_e has the dominance property K, we can apply the product theorem 1.64 to $C_e \times D_{-f} = -F_{-1}$ to obtain $C_e \times D_{-\infty}'' = -F_{-1}$, for all $e > 0$. Applying the sum theorem 1.61 to this equality yields $C_\infty' \times D_{-\infty}'' = -F_{-1}$.

2.4. **The classes between L_0 and $-L_\infty$.** We consider next results involving the classes L_0, $F\cap-F_{-1}$, F, $-F_{-1}$, $F\cup-F_{-1}$, $-L_\infty$, so far as they have not been previously treated. We have:

2.41. *$F\times P=P$ for every class P of System* I.

Since the sequence $\alpha=\{1\}$ is of F, it is obvious that $F\times P\supset P$. On the other hand, since all classes P of System I have the property Δ_0'', $F\times P\subset P$.

2.42. *$-F_{-1}\times P=-F_{-1}$ for any class P contained in or equal to F.*

Since $-F_{-1}\times F\subset-F_{-1}$, it follows that $-F_{-1}\times P\subset-F_{-1}$ for all P contained in F. On the other hand suppose $\gamma=\{c_n\}$ is of $-F_{-1}$. Select a $\beta_0=\{b_{0n}\}$ of P, and the sequence n_k so that $c_{n_k}<b_{0k}/k$. Let $b_n=b_{0n}$ if $n\neq n_k$, and $b_n=b_{0k}$ if $n=n_k$. Then if $a_n=c_n/b_n$, it follows that $\liminf_n a_n=0$, that is, α is of $-F_{-1}$. But for the classes P of System I, $\beta=\{b_n\}$ is of P if β_0 is.

Since $-F_{-1}\subset F\cup-F_{-1}\subset-L_\infty\subset D''_{-\infty}$, we can conclude from 2.33 and 2.42 that:

2.43. $-F_{-1}\times P=(F\cup-F_{-1})\times P=(-L_\infty)\times P=D''_{-\infty}\times P=-F_{-1}$ *for all classes P between and including C_0'' and C'_∞.*

2.44. *$L_0\times P=P$ for all classes P between and including C_0'' and L_0, and also between and including $D''_{-\infty}$ and D'_{-0}.*

Since $L_0\subset F$, and $F\times P=P$ for all P it follows that $L_0\times P\subset P$ for all P. An analysis of the statement $L_0\times P\supset P$ shows that it is equivalent to the fact that the classes P have the dominance property[7] K_0, namely, that if $\alpha=\{a_n\}$ is of P, then there exists in P an $\alpha_0=\{a_{0n}\}$ such that $\lim_n a_n/a_{0n}=0$, which in turn is related to the "no last absolutely convergent or divergent series" theorem. The proof for $C_1=C$ and consequently for C_e can be based on the Abel-Dini Theorem for convergent series.[8] Let $\gamma=\{c_n\}$ be of C and let $\rho=\{r_n\}=\sum_n^\infty c_m$. Then $\gamma\times\rho^{-1+e}$ is of C for $e>0$. If we set $\alpha=\gamma\times\rho^{-1+e}$, then $\beta=\rho^{1-e}$ which is of L_0. For the other classes between and including C_0'' and C'_∞, we can apply 1.61 and 1.64, since L_0 has the property K. For $L_0\times L_0\subset L_0$, we take $\alpha=\beta=\gamma^{1/2}$. For the class D_{-1}, we utilize in a similar way the Abel-Dini Theorem for divergent series, that is, define $\alpha=\gamma\times\sigma$, where $\sigma=\{(\sum_1^n c_m)^{-1}\}$ so that $\beta=\sigma^{-1}$ which is of L_0. The extension to D_{-e} and the other classes between $D''_{-\infty}$ and D'_{-0} is then similar to that for the C.

2.45. *$(F\cap-F_{-1})\times P=P$ for all P between C_0'' and L_0 inclusive and $D''_{-\infty}$ and D'_{-0} inclusive.*

This is a corollary to 2.44 and 2.41 in view of the fact that $F\supset F\cap-F_{-1}\supset L_0$.

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[7] See Moore, loc. cit. p. 44.

[8] See Knopp, loc. cit. p. 293.

These theorems while they cover a wide range still leave a few particular cases. We have:

2.46. $-L_\infty \times L_0 = -F_{-1}$; $-L_\infty \times (F\cap -F_{-1}) = -L_\infty$.

From the definitions it is obvious that $-L_\infty \times L_0 \subset -F_{-1}$, for if $\alpha = \{a_n\}$ is of $-L_\infty$, then a_n contains a bounded subsequence, so that if $\beta = \{b_n\}$ is of L_0, then $\lim\inf_n a_n b_n = 0$. On the other hand suppose that $\gamma = \{c_n\}$ is of $-F_{-1}$. Let $\alpha_0 = \{a_{0n}\}$ be any element in L_0 and select an increasing sequence n_k so that $c_{n_k} \leqq a_{0k}$. Then define $\alpha = \{a_n\}$ so that $a_n = a_{0n}$ for $n \neq n_k$ and $a_n = a_{0k}$ for $n = n_k$. Then $b_{n_k} = c_{n_k}/a_{n_k} \leqq 1$ so that β is of $-L_\infty$ and α is of L_0.

For the second part, it is again obvious that $-L_\infty \times (F\cap -F^{-1}) \subset -L_\infty$. On the other hand let $\gamma = \{c_n\}$ be of $-L_\infty$, and let $\{c_{n_k}\}$ be a bounded subsequence of $\{c_n\}$. Let $a_n = c_n$ if $n = n_k$ and $a_n = c_n n$ if $n \neq n_k$. Then $b_n = 1$ for $n = n_k$ but $b_n = 1/n$ for $n \neq n_k$ so that α is of $-L_\infty$ and β is of $F\cap -F_{-1}$.

2.47. $L_0 \times (F\cup -F_{-1}) = -F_{-1}$.

This is a consequence of 2.42, namely, $L_0 \times (-F_{-1}) = -F_{-1}$, and 2.46, namely, $L_0 \times (-L_\infty) = -F_{-1}$, and the fact that $-F_{-1} \subset F\cup -F_{-1} \subset -L_\infty$.

2.48. $(F\cup -F_{-1}) \times (F\cap -F_{-1}) = -F_{-1}$.

We have $(F\cap -F_{-1}) \times F = F\cap -F_{-1}$ and $(F\cap -F_{-1}) \times -F_{-1} = -F_{-1}$, by 2.41 and 2.42. By addition we then get 2.48.

2.49. $(F\cap -F_{-1}) \times (F\cap -F_{-1}) = F\cap -F_{-1}$.

This is obvious.

2.5. **The classes** V **and** 0. We finally consider the classes V and 0 and note:

2.51. $P \times V = V$ *for all* P *of System* I *excepting* 0.

2.52. $P \times 0 = 0$ *for all* P *of System* I.

These are self-evident.

It may however be of interest to apply Theorems 1.52 to these results in order to obtain corresponding division theorems. Thus

$$\begin{aligned} 0/0 &= -(0 \times -0_{-1})_{-1} = -(0 \times V)_{-1} = V, \\ V/V &= -(V \times V_{-1})_{-1} = -(V \times 0)_{-1} = V, \\ V/0 &= -(0 \times -V_{-1})_{-1} = (0 \times 0)_{-1} = V, \\ 0/V &= -(V \times -0_{-1})_{-1} = -(V \times V)_{-1} = 0. \end{aligned}$$

Of these $V/0 = V$ is an example available for 1.510 and 1.512.

We have collected all of the results of this part in the form of a table, indicating in each place the particular theorem on which the entry is based. Because of the fact that $P \times Q = Q \times P$, it is obvious that it is sufficient to fill in only one half of the entries, as the re-

	0	C''_0	C'_e	C_e	C''_e	C'_f	C_f	C'	C'_∞	L_0	$F\cap -F_{-1}$	F
0	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.	0 2.52	0 2.52	0 2.52	0 2.52
C''_0		C''_0 2.24	C''_0 2.24	C''_0 2.24	C''_0 2.24	C''_0 2.24	C''_0 2.24	C'' 2.	C''_0 2.24	C''_0 2.44	C''_0 2.45	C''_0 2.41
C'_e			$C'_{e/2}$ 2.23	$C'_{e/2}$ 2.23	$C'_{e/2}$ 2.23	C'_g 2.23	C'_g 2.23	C'_g 2.2	C'_∞ 2.24	C'_e 2.44	C'_e 2.45	C'_e 2.41
C_e				$C_{e/2}$ 2.21	$C''_{e/2}$ 2.23	C'_g 2.23	C_g 2.21	C''_g 2.23	C'_∞ 2.24	C_e 2.44	C_e 2.45	C_e 2.41
C''_e					$C''_{e/2}$ 2.23	C'_g 2.23	C''_g 2.23	C''_g 2.23	C'_∞ 2.24	C''_e 2.44	C''_e 2.45	C''_e 2.41
C'_f						$C'_{f/2}$ 2.23	$C'_{f/2}$ 2.23	$C'_{f/2}$ 2.23	C'_∞ 2.24	C'_f 2.44	C'_f 2.45	C'_f 2.41
C_f							$C_{f/2}$ 2.21	$C''_{f/2}$ 2.23	C'_∞ 2.24	C_f 2.44	C_f 2.45	C_f 2.41
C''_f								$C''_{f/2}$ 2.23	C'_∞ 2.24	C''_f 2.44	C''_f 2.45	C''_f 2.41
C'_∞									C'_∞ 2.24	C'_∞ 2.44	C'_∞ 2.45	C'_∞ 2.41
L_0										L_0 2.44	L_0 2.45	L_0 2.41
$F\cap -F_{-1}$											$F\cap -F_{-1}$ 2.49	$F\cap -F_{-1}$ 2.41
F												F 2.41
$-F_{-1}$												
$F\cup -F_{-1}$												
$-L_\infty$												
$D''_{-\infty}$												
D'_{-f}												
D_{-f}												
D''_{-f}												
D'_{-e}												
D_{-e}												
D''_{-e}												
D'_{-0}												
V												

$0<e<f.\quad g=ef/(e+f).\quad g'=ef/(f-e).$

$-F_{-1}$	$F\cup -F_{-1}$	$-L_\infty$	$D''_{-\infty}$	D'_{-f}	D_{-f}	D''_{-f}	D'_{-e}	D_{-e}	D''_{-e}	D'_{-0}	V
0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52	0 2.52
$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.31	$-F_{-1}$ 2.28	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	$-F_{-1}$ 2.26	$-F_{-1}$ 2.26	$-F_{-1}$ 2.26	$-F_{-1}$ 2.27	$-F_{-1}$ 2.21	$-F_{-1}$ 2.27	D'_{-0} 2.32	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	$-F_{-1}$ 2.26	$-F_{-1}$ 2.26	$-F_{-1}$ 2.26	$-F_{-1}$ 2.27	$-F_{-1}$ 2.27	$D''_{-\infty}$ 2.29	D'_{-0} 2.32	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	$-F_{-1}$ 2.26	$-F_{-1}$ 2.26	$-F_{-1}$ 2.26	$-F_{-1}$ 2.27	$D''_{-\infty}$ 2.29	$D''_{-\infty}$ 2.27	D'_{-0} 2.32	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	$-F_{-1}$ 2.27	$-F_{-1}$ 2.27	$-F_{-1}$ 2.27	$D'_{-g'}$ 2.25	$D'_{-g'}$ 2.25	$D'_{-g'}$ 2.25	D'_{-0} 2.32	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	$-F_{-1}$ 2.27	$-F_{-1}$ 2.27	$D''_{-\infty}$ 2.29	$D'_{-g'}$ 2.25	D_{-g} 2.21	$D''_{-g'}$ 2.25	$D'_{-0},$ 2.32	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	$-F_{-1}$ 2.27	$D''_{-\infty}$ 2.29	$D''_{-\infty}$ 2.29	$D'_{-g'}$ 2.25	$D''_{-g'}$ 2.25	$D''_{-g'}$ 2.25	D'_{-0} 2.32	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.43	$-F_{-1}$ 2.43	$-F_{-1}$ 2.33	D'_{-f} 2.25	D'_{-f} 2.25	D'_{-f} 2.25	D'_{-e} 2.25	D'_{-e} 2.25	D'_{-e} 2.25	D'_{-0} 2.25	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.47	$-F_{-1}$ 2.46	$D''_{-\infty}$ 2.44	D'_{-f} 2.44	D_{-f} 2.44	D''_{-f} 2.44	D'_{-e} 2.44	D_{-e} 2.44	D''_{-e} 2.44	D'_{-0} 2.44	V 2.51
$-F_{-1}$ 2.42	$-F_{-1}$ 2.48	$-F_{-1}$ 2.47	$D''_{-\infty}$ 2.45	D'_{-f} 2.45	D_{-f} 2.45	D''_{-f} 2.45	D'_{-e} 2.45	D_{-e} 2.45	D''_{-e} 2.45	D'_{-0} 2.45	V 2.51
$-F_{-1}$ 2.41	$F\cup -F_{-1}$ 2.41	$-L_\infty$ 2.41	$D''_{-\infty}$ 2.41	D'_{-f} 2.41	D_{-f} 2.41	D''_{-f} 2.41	D'_{-e} 2.41	D_{-e} 2.41	D''_{-e} 2.41	D'_{-0} 2.41	V 2.51
V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
		V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
			V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
				V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
					V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
						V 2.12	V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
							V 2.12	V 2.12	V 2.12	V 2.12	V 2.51
								V 2.12	V 2.12	V 2.12	V 2.51
									V 2.12	V 2.12	V 2.51
										V 2.12	V 2.51
											V 2.51

mainder can be obtained by reflection on the diagonal. We note that System I is a closed system under multiplication. Also we recall that with this table it is possible to make a corresponding table of division theorems by using 1.52.

III. A GENERAL MULTIPLICATION THEOREM

3.1. **On classes $(P-Q)\times(R-S)$.** In the preceding part, we made a study of the classes $P\times Q$, where P and Q are any two classes chosen from System I. We noted that the theorems of that part gave rise to theorems relative to the class $P\times Q$ if P and Q are both in System II, because of the fact that if P is in I, then P_{-1} is in II, and from $P\times Q=R$ it follows that $P_{-1}\times Q_{-1}=R_{-1}$. A natural question is to characterize $P\times Q$ if P is in System I and Q in System II. The answer to this happens to be trivial. For we have:

3.11. THEOREM. *If P is in System* I *and Q in System* II, *then $P\times Q=V$, unless P or Q is the null class* 0, *in which case $P\times Q=0$.*

To prove this theorem, it is obviously sufficient to consider only the case in which $P=C_0''$ and $Q=C_{-0}''$. Further since $P\times Q$ is contained in V for any P and Q, we need only show that if $\gamma=\{c_n\}$ is any sequence of V, then there exist $\alpha=\{a_n\}$ of C_0'' and $\beta=\{b_n\}$ of C_{-0}'', such that $c_n=a_nb_n$. We note that if α is any sequence of C_0'' and if we define $\beta=\gamma\times\alpha^{-1}$ then β will be of C_{-0}'', provided γ is bounded from zero, that is, in F_{-1}. Also if β is of C_{-0}'' and $\alpha=\gamma\times\beta^{-1}$, then α is of C_0' if γ is bounded from infinity, that is, in F. Assume then that γ is in $-F_{-1}\cap -F$. Then divide the sequence of integers n into two mutually exclusive ordered sets n_k' for which $c_n\geqq 1$ and n_k'' for which $c_n<1$. Let α_0 be any element of C_0'' and β_0 any element of C_{-0}''. Define $\alpha=\{a_n\}$ as follows:

$$a_n = a_{0k} \quad \text{if} \quad n = n_k', \qquad a_n = c_n/b_{0k} \quad \text{if} \quad n = n_k'';$$

so that

$$b_n = c_n/a_{0k} \quad \text{if} \quad n = n_k', \qquad b_n = b_{0k} \quad \text{if} \quad n = n_k''.$$

Then obviously α is of C_0'' and β is of C_{-0}''.

The System I (as well as II) defines also classes by taking differences, particularly the class $P-Q$ where P properly contains Q. Since if P is in I then $-Q$ is in II, and conversely, the classes $P-Q$ are really products of a class in I by one in II. There arises the question of the characterization of the classes $(P-Q)\times(R-S)$. If we note that $P-Q$ is in fact $P\cap -Q$, then if a product theorem similar to 1.51 were universally applicable, we might guess that

$$\text{(M):}\qquad (P-Q)\times(R-S)=(P\times R)\cap(-Q\times -S).$$

If $Q=S=0$, or $P=R=V$, the identity is trivial. If we take $Q=0$, $R=V$ then the guess would give

$$P\times -S=(P\times V)\cap(-0\times -S)=V$$

since $P\neq 0$ and $S\neq V$. If P and S are in I, then $-S$ is in II and this is Theorem 3.11. The object of this part is to determine limitations on the classes P, Q, R and S so that the equality (M) is valid in System I. Corresponding results for System II follow. Throughout this part it will be assumed that *P properly contains Q and R properly contains S in the differences $P-Q$, $R-S$.*

The method of procedure will be to find conditions for the validity of the equations

$$(P-Q)\times R=P\times R\quad\text{and}\quad(P-Q)\times -S=-Q\times -S,$$

which are the special cases of the general equation for $Q=0$ and $R=V$, respectively, and from these to derive the general equality.

It is however not necessary to prove both of these two equalities, since we have:

3.12. THEOREM. *A theorem of the form $(P-Q)\times R=P\times R$ has as consequence a theorem of the form $(P'-Q')\times -S'=-Q'\times -S'$, and conversely.*

For if $(P-Q)\times R=P\times R$, then $(P_{-1}-Q_{-1})\times R_{-1}=P_{-1}\times R_{-1}$. Now if P is in I, then P_{-1} is in II, and $-P_{-1}$ is a member of I. If we set $P'=-Q_{-1}$, $Q'=-P_{-1}$, $S'=-R_{-1}$, then obviously Q' is properly contained in P' and our identity becomes

$$(P'-Q')\times -S'=-Q'\times -S'.$$

Because of the dual nature of these two equalities, we shall refer to the process of substitution involved as a dualizing process.

We might also note that the Systems I and II are combinations of half of the System I, namely,

$$0,\ C_0'',\ C_\theta',\ C_\theta,\ C_\theta'',\ C_\infty',\ L_0,\ F\cap -F_{-1},\ F.$$

If we denote this set of classes by Ia, the balance by Ib, and similarly for II, then

$$3.13\qquad \mathrm{Ib}=-(\mathrm{Ia})_{-1},\quad \mathrm{IIa}=(\mathrm{Ia})_{-1},\quad\text{and}\quad \mathrm{IIb}=(\mathrm{Ib})_{-1}=-\mathrm{Ia},$$

so that

$$\mathrm{I}=(\mathrm{Ia},\mathrm{Ib})=(\mathrm{Ia},-(\mathrm{Ia})_{-1})\quad\text{and}\quad\mathrm{II}=((\mathrm{Ia})_{-1},-\mathrm{Ia}).$$

3.2. On the relation $(P-Q)\times R=P\times R$. It is obvious that if Q is properly contained in P, then $(P-Q)\times R$ is contained in $P\times R$. Hence we need to derive only sufficient conditions so that $(P-Q)\times R$ contains $P\times R$. In order to state a general theorem easily, we define a dominance property related to two classes, namely:

3.21. DEFINITION. The pair of classes P, Q with P properly containing Q have the *dominance property* K_2 relative to each other, if for any element β of Q there exists an element α of $P-Q$ which ultimately dominates β, that is, $a_n\geqq b_n$ for $n>n_0$.

Also dually:

3.211. DEFINITION. The pair of classes P, Q with P properly containing Q have the dominance property $\overline{K}_2$ relative to each other, if for every α of $-P$, there exists a β of $P-Q$ such that β is ultimately dominated by α.

We then have the following theorem.

3.22. THEOREM. *If R has the dominance property Δ_0'', and P and Q have the dominance property K_2 with respect to each other, then $(P-Q)\times R\supset P\times R$, and so $(P-Q)\times R=P\times R$.*

The proof of this is almost self-evident. Suppose δ is of $P\times R$, that is, there exists α of P and γ of R such that $\delta=\alpha\times\gamma$. If α is of $P-Q$, no further steps are necessary. If however α is of Q let β be of $P-Q$ ultimately dominating α and set $\gamma_1=\delta\times\beta^{-1}=\alpha\times\gamma\times\beta^{-1}$. Then γ_1 is ultimately dominated by γ and so of R, that is, $\delta=\beta\times\gamma_1$ is of $(P-Q)\times R$.

In order to apply this theorem to the case where P, Q and R are classes in System I, we show:

3.23. *If P and Q are any two classes of System* I, *P properly containing Q, and excluding the case where P is $F\cup-F_{-1}$, then P and Q have the dominance property K_2 relative to each other.*

The method of demonstrating this property for any two classes of System I chosen from Ia, that is, those lying between 0 and F inclusive, is illustrated by the following special case:

3.231. $P=C_e$, $Q=C_e'$. Let α be any element of C_e'. Then since by 1.4, C_e properly contains C_e', there exists an element β_0 of C_e-C_e'. Define β so that $b_n=b_{0n}$ if $a_n\leqq b_{0n}$, and $b_n=a_n$ if $a_n\geqq b_{0n}$. Then since $\beta\geqq\beta_0$, it follows that β is not of C_e'. On the other hand since

$$\sum_n b_n^e=\sum_{n'} b_{0n'}^e+\sum_{n''} a_n^e<\sum b_{0n}^e+\sum a_n^e$$

it follows that β is in C_e.

It is obvious that in this proof we used the fact that $-Q$ has the property Δ_0' in reverse, which is true for all Q in System I. Further we have used the fact that if α and β are sequences of P, then γ, the greater of α and β, is also in P. This holds for all of the classes of System Ia, excepting $F\cap -F_{-1}$. However the procedure above is still effective in this case, since if β is of Q and Q is properly contained in $F\cap -F_{-1}$, then α is of L_0. If β is of $F\cap -F_{-1}$, and α is of L_0, then the greater of α and β is still in $F\cap -F_{-1}$. Hence we have the theorem:

3.232. THEOREM. *Any two classes P, Q with P properly containing Q from System* Ia *have the property K_2 relative to each other.*

This is no longer true if we continue in the System I; we have:

3.233. *The class $F\cup -F_{-1}$ does not have the property K_2 relative to $-F_{-1}$, or relative to F, but it does have the property K_2 relative to $F\cap -F_{-1}$, and so relative to all the other classes of System* I.

Let α be an element of $-F_{-1}$ for which both lim sup $a_n = \infty$, and lim inf $a_n = 0$. Then there exists in $F\cup -F_{-1}$ no dominating element belonging to $-(-F_{-1})$, that is, F_{-1}. Similarly if α is in F with lim inf $a_n > 0$, there does not exist in $(F\cup -F_{-1}) - F$, that is, in $-F_{-1}\cap -F$, an element dominating α. On the other hand let α belong to $F\cap -F_{-1}$, and let $\lim_k a_{n_k} = 0$, with $a_{n_k} \leqq 1$. Define $\beta = \{b_n\}$ by the conditions $b_n = a_n$ for $n \neq n_k$, or for $n = n_k$, k even, and $b_n = 1/a_n$ with $n = n_l$, k odd. Then β is of $-F_{-1}$, dominates α, but is not in $F\cap -F_{-1}$.

For the remaining members of System I, we demonstrate the property K_2 between adjacent classes, from which it then follows in general. We consider the following cases:

3.234. *$-L_\infty$ has the property K_2 relative to $-F_{-1}$, F, and $F\cup -F_{-1}$, and consequently all classes of System* I.

Suppose that α belongs to F and lim inf $a_n > 0$, that is, α is not in $-F_{-1}$. Then define β: $b_n = a_n$ for n even, $b_n = na_n$ for n odd. Then β is in $-L_\infty$, but not in F or $-F_{-1}$.

Suppose α belongs to $-F_{-1}$. Let n' be the set of integers for which $a_n \leqq 1$. Divide the integers n' into two infinite subsets n'', n''' so that $\lim_{n''} a_{n''} = 0$. Then define $b_n = a_n$ for $n \neq n'$, $b_n = 1/a_n$ for $n = n''$ and $b_n = 1$ for $n = n'''$. Then β is of $-L_\infty$, but not in F nor $-F_{-1}$.

3.235. *$D''_{-\infty}$ has the K_2 property relative to $-L_\infty$.*

Let α be of $-L_\infty$, that is, lim inf $a_n = a_0 < \infty$ and suppose that lim $a_{n_k} = a_0$. It is obviously sufficient to assume $a_0 > 0$. Suppose β_0 is any element of $D''_{-\infty}$ not belonging to $-L_\infty$, for example, $n \exp(1/(\log n)^{1/2})$. Define $\beta = \{b_n\}$ by the conditions $b_n = na_n$ for $n \neq n_k$, $b_n = b_{0k}a_n$ for $n = n_k$. Then β is in $D''_{-\infty}$ but $\lim_n b_n = \infty$.

3.236. *All pairs of classes P, Q in the D group of System* I *have the* K_2 *property relative to each other.*

We prove this by showing that the theorem holds for the following cases: (a) D_{-e}, D_{-f} with $e<f$, (b) D_{-1}, D'_{-1}, and (c) D''_{-1}, D_{-1}, since obviously the case of any two pairs from the D group can be deduced from these three cases.

(a) Suppose that α is of D_{-f}, that is, $\sum_n a_n^{-f} = \infty$. Without loss of generality it is possible to assume that $a_n \geqq 1$. Let $\sigma = \{s_n\} = \{\sum_1^n a_m^{-f}\}$ and let $\beta = \alpha \times \sigma^{1/e}$. Then since $\beta^{-f} = \alpha^{-f} \times \sigma^{-f/e}$ and $f/e > 1$ the Abel-Dini Theorem assures us that β^{-f} is of C_1, that is, β belongs to C_{-f}. On the other hand

$$\beta^{-e} = \alpha^{-e} \times \sigma^{-1} \geqq \alpha^{-f} \times \sigma^{-1}$$

since $f > e$, and $a_n \geqq 1$. Now $\alpha^{-f} \times \sigma^{-1}$ is in D_1, that is, β is of D_{-e}. Obviously β dominates α.

(b) Suppose α is in D'_{-1}, that is, there exists an $f > 1$, so that α is of D_{-f}. Using the same notation as in (a). we now define $\beta = \alpha \times \sigma$. Then if $g > 1$, $\beta^{-g} = \alpha^{-g} \times \sigma^{-g} \leqq \alpha^{-1} \times \sigma^{-g}$ is of C_1, that is, β is in C_{-g} for all $g > 1$, and so in $C''_{-1} = -D'_{-1}$. On the other hand β is in D_{-1} and dominates α.

(c) Suppose α is in D_{-1}. Using the same notation as in (a) with $f = 1$ and assuming $a_n \geqq 1$, we set $b_n = a_n s_n \log^2 s_n$, that is, $\beta = \alpha \times \sigma \times \log^2 \sigma$. Then by an extension of the Abel-Dini Theorem,[9] β is of C_{-1}. On the other hand β is of D_{-f}, for $f < 1$. For if $f < g < 1$

$$\beta^{-f} = \alpha^{-f} \times \sigma^{-f} \times \log^{-2f} \sigma \geqq \alpha^{-1} \times \sigma^{-f} \times \log^{-2f} \sigma \geqq \alpha^{-1} \times \sigma^{-g}$$

ultimately, since $\lim_n s_n^{g-f} \log^{-2f} s_n = \infty$.

The fact that V has the K_2 property with respect to any class in System I (or System II) is trivial.

With respect to the property $\overline{K}_2$, we have:

3.237. *If P and Q are any two classes of System* I, *with P properly containing Q, and* (P, Q) *is neither* $(F, F \cap -F_{-1})$ *nor* $(-F_{-1}, F \cap -F_{-1})$, *then P and Q have the* $\overline{K}_2$ *property relative to each other, that is, for every element* α *of* $-P$, *there exists an element* β *of* $P-Q$, *such that* α *ultimately dominates* β.

This follows immediately from the preceding results by the dualizing process.

As a consequence of these results, we are able to state the following theorems:

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[9] See T. H. Hildebrandt, *Remarks on the Abel-Dini Theorem,* Amer. Math. Monthly vol. 49 (1942) p. 443.

3.241. *The relation* $(P-Q)\times R=P\times R$ *provided* P, Q *and* R *are classes of System* I, *excepting possibly* $(P, Q)=(F\cup -F_{-1}, -F_{-1})$ *or* $(F\cup -F_{-1}, F)$.

3.242. *The relation* $(P-Q)\times -S=-Q\times -S$, *with* P *properly containing* Q, *holds for all classes of System* I, *excepting possibly* $(P, Q)=(F, F\cap -F_1)$ *or* $(-F_{-1}, F\cap -F_{-1})$.

For the exceptional cases we have:

3.243. *The relation* $((F\cup -F_{-1})-(-F_{-1}))\times R=(F\cup -F_{-1})\times R$ *is not valid for any* R *of System* I *excepting* 0 *and* V.

For if α belongs to $(F\cup -F_{-1})-(-F_{-1})$, then $\infty>\limsup a_n>\liminf a_n>0$. Consequently $((F\cup -F_{-1})-(-F_{-1}))\times R=R$. But from the final table of Part II it is obvious that $(F\cup -F_{-1})\times R=R$ only for $R=0$ and V.

Applying 3.12 we have:

3.244. *The relation* $((F-(F\cap -F_{-1})\times -S=-(F\cap -F_{-1})\times -S$ *is valid for no* S *excepting the classes* 0 *and* V *in System* I.

On the other hand we have:

3.245. *The relation* $((F\cup -F_{-1})-F)\times R=(F\cup -F_{-1})\times R$ *holds for all* R *of System* I *excepting* F.

By Part II, $(F\cup -F_{-1})\times R=-F_{-1}$, for any R between C_0'' and $F\cap -F_{-1}$ inclusive. On the other hand, the class $(F\cup -F_{-1})-F$ is the class of sequences α such that $\limsup a_n=\infty$ and $\liminf a_n=0$. Let α be any element of $-F_{-1}$ and β_0 any element of C_0''. Select the integers n_k so that $c_{n_k}<b_{0k}$. Let $b_n=b_{0n}$ for $n\neq n_k$, $b_n=(b_{0k})^{1/2}$ for $n=n_k$ and k even, $b_n=c_n/k$ for k odd. Then for k odd $a_{n_k}=c_n/b_n<(b_{0k})^{1/2}$ which approaches zero with k, while for k even $a_{n_k}=k$ and $b_{n_k}<b_{0k}/k$. It follows that β is of C_0'' and α of $(F\cup -F_{-1})-F$, so that $((F\cup -F_{-1})-F)\times C_0''\supset -F_{-1}$. The same type of proof works for all classes R between and including C_0'' and $F\cap -F_{-1}$.

For R between and including $-F_{-1}$ and V we have $(F\cup -F_{-1})\times R=V$. It is consequently sufficient to prove that $((F\cup -F_{-1})-F)\times(-F_{-1})\supset V$. Let γ be any element of V and suppose in the first place that $\limsup c_n=\infty$, that is, there exists a subsequence such that $\lim_k c_{n_k}=\infty$. For $n\neq n_k$, take a_n and b_n arbitrary with $a_nb_n=c_n$. For $n=n_k$ and k even, take $a_n=c_n^2$, so that $b_n=1/c_n$. For $n=n_k$ and k odd, take $a_n=1/c_n$, $b_n=c_n^2$. Then both α and β are of $(F\cup -F_{-1})-F$ and so also of $-F_{-1}$. Suppose next that $\infty>\limsup c_n>0$, so that there exists a sequence c_{n_k} such that $\lim_k c_{n_k}=c_0>0$. Take a_n and b_n arbitrary for $n\neq n_k$, with $a_nb_n=c_n$. For $n=n_k$, k even, take $a_n=kc_n$, so that $b_n=1/k$, and for $n=n_k$, k odd, take $a_n=c_n/k$, so that $b_n=k$. Then α and β have the desired properties. Finally if $\lim c_n=0$, a disposition similar to the first case will work.

For $R=F$ we have $((F\cup -F_{-1})-F)\times F=(F\cup -F_{-1})-F$ but $(F\cup -F_{-1})\times F=F\cup -F_{-1}$. This completes the proof of the theorem. As a corollary we have:

3.246. *The relation* $((-F_{-1})-(F\cap -F_{-1}))\times -S=(-F_{-1})\times -S$ *holds for all* S *of System* I *excepting* $-F_{-1}$.

3.3. **Decomposition theorems.** It will be useful later to have certain results on the decomposition of sequences of a class into mutually exclusive subsequences belonging to the class. We have:

3.31. THEOREM. *If* α *is any element of a class* P *of System* I (*or System* II) *then there exist two complementary infinite subsequences of* α: $\alpha'=\{a_{n'}\}$ *and* $\alpha''=\{a_{n''}\}$ *so that each subsequence belongs to* P.

For the classes which are characterized by a convergence property, this theorem is obvious, since every subsequence of any element of such a class has the same convergence property. The same is true of the class L_0 and F. For an element α of $F\cap -F_{-1}$, we select a subsequence $\{a_{n_k}\}$ such that $\lim_k a_{n_k}=0$, decompose it into two sequences having the same property and include the remaining elements of the original sequence in either of these subsequences. The same procedure is obviously applicable to $-F_{-1}$, and consequently to $F\cup -F_{-1}$. For an element α of $-L_\infty$, we use a similar procedure, excepting that in this case, we utilize a subsequence a_{n_k} such that $\lim_k a_{n_k}=a_0<\infty$. For the classes of divergent character, D_{-e} and D'_{-e}, the following procedure is effective. Select a subsequence n_k such that

$$\sum_{n_{k-1}\leqq n<n_k} a_n^{-e} > 1.$$

Take $n=n'$ if $n_{k-1}\leqq n<n_k$, k even, and $n=n''$ for k odd. For the classes D''_{-e} we modify this procedure slightly. Let f_k approach e monotonely decreasing, and select n_k by the condition

$$\sum_{n_{k-1}\leqq n<n_k} a_n^{-f_k} > 1.$$

Then the same decomposition as in the preceding case will give two subsequences each in D''_{-e}.

For System II the result follows from the one-to-one correspondence between sequences α and α^{-1}.

For products of classes we have a similar result, namely:

3.32. THEOREM. *If* P *and* Q *are any two classes chosen either from*

System I *or* II, *and* α *belongs to* $P\cap Q$, *then there exist complementary subsequences* $\{a_{n'}\}$ *and* $\{a_{n''}\}$ *of* α *each of which belongs to* $P\cap Q$.

The proof of this theorem depends upon the following lemma valid for all classes or properties of System I and System II:

3.33. Lemma. *If* α_0 *is any sequence having the property* P *and* α_1 *is a subsequence of a sequence having the property* P *then any combination of* α_0 *and* α_1 *into a single sequence has property* P.

Obviously this includes the case when the sequence α_1 has the property P.

We return to the proof of 3.32. Let α belong to $P\cap Q$. Then there exist $\alpha'=\{a_{n'}\}$ and $\alpha''=\{a_{n''}\}$, complementary subsequences of α each belonging to P. By 3.33 one of these sequences must belong to Q. Suppose it is the first. Then this can be divided into two subsequences $\alpha'''=\{a_{n'''}\}$ and $\alpha^{\text{iv}}=\{a_{n^{\text{iv}}}\}$ both of which belong to Q and one of which belongs to P. Let it be the first. Then α''' and the rearrangement of $\{a_{n''}\}\{a_{n^{\text{iv}}}\}$ in proper order gives by 3.33 two complementary subsequences each belonging to both P and Q. For α'' is of P and α^{iv} is a subsequence of a sequence of P, while α^{iv} is of Q and α'' is a subsequence of a sequence of Q. As a corollary we have:

3.34. *If* P *and* Q *are classes of System* I, *such that* P *properly contains* Q, *then any sequence of* $P-Q$ *can be divided into two complementary subsequences both of which belong to* $P-Q$.

We finally make the following two observations:

3.35. *The classes* P *of the System* Ia *with the exception of* $F\cap -F_{-1}$ *have the property*: *If* α *is of* P *then every subsequence of* α *has the property* P.

This holds also for the classes of System IIa, excepting $F_{-1}\cap -F$.

3.36. *The classes* P *of the System* Ib *with the exception of* $F\cup -F_{-1}$ *have the property*: *If* α' *is a subsequence of a sequence of* P, *which has the property* P, *then the sequence* β *obtained by altering in any way the elements of* α *complementary to* α' *will have property* P.

3.4. **The multiplication theorem.** We are now in position to prove the following theorem.

3.41. Multiplication Theorem. *If* P, Q, R *and* S *are of System* I, *neither* P *nor* R *is* $F\cup -F_{-1}$, *and* Q *nor* S *is* $F\cap -F_{-1}$, P *properly contains* Q, *and* R *properly contains* S, *then* $(P-Q)\times(R-S)=(P\times R)\cap(-Q\times -S)$.

As we have already noted that $(P-Q)\times(R-S)\subset(P\times R)$

$\frown(-Q\times -S)$, only the inverse inclusion needs to be shown. Since neither P nor R are $F\cup -F_{-1}$ and Q nor S are $F\frown -F_{-1}$, the application of Theorems 3.241 and 3.242 yields

$$P\times R=(P-Q)\times R=P\times(R-S),$$
$$-Q\times -S=(P-Q)\times -S=-Q\times(R-S).$$

If then γ is a member of $(P\times R)\frown(-Q\times -S)$, it also belongs to the four classes $(P-Q)\times R$, $(P-Q)\times -S$, $P\times(R-S)$, $-Q\times(R-S)$, that is, we can write

$$\gamma=\alpha_1\times\beta_1=\alpha_2\times\beta_2=\alpha_3\times\beta_3=\alpha_4\times\beta_4$$

where α_1 and α_2 are of $P-Q$, α_3 is of P, α_4 is of $-Q$, β_1 is of R, β_2 is of $-S$, and β_3 and β_4 are of $R-S$. Obviously the theorem is proved unless β_1 is of S, β_2 is of $-R$, α_3 is of Q and α_4 is of $-P$. We shall consequently make these assumptions. We break up the proof into cases.

Case I. *P and R are both of System* Ib *with $F\cup -F_{-1}$ omitted.* We assume then

$$\gamma=\alpha_2\times\beta_2 \text{ with } \alpha_2 \text{ of } P-Q,\ \beta_2 \text{ of } -R,$$
$$\gamma=\alpha_4\times\beta_4 \text{ with } \alpha_4 \text{ of } -P,\quad \beta_4 \text{ of } R-S.$$

By 3.34, the sequence α_2 can be divided into two subsequences, α_2' and α_2'', such that both are in $P-Q$. Consider the corresponding sequences β_4' and β_4''. Since β_4 is of R, by 3.33, not both β_4' and β_4'' can be of $-R$, and since β_4 is of $-S$ not both β_4' and β_4'' can be of S. By considering all possibilities it develops that we have either β_4' or β_4'' is of $R-S$, or one is of $-R$ and the other of S.

Suppose then that β_4' is of $R-S$. Then we define $\alpha=(\alpha_4',\alpha_2'')$, $\beta=(\beta_4',\beta_2'')$ arranged in proper order. Then since α_2'' is of $P-Q$, and P is in Ib, and not $F\cup -F_{-1}$, α is of P by 3.36. Since α_2'' is of $-Q$, and α_4' is a subsequence of α_4 which is of $-P$ and so of $-Q$, α is of $-Q$ by 3.33. A repetition of this reasoning shows that β is of $R-S$.

In case one of the β_4', β_4'' is of $-R$ and the other of S suppose that β_4' is of $-R$. Then by 3.237 there exists a sequence $\beta_0'\leqq\beta_4'$ of $R-S$, if R is in the $\overline{K}_2$ relation to S, which is true since S is not the class $F\frown -F_{-1}$. Define $\alpha=(\alpha_4'\times\beta_4'\times\beta_0'^{-1},\alpha_2'')$ and $\beta=(\beta_0',\beta_2'')$. Then β is of $R-S$ as above. Further α is of P as above. On the other hand $\alpha'\geqq\alpha_2'$, and consequently is a subsequence of a sequence having the property $-P$ and so of $-Q$. Hence α is of $-Q$.

Case II. *P is in* Ia, *R in* Ib *with $F\cup -F_{-1}$ omitted.* Then take

$$\gamma=\alpha_2\times\beta_2 \text{ with } \alpha_2 \text{ of } P-Q,\ \beta_2 \text{ of } -R,$$
$$\gamma=\alpha_3\times\beta_3 \text{ with } \alpha_3 \text{ of } Q,\ \beta_3 \text{ of } R-S.$$

Proceed as in Case I, that is, break up $\alpha_2=(\alpha_2', \alpha_2'')$ each of $P-Q$ and assume in the first place that β_3' is of $R-S$. Then define $\alpha=(\alpha_3', \alpha_2'')$ and $\beta=(\beta_3', \beta_2'')$. Then α is of P since α_2'' has property P and α_3' is a subsequence of a sequence of P. α_2'' is of $-Q$. Then any sequence added to α_2'' will give a sequence which is still in $-Q$, if Q is in Ia and not $F\cap -F_{-1}$. Then α is of $-Q$. The proof that β is of $R-S$ is the same as in Case I above.

If one of the pair β_3', β_3'' is of $-R$ and the other of S, we assume β_3' of S. Then there exists a $\beta_0' \geqq \beta_3'$ of $R-S$, since R has the K_2 property relative to S, when R is not $F\cup -F_{-1}$. Let $\alpha=(\alpha_3' \times \beta_3' \times \beta_0'^{-1}, \alpha_2'')$, $\beta=(\beta_0', \beta_2'')$. Obviously β is of $R-S$. On the other hand since $\alpha'=\alpha_3' \times \beta_3' \times \beta_0'^{-1} \leqq \alpha_3'$, and α_3' is of Q, it follows that α is of P. Further α is of $-Q$ as above.

Case III. *P is in* Ib *with* $F\cup -F_{-1}$ *omitted, R in* Ia. This follows by parity from Case II.

Case IV. *P is in* Ia, *R in* Ia. This can be deduced from Case I by the dualization process. We can also proceed directly. We take

$$\gamma = \alpha_1 \times \beta_1 \text{ with } \alpha_1 \text{ of } P - Q,\ \beta_1 \text{ of } S,$$
$$\gamma = \alpha_3 \times \beta_3 \text{ with } \alpha_3 \text{ of } Q,\ \beta_3 \text{ of } R - S,$$

and follow Case II.

This concludes the proof of the theorem. We might point out that the reasoning in Case II still applies if P is any class in System I, Q is in Ia with $F\cap -F_{-1}$ omitted, R in Ib with $F\cup -F_{-1}$ omitted, and S any class in I; while Case IV applies if P and R are in System I, Q and S are in System Ia with $F\cap -F_{-1}$ omitted, and R has the K_2 relation to S. By a careful analysis of the sufficient conditions involved, it is possible to extend the multiplication theorem to cases in which P or R are the class $F\cup -F_{-1}$, and Q or S the class $F\cap -F_{-1}$. A complete analysis requires the use of the Table of Part II and is rather tedious. We therefore conclude with a combination of classes where the multiplication theorem is not valid, namely:

3.42. THEOREM. *If P is the class* $F\cup -F_{-1}$, *and Q is the class* $-F_{-1}$, *then for no R properly containing S is* $(P-Q)\times(R-S)=(P\times R)\cap(-Q\times -S)$; *and dually the equality does not hold if Q is the class* $F\cap -F_{-1}$ *and P the class F, R and S any.*

This follows at once from a reference to the table of Part II.

BOOK REVIEWS

The theory of rings. By Nathan Jacobson. (Mathematical Surveys, no. 2.) New York, American Mathematical Society, 1943. 150 pp. $2.25.

In recent years the theory of rings has been one of the centers of most vigorous mathematical life. Although originally an outgrowth of the theory of algebras, it has made itself completely independent of its origin. This was necessitated by two considerations. Firstly, it appeared that the special hypotheses inherent to the theory of algebras were not needed for the greater part of the theory of rings. The latter theory, thus stripped of unnecessary encumbrances, became more general and at the same time simpler, clearer and more elegant. Secondly, there are some important applications of the theory of rings that do not fit into the framework of the theory of algebras, such as the applications to the rings of endomorphisms of abelian operator groups.

In the book under consideration a comprehensive account is given of the theory of rings. In view of the fact that mathematicians from all over the world have contributed toward the growth of the theory and that their results are scattered over all the international mathematical periodicals, the collection of the material is by itself no mean task (Jacobson's bibliography covers eight pages of fine print). But it is even more important and difficult to obtain a unified treatment of such a host of different methods and points of view. In this Jacobson has succeeded admirably by means of the methodological principle which he uses. The knowledge obtained in the abstract theory of rings is first applied to the study of abelian operator groups over rings (Emmy Noether's representation moduli). The structural theory of abelian operator groups is next used for an investigation of their rings of endomorphisms. The cycle is closed by means of the following theorem which permits the application of the theory of endomorphism rings to the abstract theory of rings: If R is an abstract ring with an identity element, we denote by a_r (by a_l) for a in R the linear transformation which maps the element x in R onto xa (onto ax); if we consider the additive group R_+ of R as an operator group with respect to the a_l, the ring of the a_r is just the full endormorphism ring and it is essentially the same as the original abstract ring R.

It would lead us too far to give a halfway complete description of the wealth of material covered in this book. It must suffice to mention some of the more salient facts.

In the first chapter the generalities of the theory of operator groups are developed, including a lucid treatment of the uniqueness theorem for direct decompositions which is based on Fitting's Lemma. These two results are used continuously in the sequel.

The general structure theory of rings has been developed solely on the basis of the descending chain condition for ideals. This is made possible by proving C. Hopkins' two theorems which assert the existence of the radical (and its nilpotency) in such rings and the validity of the ascending chain condition for ideals, provided the ring contains an identity. Principal ideal domains enjoy many surprising properties. Most significant are the elementary divisor theorem of Jacobson-Teichmüller and the corresponding invariance theorem of Nakayama.

The structure of finitely generated abelian groups over principal ideal rings is elucidated by proving that such groups are direct sums of indecomposable cyclic groups, and that their structure is completely determined by the bounds of the orders of these cyclic direct summands which are thus shown to be characteristic invariants.

The whole book culminates in the chapter on multiplicative ideal theory. Almost every result is applied here and many concepts find an illustration in this theory. It is shown that for the two-sided ideals in an order one may obtain unique factorization in the classical sense, including the commutativity of multiplication, and that under comparatively simple necessary conditions on the orders in the ring one may construct Brandt's groupoid of ideals. This arithmetic theory of ideals may be considered, in a way, as the conclusion of a development that started with Emmy Noether's famous five axioms and with Dickson's *Algebras and their arithmetics.*

Numerous applications are given in the book: the ring which is obtained by adjoining a semi-linear transformation to a field, the proof of the complete reducibility of a finite group of semi-linear transformations provided the characteristic of the field is not a divisor of the subgroup of ordinary linear transformations, R. Brauer's group of algebras over a field, the author's Galois theory of non-commutative fields.

This very excellent treatment of the theory of rings is more than a compendium. For not only does it offer a résumé of the contents of this theory, but it gives at the same time a very instructive introduction into the working methods used here which will be helpful to others than the algebraists. Since the book is, apart from a very few exceptions, quite self-contained, not much previous knowledge is needed, although it seems desirable to be familiar with the ways of thinking practised in abstract algebra. Thus the book should not only be indispensable to every worker in the theory of rings, but may also be used in connection with an introductory course in abstract algebra.

REINHOLD BAER

An essay on the psychology of invention in the mathematical field. By Jacques Hadamard. Princeton University Press, 1945. 143 pp. $2.00.

Professor Hadamard points out at the beginning of his little book that he is handicapped in the study it deals with by not being a psychologist. Perhaps I should point out that I am handicapped in reviewing him by being neither a psychologist nor a mathematician. But as he bravely goes on, so must I; both of us converging on that question of extraordinary interest in the history of ideas: How do great discoveries and inventions come about?

Hadmard's answer—limited, of course, to the mathematical field—is based on a variety of evidence: the testimony of contemporary mathematicians, the writings of previous psychologists, philosophers and scientists, the interpretation of certain characteristics (logical or intuitive) in the work of famous discoverers and, finally, the author's own minute introspection.

From a careful analysis and comparison of these diverse materials, Professor Hadamard concludes that the general pattern of invention, or, as it might also be put, of original work, is three-fold: conscious study, followed by unconscious maturing, which leads in turn to the moment of insight or illumination. Thereupon another period of conscious work ensues, the purpose of which is to achieve a synthesis of several elements: the novel idea, its logically deduced consequences including proof, and the traditional knowledge to which the new item is added.

Hadamard's investigation, modest and tentative as are its results, seems to me of capital importance in the realm of criticism and cultural history. For what he has done is to show that the human mind tends to behave much the same way whenever it invents, whether in mathematical or in poetic form—a conclusion which does not deny differences of temperament. Our author, on the contrary, is at pains to distinguish among types of mathematical geniuses. He classes them as logical or intuitive, concrete or abstract, yet with enough flexibility to allow for deceptive appearances and for the overlapping of categories. But it is clear in the end that in any process of creation there lurks a mystery—a mystery at least equal to that of thinking itself.

It is worth noting that Hadamard is ever ready to accept as sidelights on his subject the reports of a Mozart or a Paul Valéry on their respective arts. In other words, the customary distinction between mathematics, "cold," "logical," "precise," and the fine arts, presumably hot, chaotic, and gaseous, disappears on a really close view of what goes on. Historical examples show how mathematical genius has leaped ahead of funded truth and thereby become unintelligible to its contemporaries (Galois), just like the "modern" artists of any period; and again, how defiance of common sense and logic has occasionally led to a magnificent expansion of the field (Cardan) though incurring the charge of madness, just like a revolutionary artist, a Blake or a Berlioz. The recurrent adjective "beautiful," which mathematicians have been fond of using since Poincaré, is therefore not misplaced. Rather, it serves as a reminder of the real bond uniting mathematical (that is, scientific) and artistic creation.

So useful and substantial is Professor Hadamard's book, and so compact as well, that one feels a certain reluctance to isolating a few defects in its composition. Yet in the interests of further research, this must be done. A first and usually superficial but constant fault is its misuse of English words cognate with French. Since the author thanks his publishers for their help, I presume they made suggestions as to his English; but these were inadequate in kind and number, leaving the text here and there ambiguous for any reader who does not guess the original thought in the author's mind.

A possibly related error, but more serious, is the misinterpretation of what William James says in a passage quoted from *A pluralistic universe.* Oddly enough, Hadamard is blaming James's narrow conception of logic as verbalized thought when, in fact, the impugned passage precisely asserts James's independence of that conception.

Still with respect to sources and quotations, I regret that Professor Hadamard's attempt at covering "the literature" should have led him to deal with a good many second-rate French writers of the late 19th century. Souriau and Paulhan, even Ribot, Fouillée and LeDantec, are scarcely impressive, either in reputation or in actual utterance. Nor does one find much new light in the suggestions culled by the author from contemporaries such as Valéry and A. E. Housman. Again, Graham Wallas in his *Art of thinking* was only a high class popularizer; his ideas come from James and others who should be preferred as authorities. Similarly, it would have been better to quote Freud on the unconscious instead of Dr. de Saussure. This methodological error only *thickens* the essay; in the room occupied by the epigoni, one could have had the weightier words of the great artists and psychologists, matching the scientists, whom Hadamard is careful to take only from the first rank.

Both because of these minor blemishes and for the sake of wider influence, it is to be hoped that the author will return to his theme and at once revise and expand his essay. There exists, outside mathematics, a considerable literature of "creators' confessions" in which he would find support and amplification of his results.

JACQUES BARZUN

Table of arc sin x. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1945. 19+124 pp. $3.50.

The principal table lists the values of arc sin x to twelve decimal places at intervals of .0001 for the range $0 \leqq x \leqq .9890$ and at intervals of .00001 for $.9890 \leqq x \leqq 1$. The second central difference is also tabulated. The methods and accuracy of interpolation are discussed in the introduction. For example, using the Gregory-Newton formula through second differences one may obtain to twelve decimals the arc sine of an argument (in a suitable range) given to seven decimals. This operation may be carried out conveniently on a ten-bank calculating machine. There are six short auxiliary tables which facilitate interpolation.

E. R. LORCH

Tables of associated Legendre functions. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1945. 46+306 pp. $5.00.

Work on these tables was begun in 1940 to meet an urgent need for a table to six significant figures at intervals of 0.1. As here presented, the tables are subject to difficulties in interpolation in the neighborhood of certain values of the argument. The Mathematical Tables Project hopes to carry out a subtabulation program which will eliminate these difficulties. Since the date of completion of this program is very uncertain, the tables so far completed are herewith made available.

There are fourteen principal and five supplementary tables. For example, table I gives the values of $P_n^m(\cos\theta)$ for values of n from 1 to 10, of m from 1 to 4 $(m \leqq n)$ and of θ from 0° to 90° in intervals of 1°; results are given to six significant figures. Also tabulated are $dP_n^m(\cos\theta)/d\theta$, $P_n^m(x)$, $1 \leqq x \leqq 10$; $Q_n^m(x)$, $1 < x \leqq 10$; its derivative; and other variations of these functions involving pure imaginary arguments and half integral values of n. Dr. Lowan has written an introduction detailing the methods of computation and of checking the manuscript. There is a valuable bibliography listing cognate material.

E. R. LORCH

Note. In the November, 1945, issue of this BULLETIN appear reviews of *Mathematical theory of optics* by R. K. Luneberg and of the *Supplementary notes* which accompany the book. Inadvertently M. Herzberger was credited with the authorship of all four notes. N. Chako and A. A. Blank were, in fact, the authors of the first note.

NOTES

In accordance with regulations drawn up by the Office of War Mobilization and Reconversion, in cooperation with the Army and Navy, it is now possible to request the release from the services of persons who are qualified to serve as instructors in the colleges and universities. Requests must be made through the U. S. Office of Education. Details regarding the procedures to be followed in making requests are contained in a Bulletin, issued by the Office of Scientific Personnel, a copy of which may be secured by writing to the Secretary of the Society.

In connection with Netherlands-America week held in Amsterdam during the first week of December, addresses by Professor H. M. Gehman of the University of Buffalo and Professor Gabor Szegö of Stanford University were given in the Science Section.

A symposium on mathematical statistics and probability was held at the University of California at Berkeley January 28–30, 1946.

The following persons have received pre-doctoral fellowships from the National Research Council. The institutions given are those at which the individuals will study. E. J. Akutowicz, Harvard University; B. Altshuler, New York University; M. G. Arsove; S. I. Askovitz, University of Pennsylvania; Richard Bellman, Princeton University; R. H. Bowker, Columbia University; Herman Chernoff, Brown University; R. M. Cohn, Columbia University; J. Dugundje, Massachusetts Institute of Technology; B. R. Gelbaum, Harvard University; A. W. Goodman, Columbia University; William Gustin, University of California at Los Angeles; L. A. Henkin, Princeton University; I. I. Hirschman, Brown University; P. Olum, Harvard University; E. Paulson, Columbia University; I. H. Rose, Harvard University; W. R. Scott, Ohio State University; Daniel Zelinsky, University of Chicago.

The Department of Mathematics of Harvard University is receiving applications for Benjamin Peirce Instructorships for the academic year 1946–1947. A candidate should ordinarily have the doctorate or its equivalent. Applications and requests for further information may be sent to the Chairman of the Department.

Professor Subrahmanyan Chandrasekhar of the University of Chicago has been awarded the Sir Cattanmanchi Ramalinga Reddy National Prize for Mathematics by Andhra University of India.

Associate Professor Garrett Birkhoff of Harvard University and Professor Oscar Zariski of the University of Illinois have been elected members of the Academico Correspondiente of the Academia Nacional de Ciencias Exactas Físicas y Naturales de Lima.

Professor W. C. Brenke of the University of Nebraska has retired with the title professor emeritus.

Dean L. P. Eisenhart of Princeton University has retired with the title professor emeritus.

Professor F. L. Hitchcock of Massachusetts Institute of Technology has retired with the title professor emeritus.

Dr. D. G. Humm has been elected president of the Southern California Academy of Criminology.

Professor R. E. Langer of the University of Wisconsin has been elected vice president of the American Association for the Advancement of Science.

Professor J. H. M. Wedderburn of Princeton University has retired with the title professor emeritus.

Mr. Walter Pitts of the Radiation Laboratory, Massachusetts Institute of Technology, has been awarded a Guggenheim Fellowship.

Professor J. G. Van der Corput has been appointed to a professorship at the University of Amsterdam.

Dr. R. H. Bing of the University of Texas has been promoted to an assistant professorship.

Assistant Professor F. L. Brooks of Kent State University, Kent, Ohio, has been promoted to an associate professorship.

Dr. W. B. Caton of the Illinois Institute of Technology has been appointed to an assistant professorship at Duluth State Teachers College.

Assistant Professor Claude Chevalley of Princeton University has been promoted to an associate professorship.

Assistant Professor Helen E. Clarkson of Hardin-Simmons University has been appointed to an adjunct professorship at the University of South Carolina.

Assistant Professor G. F. Cramer of Tulane University has been promoted to an associate professorship.

Dr. M. M. Day of the University of Illinois has been promoted to an assistant professorship.

Assistant Professor Douglas Derry of the University of Saskatchewan has been promoted to an associate professorship.

Assistant Professor J. L. Dorroh of Louisiana State University has been promoted to an associate professorship.

Professor Harold Hotelling of Columbia University has been appointed to a professorship at North Carolina State College.

Professor W. V. Houston of the California Institute of Technology has been elected president of the Rice Institute.

Dr. Rufus Isaacs has been appointed to an assistant professorship at the University of Notre Dame.

Mr. O. C. Kreider has been appointed to an assistant professorship at Iowa State College.

Dr. W. G. Madow has been appointed visiting professor of statistics at the University of São Paulo, São Paulo, Brazil.

Dr. Szolem Mandelbrojt has been appointed to a professorship at the Rice Institute.

Dr. C. W. Mendel of the University of Illinois has been promoted to an assistant professorship.

Assistant Professor H. J. Miles of the University of Illinois has been promoted to an associate professorship.

Assistant Professor A. K. Mitchell of Trinity College, Hartford, Connecticut, has accepted a position with the United Aircraft Corporation.

Dr. E. W. Montroll of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at the University of Pittsburgh.

Assistant Professor Hermance Mullemeister of the University of Washington has been promoted to an associate professorship.

Dr. A. M. Peiser has been appointed head of the Statistics Research Group at the Langley Field laboratory of the National Advisory Committee for Aeronautics.

Dr. I. E. Perlin has been appointed to an assistant professorship at the Georgia School of Technology.

Dr. J. W. Peters of the University of Illinois has been promoted to an assistant professorship.

Dr. A. L. Putnam of Yale University has been appointed to an assistant professorship at the University of Chicago.

Dr. Moses Richardson of Brooklyn College has been promoted to an assistant professorship.

Associate Professor F. A. Rickey of Louisiana State University has been promoted to a professorship.

Associate Professor L. V. Robinson of Emory University, Emory University, Georgia, has been appointed to an associate professorship at the University of South Carolina.

Dr. Max Shiffman of the College of the City of New York has been appointed lecturer at Princeton University.

Miss Helen F. Smith of Iowa State College of Agriculture and Mechanic Arts has been promoted to an assistant professorship.

Mr. L. L. Steimley of the University of Illinois has been promoted to an assistant professorship.

Dr. S. L. Thorndike has been appointed to an associate professorship at Winthrop College, Rock Hill, South Carolina.

Dr. F. E. Ulrich of the Rice Institute has been promoted to an assistant professorship.

Dr. H. E. Vaughan of the University of Illinois has been promoted to an assistant professorship.

Assistant Professor P. A. White of Louisiana State University has been appointed to an assistant professorship at the University of Southern California.

Assistant Professor J. W. T. Youngs of Purdue University has been appointed to an associate professorship at Indiana University.

Professor Oscar Zariski of Johns Hopkins University, who has been for the past year at the University of São Paulo as a visiting professor, has been appointed to a professorship at the University of Illinois.

The following appointments to instructorships are announced:

University of California: Dr. Abraham Sidenberg; University of Chicago: Mr. A. R. Jacoby; University of Cincinnati: Mr. Gerhard Vater; College of the City of New York: Dr. S. F. Barber; Columbia University: Dr. E. R. Kolchin; Iowa State College: Mr. O. S. Carpenter, Mr. R. E. Carr, Mr. A. E. Engelbrecht, Mr. C. J. Maloney; Johns Hopkins University: Mr. Arthur Bernstein; University of Kentucky: Mr. C. L. Riggs, Mr. Donald Rose, Mr. F. D. White; Marquette University: Mr. James Martin; Ohio State University: Dr. V. F. Cowling; Princeton University: Dr. G. P. Hochschild; Purdue University: Dr. B. H. Arnold, Dr. W. F. Eberlein, Dr. A. L. Whiteman; University of Saskatchewan: Mr. W. Buscombe, Dr. W. C. G. Fraser, Miss Alma Hiebert; South Dakota State College: Mr. J. H. Blau; University of Tennessee: Miss Marjorie Watson; University of Texas: Mr. W. B. Coleman; Yale University: Mr. R. E. Fullerton, Mr. L. R. Norwood, Dr. B. J. Pettis.

Professor C. E. Wright of the Military College of Science, Woolwich, died October 30, 1945.

Professor Harry Bateman of the California Institute of Technology died January 21, 1946, at the age of sixty-three years. He had been a member of the Society for thirty-five years.

Professor Emeritus A. L. Candy of the University of Nebraska died July 18, 1945. He had been a member of the Society for twenty-eight years.

Professor Emeritus A. R. Congdon of the University of Nebraska died November 11, 1945.

Dr. Hans Fried of Swarthmore College died December 23, 1945, at the age of fifty-two years.

Professor Emeritus Arthur Korn of the Stevens Institute of Technology died December 21, 1945, at the age of seventy-five years. He had been a member of the Society for thirty-two years.

Professor Emeritus T. A. Pierce of the University of Nebraska died August 18, 1945. He had been a member of the Society for twenty-eight years.

Professor Emeritus C. A. Van Velzer of Carthage College died November 19, 1945, at the age of ninety-four years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Algebra and Theory of Numbers

53. A. A. Albert: *On Jordan algebras of linear transformations.*

Consider any linear space A of linear transformations $a, b, \cdots$ over a field F of characteristic not two such that $a \cdot b = 1/2(ab+ba)$ is in A for every a and b of A where ab is the associative product. Then A is called a *Jordan algebra* with respect to the operation $a \cdot b$. The radical of A is defined to be its maximal solvable ideal N and it is shown to be solvable if and only if all transformations in A are nilpotent. A trace criterion is derived in case F is nonmodular, and a Pierce decomposition relative to an idempotent is obtained. The trace condition is then used to derive the analogues of the associative algebra theorem in the case of a principal idempotent. If $N=0$, A is called semisimple and it is shown that A is a direct sum of simple algebras. All simple Jordan algebras over an algebraically closed field are obtained and the results may be combined with those of a Chicago dissertation of G. Kalisch to yield all simple Jordan algebras over any nonmodular field. (Received December 21, 1945.)

54. B. L. Brown and N. H. McCoy: *Radicals and subdirect sums.*

Suppose that in each ring $\mathfrak{R}$ there is assigned a mapping F of $\mathfrak{R}$ into the set of two-sided ideals of $\mathfrak{R}$ in such a way that if $a \rightarrow \bar{a}$ is a homomorphism of $\mathfrak{R}$ on $\overline{\mathfrak{R}}$, then $F(\bar{a}) = \overline{F(a)}$. An element a of $\mathfrak{R}$ is *F-regular* if and only if $a \in F(a)$, and an ideal is *F-regular* if and only if each of its elements is F-regular. The F-radical $\mathfrak{N}$ of $\mathfrak{R}$ is to consist of the elements of $\mathfrak{R}$ which generate F-regular two-sided ideals. It is shown that $\mathfrak{N}$ is a two-sided ideal in $\mathfrak{R}$, that $\mathfrak{R}/\mathfrak{N}$ has zero F-radical and that, under certain conditions, the F-radical of the matrix ring $\mathfrak{R}_n$ is $\mathfrak{N}_n$. The vanishing of the F-radical is necessary and sufficient that $\mathfrak{R}$ be isomorphic to a subdirect sum of subdirectly irreducible rings of zero F-radical. Various choices of F are discussed, a case of special interest being $F_1(a) = \{x + ax + \sum y_i z_i + \sum y_i a z_i;\ x,\ y_i,\ z_i \in \mathfrak{R}\}$. In the presence of the descending chain condition for right ideals, the F_1-radical coincides with the radical as defined by Jacobson (Amer. J. Math. vol. 67 (1945) pp. 300–320) and also with the various other definitions. The vanishing of the F_1-radical is necessary and sufficient that $\mathfrak{R}$ be isomorphic to a subdirect sum of simple rings, each with unit element. (Received January 30, 1946.)

55. Marshall Hall: *The sum of continued fractions.*

By proving a general theorem on Cantor point sets, it is shown that every real number modulo 1 is the sum of two continued fractions whose partial quotients do not exceed 4. The number 4 is the best possible value for this theorem. As an application it is shown that every number less than $1/32^{1/2}$ is the minimum of a quadratic form of discriminant 1, a result which contrasts with the Markoff theorem that the number of minima exceeding any number greater than 1/3 is finite. (Received January 29, 1946.)

56. L. K. Hua: *Orthogonal classification of Hermitian matrices.*

The paper is concerned with the orthogonal conjunctiveness of the Hermitian matrices H and K with complex elements. H and K are said to be conjunctive orthogonally if $K=\bar{P}HP'$ for $PP'=I$. Then $\bar{H}H$ and $\bar{K}K$ are similar and have the same elementary divisors. The nature of the elementary divisors of a matrix T expressible as such a product $\bar{H}H$ is determined and a discussion is given of the existence of an H, unique apart from orthogonal conjunctiveness, such that $\bar{H}H$ has a prescribed set of elementary divisors. Canonical forms are given. (Received December 21, 1945.)

57. R. E. Johnson and Fred Kiokemeister: *The endomorphisms of the total operator domain of an infinite module.*

Let P be a division ring, and let Ξ be a P-module with a countable P-basis. The total operator domain Δ of Ξ can be identified with an infinite matrix ring over a division ring $\bar{P}$ anti-isomorphic with P. The present paper is a study of the endomorphisms of Δ. It is shown that the only nonzero endomorphisms of Δ are the meromorphisms. Any meromorphism is complete in the sense that it preserves certain infinite sums which are defined in Δ. A representation is found for the class of all meromorphisms. The $\bar{P}$-meromorphisms are shown to be expressible as a direct sum of inner meromorphisms, while any $\bar{P}$-automorphism is itself inner. (Received January 23, 1946.)

58. G. K. Kalisch: *Completion of topological rings and fields.*

This paper deals with the possibility of completing topological rings and fields, the principal result being a proof that any topological ring can be completed (cf. in this connection D. van Dantzig, *Zur topologischen Algebra*, I, Math. Ann. vol. 107 (1932) pp. 587–626, in particular TR. 21 and TR. 23 on p. 620, his "Ringkomplettierungsaxiom"). (Received January 21, 1946.)

59. D. H. Lehmer: *Some non-existence theorems on partitions.*

According to theorems due to Euler and Rogers-Ramanujan, every positive integer n may be partitioned into distinct parts in as many ways as into odd parts; and n may be partitioned into parts differing by two or more in as many ways as into parts congruent to 1 or 4 modulo 5. The present note proves that there exists no set S of positive integers with the property that every positive integer may be partitioned into parts differing by d or more in as many ways as into parts taken from the set S either with or without repetitions allowed, except for the above mentioned cases where $d=1$ and 2. A similar theorem holds for compositions. (Received January 21, 1946.)

60. J. M. Thomas: *Division sequences.*

Formulas are developed for the Euclid and Sturm sequences defined by polynomials of $R[x]$, where R is an integrity domain. The polynomials of the sequences are expressed by certain determinants formed from the resultant by recursion relations. The results are complete in the sense that they apply either to indeterminate or to numerical coefficients for arbitrary degree differences. (Received January 16, 1946.)

61. Leonard Tornheim: *A method for determining the number of compositions of certain types.*

In the remainder on division of x^m by $x^n - a_1x^{n-1} - \cdots - a_n$, the coefficient of x^{n-1} is $\sum a_{i_1}a_{i_2}\cdots$, where the sum is over all compositions $(i_1, i_2, \cdots)$ of $m-n+1$ with every element $i \leqq n$. By assigning to $a_1, \cdots, a_n$ certain numerical values the total number of compositions with elements $i \leqq n$ of the following types are obtained: (1) all such compositions; (2) all with certain elements absent; (3) all containing certain elements a given number of times; and (4) all with (2) and (3) holding jointly. Also it is possible with this method to find the total number of partitions with parts not greater than n, to list them, and to determine how many compositions correspond to a given partition. (Received January 17, 1946.)

ANALYSIS

62. Gertrude Blanch: *On the computation of Mathieu functions.*

If a characteristic value of Mathieu's differential equation is known to within an error λ, it is possible, by fairly simple means, to correct the characteristic value and the approximate values of the Fourier coefficients defining the solution, to within an error proportional to λ^2. The precise formulas for the corrections, a systematic method of carrying out the computations, and two illustrative examples are given in this paper. Although Mathieu functions only are dealt with, the method is applicable to solutions of other types, where the coefficients of the Fourier (or power) series are determined by a three-term recurrence formula. (Received January 12, 1946.)

63. R. C. Buck: *On a class of entire functions.* I.

Let $K(a, c)$ denote the class of entire functions of order 1 and of type at most a on the whole real axis and type c on the whole imaginary axis, with $c < \pi$. This class has the property that a function $f(z)$ belonging to it is determined completely by the sequence $f(n)$, $n = 1, 2, \cdots$. The author first proves a necessary and sufficient condition that for a given sequence w_n there exist a function $f(z)$ of $K(a, c)$ such that $f(n) = w_n$. Using this, a wide variety of theorems is obtained. For example, if $f(n) = r_n e^{i\theta_n}$ and if there is an α, $c < \alpha \leqq \pi$, such that $\lim\inf\{\cos(\theta_n + n\alpha)\}^{1/n} > 0$, then $f(z) \equiv 0$, if $f(z)$ belongs to $K(a, c)$. As a special case, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two disjoint closed convex sets and if $f(n) \in \mathcal{D}_1$ for $n \in \mathcal{A}_1$, $f(n) \in \mathcal{D}_2$ for $n \in \mathcal{A}_2$ where $\mathcal{A}_1 \cup \mathcal{A}_2$ has density 1, and if $\mathcal{A}_1$ has density 1/2 and satisfies an additional condition, then $\lim\sup \log|f(iy)|/|y| \geqq \pi$. In case the w_n are close enough to integral values, more can be said. Thus, for example, it is proved that no function of $K(0, 0) = K_0$ can take prime values at the integers without being constant. (Received January 17, 1946.)

64. R. C. Buck: *On a class of entire functions.* II.

The results of the previous paper are extended to more general sequences than $\{f(n)\}$, treating instead numbers F_n expressible as linear combinations of the numbers $f(n)$. Most of the theorems proved for the sequence $f(n)$ hold also for F_n. Thus, for example, if $F_n = \sum_0^n g(k)f(k)$ where $g(z)$ belongs to K_0 and $f(z)$ to $K(a, c)$ for $c < \pi$, then for any $\epsilon > 0$, $\log|F_n| > (h(0) - \epsilon)n$ for a set of maximum density at least $1 - c/\pi$. These theorems also have corollaries concerned with the solution of certain functional equations. Extension is also made to sequences based on $f(\lambda_n)$ where $\limsup|n-\lambda_n|^{1/n}<1$. Counterexamples show that for some of these theorems, the condition $\lambda_n-n=O(1)$ is not sufficient. (Received January 17, 1946.)

65. Herman Chernoff: *Complex solutions of partial differential equations.* II.

The author considers certain "classes" $\mathcal{C}$ of complex solutions of equations $\Delta u+Au_x+Bu_y+Cu=0$. (For a more detailed discussion of "classes" see Bergman, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155 and vol. 57 (1945) pp. 299–331.) He defines in the usual way the index of points where $u(x, y)=\alpha$. A function u which has the property that the sum of the indices in the circle $|z|<r$ is 0 or 1 for each value of α and for all $r<\rho$ is said to be *pseudo-simple in the circle* $|z|<\rho$. Generalizing the classical results on simple functions of a complex variable, the author proves the following inequalities: if $u(x, y)=\sum A_{mn}z^m\bar{z}^n$ $(z=x+iy,\ \bar{z}=x-iy)$ belongs to a class $\mathcal{C}$ and is pseudo-simple in $|z|<\rho$ then $|C_m(r)|\leqq k(m)r^{m-1}C_1(r)$ for $0<r<\rho, m=2, 3, 4, \cdots$. Here $k(m)$ are fixed positive numbers ($k(2)=2$, $k(3)=3$, $k(4)=4.2858$, $k(5)=5.9158$, and so on). Furthermore $C_m(r)=\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}A_{\nu 0}\Gamma(\nu+1)H(m, n, \nu)r^n/\pi^{1/2}\Gamma(\nu+1/2)$ where $H(m, n, \nu)$ are independent of the particular solution u but depend on the class $\mathcal{C}$. (Received January 26, 1946.)

66. Paul Civin: *Polynomial dominants.*

Let $f(x)$ be a continuous function of period 2π, and let $F_n(x)$ be a trigonometric polynomial of degree n which dominates $f(x)$ at the points $x_j^n=2j\pi/(2n+1)$ for $j=0, \cdots, 2n$, that is, $F_n(x_j^n)\geqq|f(x_j^n)|$. Relations of inequality are obtained between various integral means of the functions $f(x)$ and $F_n(x)$. (Received December 26, 1945.)

67. Evelyn Frank: *The location of the zeros of polynomials with complex coefficients.*

This paper develops a rational process for obtaining by successive approximation the zeros of polynomials $P(z)$ with complex coefficients. The process depends upon: (1) If w' is a pure imaginary zero (∞ or 0 included) of $f_r(w)=P(r(1-w)/(1+w))$, $r>0$, then $z'=r(1-w')/(1+w')$ is a zero of $P(z)$. (2) The pure imaginary zeros w' of $P_r(w)=(1+w)^nf_r(w)$ are precisely the zeros of $D_r(w)$, greatest common divisor of $P_r(w)$ and $Q_r(w)=[P_r(w)\pm\bar{P}_r(-w)]/2$ (plus if degree of $P_r(w)$ odd, minus if even; $\bar{P}_r(w)$ obtained by replacing coefficients of $P_r(w)$ by their complex conjugates). (3) By the euclidean algorithm for determining $D_r(w)$, a continued fraction expansion for $Q_r(w)/P_r(w)$ results, in which the number of negative coefficients $c_p(r)$ is precisely the number of zeros of $P(z)$ within $|z|=r$ (Frank, *On the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 144–157). (4) Hence, if r is varied and changes in signs of the $c_p(r)$ noted, the values of the moduli of the zeros of $P(z)$ can be approximated as closely as desired and simultaneously the corresponding approximation to the zeros themselves can be obtained from the last quotient. The computation can be reduced to synthetic divisions and cross-multiplication processes. (Received December 17, 1945.)

68. R. E. Fullerton: *Compactness in certain Lebesgue spaces.*

Let m be a completely additive measure function defined over a σ-field $\mathcal{J}$ of subsets of a set R. $A(R)$ and $V^p(R)$ will denote the Banach spaces of completely additive set functions which are respectively absolutely continuous and of bounded p-variation over $\mathfrak{J}$. Furthermore let $\mathfrak{J}$ be compact in the topology defined by the metric $(e_1, e_2)=M[(e_1\cup e_2)-(e_1\cap e_2)]$; $e_1, e_2\in\mathfrak{J}$. It is shown that a necessary and sufficient condition that a bounded set in $A(R)$ or $V^p(R)$ be compact is that the function of the set be equi-absolutely continuous in the norm, i. e., if $G_1\subset A(R)$ $\lim_{m(E)\to 0} \mathrm{var}_E f(e)=0$ uniformly for $f\in G_1$ and if $G_2\subset V^p(R)$ $\lim_{m(E)\to 0} [p-\mathrm{var}_E f(e)]=0$ uniformly for $f\in G_2$. As a corollary it is shown that weak compactness and compactness of sets in the Lebesgue space $L(R)$ are equivalent. In $L^p(R)$ weak compactness and equi-absolute continuity are necessary and sufficient for compactness. Further corollaries give the known condition for compactness in l^p (Cohen and Dunford, Duke Math. J. vol. 3) and the equivalence of weak compactness and compactness in l. (Received January 18, 1946.)

69. Leonard Greenstone: *Mapping by anaytic functions of multiply connected domains.*

Given an arc $\mathfrak{a}$ which lies entirely in some finitely connected domain $\mathfrak{G}$, and let $\mathfrak{G}$ be mapped conformally on $\mathfrak{B}$ so that $\mathfrak{a}$ is mapped into $\mathfrak{p}\subset\mathfrak{B}$. Suppose that the boundary of $\mathfrak{B}$, $\mathfrak{b}$, is sufficiently smooth so that (1) a circle of radius τ can roll freely along $\mathfrak{b}$ for some sufficiently large segment $\mathfrak{b}^*\subset\mathfrak{b}$, and (2) no point of $\mathfrak{p}$ is further from $\mathfrak{b}$ than a distance η, then there exists a constant c, which is independent of $\mathfrak{B}$, such that if $L(\mathfrak{p})$ is the Euclidean length of $\mathfrak{p}$, $L(\mathfrak{p})\leqq 2\pi^{1/2}\eta c(1+\eta/2\tau)$. c is the non-Euclidean length of $\mathfrak{a}$, that is, $c=\int_{\mathfrak{a}}(K_{\mathfrak{G}}(z,\bar{z}))^{1/2}|dz|$, $K_{\mathfrak{G}}$ being the kernel function of $\mathfrak{G}$ (Bergman, *Partial differential equations, advanced topics*, Brown University, 1941). If, instead of (1), it is only assumed that for at least one point of $\mathfrak{b}^*$ it is possible to place a circle of radius τ tangent to $\mathfrak{b}$ and which lies outside $\mathfrak{B}$, then if (2) is retained and h is defined as the Euclidean distance between the ends of $\mathfrak{p}$, $L(\mathfrak{p})\leqq 2\pi^{1/2}c(\eta+(\eta^2+h^2)/2\tau)$. Let $\zeta\in\mathfrak{b}$, $\zeta=\xi+i\eta$, then if $f(z)$ is a function analytic in $\mathfrak{B}$, whose real part $f_1(x, y)$ and whose derivatives are single valued in $\mathfrak{B}$, $f(z)=i\int_{\mathfrak{b}}f_1(\xi, \eta)(\int^z K_{\mathfrak{B}}(z_1, \bar{\zeta})dz_1)d\bar{\zeta}+c^*$, c^* constant. This formula identifies the second derivative of the Green's function for $\mathfrak{B}$ with the kernel function for $\mathfrak{B}$ multiplied by -2π. (Received January 12, 1946.)

70. J. D. Hill: *Summability of the sequence of Rademacher functions.*

G. G. Lorentz has recently shown (Math. Zeit. vol. 49 (1944) pp. 724–733) that if the real sequence $\{s_k\}$ is bounded, then the sequence $\{s_kR_k(y)\}$ is summable to zero a.e. by the Cesàro method (C, α) for each $\alpha>0$, where $R_k(y)$ is the kth Rademacher function. By means of results given recently by the author (Ann. of Math. vol. 46 (1945) pp. 556–562) this note points out generalizations of the Lorentz result to other methods of summability, on the one hand, and to classes of sequences more general than bounded ones, on the other. Further results of a related character are also established. (Received December 28, 1945.)

71. Walter Kohn: *Distortion by simple functions.* Preliminary report.

The strip-method of Groetsch (Berichte der Saechsischen Akademie der Wissenschaften vol. 80 (1927) p. 367) is used to obtain the following properties of simple functions of the type $w=z+a_2z^2+\cdots$ and convergent in $|z|<1$. (I) Denote the region $|z|<1$ by D and its map by Δ. Let $\rho(\psi)$ be the shortest distance on arg $(w)=\psi$ from $w=0$ to the boundary of Δ. Then $\int_0^\pi\{\rho(\psi)+\rho(\pi+\psi)\}d\psi\geqq 2\pi$, so that the average length of the segments through $w=0$ is greater than or equal to 2. (II) Inequalities are obtained for the maps of circular coronas. In particular, as the inner circle shrinks to a point, there results: $\int_D\{|w'(z)|^2/|w(z)|^2-1/|z|^2\}da\geqq 0$, where da is the element of area in D. (III) For bounded functions, $|w|\leqq M$, $|z|<1$, the distortion theorem $r/(1+r)^2\leqq|w|\leqq r/(1-r)^2$, $|z|=r$, is improved. One obtains for the variation of $|w(z)|$ on $|z|=r$: $(M/2)\{P(r)-(P(r)^2-4)^{1/2}\}\leqq|w|_{\min}\leqq r$; $r\leqq|w|_{\max}\leqq(M/2)\cdot\{P(-r)-(P(-r)^2-4)^{1/2}\}$ where $P(r)=M(r+1/r)+2(M-1)$. These bounds are exact. (Received January 19, 1946.)

72. Harry Pollard: *Integral transforms.*

Integral transformations of L^2 with kernels of type $H(s\pm t)$ are studied with respect to (i) spectral properties, (ii) inversion formulas, (iii) the iterated transformations. By use of methods of Carleman and Stone one can obtain new inversion formulas for the Weierstrass and iterated Laplace transformations. Of particular interest is the resolution of the identity of the Laplace transform, since it has apparently escaped the attention of workers in the field that this transform is a bounded self-adjoint transformation of Hilbert space with a continuous spectrum. (Received January 19, 1946.)

73. Harry Pollard: *Necessary and sufficient conditions for the representation of an analytic function as a Laguerre series.* Preliminary report.

Necessary and sufficient conditions are obtained for a function analytic in a parabola to be representable by a complex Laguerre series. This complements the corresponding result of Hille for Hermite series (Trans. Amer. Math. Soc. vol. 47 (1940) p. 80). The proof of the necessity parallels that of Hille, although the technical details are more complicated. The sufficiency proof is of a different kind from Hille's, depending on a simple though apparently new formula connecting the Laguerre and Hermite polynomials. (Received January 19, 1946.)

74. Menahem Schiffer: *On the kernel function of an orthonormal system.*

Every function $f(z)$, analytic in a domain D and with finite $\iint_D|f(z)|^2dxdy$, satisfies the integral equation $(^*)f(z)=\iint_D f(\xi)K(z,\bar\xi)d\xi d\eta$ where $K(z,\xi)$ is the kernel function defined by Bergman (*Über die Kernfunktion eines Bereiches*, J. Reine Angew. Math. vol. 169 (1933) pp. 1–42). The kernel function is shown to be determined uniquely by this property. Then if $g(z,\xi)$ denotes Green's function of D, $-(2/\pi)\partial^2g(z,\xi)/\partial z\partial\bar\xi$ is proved, by integration by parts, to have this characteristic property and to coincide, therefore, with $K(z,\bar\xi)$. From Hadamard's variation formula for Green's function the variational formula $\delta K(z,\bar\xi)=\oint K(z,\bar t)K(t,\bar\xi)\delta n ds$ is derived. If $\phi(z,\xi)$ is an analytic function of z which maps D upon the unit circle slit along concentric circular arcs such that $z=\xi$ corresponds to the center, and $\gamma(z,\xi)=-\log|\phi(z,\xi)|$, the kernel $K^*(z,\bar\xi)=-(2/\pi)\partial^2\gamma(z,\xi)/\partial z\partial\bar\xi$ satisfies equation $(^*)$ with respect to every $f(z)$ which is square integrable in D. $K^*(\xi,\xi)^{-1}$ is the minimum value of all integrals $\iint_D|f'(z)|^2dxdy$ where $f(z)$ is analytic in D and $f'(\xi)=1$. If D is mapped by univalent functions $g(z)=1/(z-\xi)+a_0+a_1(z-\xi)+\cdots$ upon an infinite domain Δ, the complement Δ^+ of Δ has an area not greater than $\pi^2K^*(\xi,\bar\xi)$. (Received January 14, 1946.)

75. I. E. Segal: *Topological groups in which multiplication on one side is differentiable.*

It is shown that if left multiplication is of class C^1 on a topological group which is a manifold of class C^1, then right multiplication is likewise of class C^1. The proof strongly utilizes the existence of Haar measure. The method does not involve introduction of canonical coordinates, and is logically independent of the theory of Lie groups. Combining the foregoing result with either a result of P. A. Smith (Ann. of Math. vol. 44 (1943) pp. 480–513) or a result of G. Birkhoff (Trans. Amer. Math. Soc. vol. 43 (1938) pp. 61–101), it follows that if a group has a neighborhood of the identity which is a manifold of class C^1 and on which left multiplication is of class C^1, then it is a Lie group. Previously the weakest condition which assured that a group be a Lie group was that due to Smith and required in addition to the preceding that right multiplication satisfy a Lipschitz condition. (Received January 21, 1946.)

76. H. J. Zimmerberg: *A class of definite integral systems.*

This paper treats a vector integral system of the form $y(x)=\lambda\int_a^b K(x, t)y(t)dt$, where the matrix $K(x, t)=H(x, t)S(t)$ and the discontinuities of the elements of the matrix $H(x, t)$ are regularly distributed, under the following conditions: (1) the matrix $S(x)$ is hermitian on ab; (2) the matrix $K_1(x, t)\equiv S(x)K(x, t)=K_1^*(t, x)$ on $a\leqq x, t\leqq b$, the $*$ denoting the conjugate transpose operation; (3) the hermitian functional $J(g)\equiv\int_a^b\int_a^b g^*(x)K_1(x, t)g(t)dxdt\geqq 0$ for arbitrary vectors $g(x)$ whose components are continuous on ab. These systems differ from the definitely self-conjugate adjoint integral systems of Wilkins (Duke Math. J. vol. 11 (1944) pp. 155–166) in that the above condition (3) replaces a corresponding definiteness assumption on the matrix $S(x)$. Such systems, which include the integral system to which an H-definitely self-conjugate adjoint differential system (Reid, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 381–425) is equivalent, are shown to possess fundamental properties similar to those previously established for definitely self-conjugate adjoint integral systems. Furthermore, it is shown that there is no restriction on the character of the characteristic solutions or the corresponding characteristic values in assuming that an integral system of the above form satisfies conditions (1) and (2). (Received December 26, 1945.)

77. H. J. Zimmerberg: *On a self-adjoint differential system of even order.*

The results of Reid (Trans. Amer. Math. Soc. vol. 52 (1942) §11) are extended to a self-adjoint system $F(u)=\lambda G(u)$, $U_\sigma(u;\ \lambda)\equiv U_\sigma^0(u)+\lambda U_\sigma^1(u)=0$ $(\sigma=1, \cdots, 2n)$, where $F(u)$ and $G(u)$ are differential operators of the form $F(u)=\sum_{\nu=0}^{n}(f_\nu(x)u^{(\nu)})^{(\nu)}$, $G(u)=\sum_{\mu=0}^{n-1}(g_\mu(x)u^{(\mu)})^{(\mu)}$, $f_n(x)\neq 0$, while for arbitrary values of λ the $U_\sigma(u;\ \lambda)$ are independent linear forms in the end values of $u, u', \cdots, u^{(2n-1)}$ at $x=a$ and $x=b$ for which $U^1(u)$ involves only the end values of $u, u', \cdots, u^{(n-1)}$. It is shown that such systems are equivalent to certain types of boundary value problems associated with the second variation of an isoperimetric problem of Bolza in the calculus of variations. Various assumptions of definiteness for these systems are also considered. (Received December 3, 1945.)

78. M. A. Zorn: *Derivatives and Fréchet differentials.*

It is shown that, for complex Banach spaces, the continuity of the Gâteaux differential in its increment variable makes the G-differential a Fréchet differential. Using a notion which has also been introduced by A. D. Michal this may be re-expressed in the form: If a function on an open set of a complex Banach space to a complex Banach space has a derivative, it possesses a Fréchet differential. (Received January 22, 1946.)

APPLIED MATHEMATICS

79. R. J. Duffin: *Nonlinear networks.* II.

A system of n nonlinear differential equations is studied and shown to have a unique asymptotic solution; that is, all solutions approach each other as the independent variable becomes infinite. The interest of these equations is that they describe the forced vibration of electrical networks. Consider an arbitrary linear network of inductors, resistors, and capacitors which has no undamped free modes of vibration. A given impressed force may give rise to more than one response but as time goes on there is a unique association between impressed force and response. This, of course, is well known. The main result of this note states that if the linear resistors of such a network are replaced by quasi-linear resistors then there still is this unique asymptotic association. A quasi-linear resistor is one in which the potential drop across it and the current through it are increasing functions of one another. No other sort of nonlinearity besides this type of nonlinear damping is considered. The proof is made to rest solely on well known properties of the Laplace transform and Hermitian forms. (Received January 19, 1946.)

80. Herbert Jehle: *Transformation of hydrodynamical equations of stellar dynamics.*

In abstract 51-9-170, the author pointed out a transformation of continuity equation and Bernoulli equation into a Schroedinger equation. The presence of $\bar{\sigma}$ (replacing $\hbar/m$ of wave mechanics) implies no modification of classical equations of motion, but a statement about residual velocities or "pressure function." Assume that the distribution (numbers and intensities) of excited ψ_{nlm} states (n goes up to about 10^6 for the author's choice of $\bar{\sigma}$) corresponds statistically to the distribution (in numbers and masses) of statistically independent elements of a system. It is known that if all stationary ψ_{nlm} states are filled up to a certain frequency limit with one element (particle) per state there will be an average of one element per phase space volume $(2\pi\bar{\sigma})^3$ for the inner regions. The above assumption is therefore equivalent to the assumption of an upper limit for the expectation value of density (of numbers) of statistically independent elements in six-dimensional phase space. This is a plausible assumption for systems close to statistical equilibrium; it means that too great densities in position space without large residual velocities cause aggregations of formerly independent masses into larger independent units. (Received January 21, 1946.)

81. R. S. Phillips: *rms error criterion in servo system design.*

A servomechanism is required to follow a signal from a knowledge of only the error in following. This error signal is usually a mixture of the true following error and some sort of random disturbance. The servo must make a compromise between following the original signal and not following the noise. This paper presents a method by which this compromise can be made. Assuming that the spectra of the signal and noise are known, one can then determine that servo system which minimizes the rms error in following. Actually the paper limits itself to determining the best values of control parameters when the type of control is given. It is assumed that the servo can be represented by a linear differential equation with constant coefficients and that the spectra can be approximated by rational fractions. The problem is then reduced to the evaluation of integrals of the type $I_n = (2\pi i)^{-1}\int_{-\infty}^{\infty} (g_n(x)/f_n(x) \cdot f_n(-x))dx$ where $f_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, $g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots + b_{n-1}$, and all the roots of f_n lie in the upper half-plane. The integrals are evaluated in terms of the coefficients of the polynomials f and g. The paper concludes with a table of integrals for $n \leqq 7$. (Received January 18, 1946.)

82. Harry Polachek: *Solution of the differential equations of motion of a projectile in a medium of quasi-Newtonian resistance.*

A solution is obtained in closed form for the differential equations of motion of a projectile in a medium of quasi-Newtonian resistance. The theory of differential variations as developed by Bliss, Moulton and Gronwall is applied. Use of this result may be made to solve a large class of ballistic problems, which are now being solved by indirect numerical methods. It is especially applicable to the case of projectiles traveling with velocities which are entirely outside the trans-sonic region. (Received January 14, 1946.)

83. Eric Reissner: *Stresses and small displacements in shallow spherical shells.*

A system of two simultaneous equations for two unknown functions F and w is derived for the bending and stretching of shallow segments of thin, elastic, spherical shells. In these equations F is Airy's stress function for the direct stress resultants and w the normal component of displacement. Writing $F_n = \cos n\theta f_n(r)$, $w_n = \cos n\theta g_n(r)$ the equations are such that f_n and g_n result as combinations of elementary functions and of Bessel functions of order n and argument proportional to $i^{1/2}$. The formulas obtained, particularly for the nonsymmetrical case, are appreciably simpler than the known results for spherical shells without the assumption of shallowness. (Received January 17, 1946.)

84. P. A. Samuelson: *A connection between the Bernoulli and Newton iterative processes.*

By the Bernoulli method, the root of a polynomial $f(X) = \sum_0^n a_j X^{n-j} = \prod_1^n (X - X_i) = 0$ is approximated by $Y_{t+1}/Y_t = (C_1X_1^{t+1} + \cdots + C_nX_n^{t+1})/(C_1X_1^t + \cdots + C_nX_n^t)$, where $\sum_0^n a_j Y_{t+n-j} = 0$, $Y_0 = b_0, \cdots, Y_{n-1} = b_{n-1}$, and the C's depend upon the X's and the b's. Weights w_k are sought such that $Z_{t+1}/Z_t = \sum_0^n w_k Y_{t+1-k}/\sum_0^n w_k Y_{t-k}$ is a good approximation to X_i, by virtue of the fact that coefficients of the *other* exponential terms are made to vanish. Clearly the w_k should be coefficients of $P(X)/(X - X_i)$ to give the exact root, X_i, in one step. If, instead, one uses the coefficients of $P(X)/(X - X_i^0)$, where X_i^0 is an approximate root, and sets $b_j = (X_i^0)^j$, then the newly calculated root $\overline{X}_i = Z_n/Z_{n-1} = X_i^0 - [f(X_i^0)/f'(X_i^0)]$, which is identical with the Newton approximation to a root. (Received February 1, 1946.)

85. P. A. Samuelson: *Computation of characteristic vectors.*

Wayland has shown (Quarterly of Applied Mathematics vol. 2 (1945) p. 277) that the method of Danielewsky and the method of elimination are the two most efficient known ways of computing the coefficients of the characteristic equation. This note shows: (1) the Danielewsky reduction can be interpreted as a special case of the method of elimination; (2) the characteristic vectors of a companion matrix with simple roots are given by a Vandermonde matrix and its easily derived inverse; (3) the characteristic vectors of any matrix can therefore be derived with about $2n^3$ multiplications all told, by applying the Danielewsky transformations to the vectors of the corresponding companion matrix. (Received February 1, 1946.)

86. P. A. Samuelson: *Generalization of the Laplace transform for difference equations.*

The Laplace transformation has standardized operational methods in the field of ordinary differential equations. Its efficacy hinges on the fundamental relation $L(s;\ Df)_D = sL(s;\ f)_D - f(0)$ where $L(s;\ f)_D = \int_0^\infty \exp(-st)f(t)dt$. The Laplace transform has been applied to difference equations, but it is a clumsy tool there by virtue of the fact that it does *not* satisfy a similar fundamental relation with respect to the shifting operator E. One can easily verify that the linear functional $L(s;\ f)_E = \sum_0^\infty f(i)s^{-i-1}$ does have the fundamental property $L(s;\ Ef)_E = sL(s;\ f)_E - f(0)$. This generalized transform can also be easily inverted by the calculus of residues and extended by suitably defined "convolution." Consequently, after a table of "generalized" transform pairs has been drawn up, the solution of ordinary difference equations can be derived by operational methods *exactly* like those of differential equations. The most important of these transform pairs is $y(t) = t(t-1), \cdots, (t-n+1)a^{t-n}$ and $\bar{y}(s) = (s-a)^{-n}(n-1)!$. $|s| > |a|$. (Received February 1, 1945.)

87. C. A. Truesdell: *On Sokolovsky's "momentless shells."*

V. V. Sokolovsky (Applied Mathematics and Mechanics n.s. vol. 1 (1937) pp. 291–306) has given expressions for the membrane stress resultant Fourier coefficients for surfaces of revolution whose meridians may be expressed in Cartesian coordinates in the forms: $f = kz^\mu$; $f = a \sin^c \phi$, $z = -ca\int \sin^c \phi d\phi$; $f = a \sec^c \phi$, $z = -ca\int \sec^c \phi \tan^2 \phi d\phi$. The first family has already been treated and generalized by the author. In the present note the author shows that a slight modification of his previous treatment of Nemenyi's stress functions enables us quickly to find solutions in terms of hypergeometric functions for the family of surfaces whose meridian is $f = a \sin^p \xi$, $z = -pb\int \sin^p \xi \tan^q \xi d\xi$, including Sokolovsky's second and third families of surfaces as special cases. Surfaces having meridians given by an error integral curve, $z = ap!\int_0^{kf} \exp(-t^p)dt$, are shown by the same means to have solutions in terms of Whittaker functions. (Received January 29, 1946.)

88. Alexander Weinstein: *On Stokes' stream function and Weber's discontinuous integral.*

It is shown that the stream function ψ corresponding to sources distributed with the density one over a circumference C is a many-valued function with the period $4\pi a$, where a denotes the radius of C. This fact, combined with the divergence theorem, yields a new proof for Weber's formula (J. Reine Angew. Math. vol. 75 (1873) p. 80) for the discontinuous integral $\int_0^\infty J_0(as)J_1(bs)ds$, which is equal to $1/b$ for $b>a$, and to 0 for $a>b$. (Received January 17, 1946.)

GEOMETRY

89. Reinhold Baer: *Polarities in finite projective planes.*

It is shown that every polarity in a finite projective plane possesses at least as many absolute points (points lying on their polars) as there are points on a line. This minimum may be attained. If there are more than the minimum number of absolute points, then the number of points on a line reduced by one is a square. More detailed information is available for regular polarities which have the property that any two lines which are not absolute, but carry absolute points, carry the same number of absolute points. These results are applied to prove a generalization of a theorem due to Topel which asserts that every geometry of Bolyai-Lobachevskiĭ is infinite. (January 18, 1946.)

90. T. C. Doyle: *Tensor theory of invariants for the projective differential geometry of a ruled surface.*

The differential equations of Wilczynski defining a ruled surface to within a projective transformation are expressed in the tensor form $y_{j..}=(U^2U_j{}^r+\rho\delta_j{}^r)y_r$, and from the tensor coefficients and arguments of these equations there is derived by formal tensor processes the same complete system of invariants and covariants of a ruled surface as first derived by Wilczynski, using integrational methods, in his *Projective differential geometry of curves and surfaces*, Leipzig, Teubner, 1906. One arrives at a systematic procedure for the transition from canonical to general forms and in this way the invariant equations of many of the covariant loci heretofore known only in their canonical forms are displayed. (Received December 17, 1945.)

91. H. W. Eves: *Arc chains and arc necklaces.* Preliminary report.

An arc chain is a sequence of circular arcs (called links), of arbitrary central angles and radii, trailed end to end. If the end points of the chain coincide one has an arc necklace. This paper deals with the elementary geometry of plane and spherical arc chains and necklaces. In addition to a number of new theorems, several well known results of elementary geometry are generalized to hold for arc chains and necklaces. Of particular interest are those arc chains and necklaces in which all the links lie on the circumferences of a two-parameter family of circles, for example, on the circumferences of a family of concurrent circles. Some attention is paid to arc necklaces whose vertices are concyclic. (Received January 27, 1946.)

92. D. P. Ling: *Geodesics on surfaces of revolution.*

The author investigates the number and the distribution of double points of the geodesics on the members of a broad class of surfaces of revolution. A "zoning" of these surfaces is established in a manner dictated by this distribution. It is shown that each surface falls into one or another of three subclasses according as each geodesic has infinitely many double points, a finite number of double points bounded from above for the whole set of geodesics, or a finite number which by a proper choice of the geodesic can be made arbitrarily large. Analytic means of distinguishing between these subclasses is set up, and a particular class of surfaces is given to serve as illustration and counter example. (Received December 3, 1945.)

LOGIC AND FOUNDATIONS

93. G. D. Birkhoff and Garrett Birkhoff: *Distributive postulates for systems like Boolean algebras.*

By slightly strengthening Newman's postulates for direct sums of Boolean algebras and Boolean rings, a simpler proof of sufficiency is obtained. A related set of postulates for distributive lattices is given, together with a discussion of alternative postulate systems for Boolean algebras and distributive lattices. (Received January 23, 1946.)

94. Archie Blake: *A Boolean derivation of the Moore-Osgood theorem.*

The process of proving a mathematical theorem is represented in symbolic logic by the transformation of logical expressions. This fact is illustrated in the case of the Moore-Osgood theorem, the central features of the derivation of which are shown to be representable in the *Prädikatenkalkul*. (Received January 12, 1946.)

95. Ira Rosenbaum: *Hegel's observations on the differential and integral calculus and its foundations.*

Attention is invited to an extended discussion of the differential and integral calculus and its foundations which appears in Hegel's *Science of logic*. The technical content, character, and interest of Hegel's discussion is indicated by citing authors, texts, methods, and problems with which Hegel dealt. The evidence relating to Hegel's knowledge of mathematics is presented and a picture of the development of Hegel's views is traced; relevant portions of the first and later editions of the *Logic* are compared. Hegel's relation to his contemporaries in mathematics is pointed out. The relevant literature is reviewed critically and after indicating the prevalent neglect of Hegel's discussion, it is concluded that such examination of Hegel's relation to the calculus as does exist is (1) dated, (2) incomplete and/or inadequate, (3) generally independent of earlier and contemporary work in the same field, and (4) limited in scope and point of view. An instance of the unsatisfactory state of the literature on this subject is considered. Hegel's observations on the calculus are examined, placed in their proper historical context, and his views compared with those of his predecessors, contemporaries, and successors. Evaluation from the standpoint of the modern logico-mathematical foundations of analysis is undertaken. (Received February 1, 1946.)

STATISTICS AND PROBABILITY

96. G. W. Brown and J. W. Tukey: *Some distributions of sample means.*

It is shown that certain monomials in normally distributed quantities have stable distributions with index 2^{-k}. This provides, for $k>1$, simple examples where the mean of a sample has a distribution equivalent to that of a fixed, arbitrarily large multiple of a single observation. These examples include distributions symmetrical about zero, and positive distributions. Using these examples, it is shown that any distribution with a very long tail (of average order greater than or equal to $x^{-3/2}$) has the distributions of its sample means grow flatter and flatter as the sample size increases. Thus the sample mean provides *less* information than a single value. Stronger results are proved for still longer tails. (Received January 14, 1946.)

TOPOLOGY

97. R. H. Bing: *Generalization of a theorem of Janiszewski.*

Suppose that H and K are plane sets neither of which cuts the point A from the point B, that the boundary of H is compact, that the junction of H and K is equal to

$H \cdot K$ and that H is the sum of a collection of mutually exclusive sets no one of which contains either a limit point of the sum of the others or two components of $H \cdot K$. Then $H+K$ does not cut A from B. (Received January 14, 1946.)

98. R. H. Bing: *Sets cutting the plane.*

Suppose that in the plane, K is a set with a connected complement and a bounded boundary, that W is either a countable set of points or a bounded closed set and that no component of W cuts the point A from the point B in K. Then W does not cut A from B in K. (Received January 19, 1946.)

99. C. H. Dowker: *Lebesgue dimension of a normal space.*

Let R be a normal space and let dim R be the Lebesgue dimension of R defined as follows: dim $R \leqq n$ means that every finite covering of R by open sets has a finite refinement of order not greater than $n+1$. It is shown that (1) dim $R \leqq n$ if and only if every star-finite covering of R by open sets has a star-finite refinement of order not greater than $n+1$, (2) dim $R \leqq n$ if and only if every locally finite covering of R by open sets has a locally finite refinement of order not greater than $n+1$, and (3) dim $R \leqq n$ if and only if, for each closed set A of R, each continuous mapping f of A into the n-sphere S^n can be extended to a continuous mapping g of R into S^n. (Received January 19, 1946.)

100. Erik Hemmingsen: *Dimension theory for normal Hausdorff spaces.* Preliminary report.

In normal Hausdorff spaces it is shown that the two following definitions of dimension are equivalent: (I) dim $S \leqq n$, if to each finite covering of S by open sets there is an open refinement in which at most $n+1$ sets intersect at a time; (II) dim $S \leqq n$, if to each mapping f of S into the $n+1$ simplex, σ^{n+1}, there is a mapping g of S into the boundary of σ^{n+1}, written s^n, with $f/f^{-1}(s^n) = g/f^{-1}(s^n)$. The theorem dim $M \times N \leqq$ dim $M+$dim N is proved for compact Hausdorff spaces. If $A = \bar{A} \subset S$, S normal, dim $A \leqq n$, $f(S) \subset \sigma^{n+1}$, f continuous, and if σ_0^{n+1} is an $n+1$ simplex contained in σ^{n+1} then it is shown that there is a mapping g of S in σ^{n+1} with $g(A) \subset (\sigma^{n+1} - \text{Int } \sigma_0^{n+1})$ and $g/B = f/B$ where $B = f^{-1}(\sigma^{n+1} - \text{Int } \sigma_0^{n+1})$; the sum theorem for countably many closed sets is a corollary. (Received January 14, 1946.)

101. F. B. Jones: *A characterization of a plane semi-locally-connected continuum.*

Let space be a simple closed surface S. If G is a collection of mutually exclusive simple domains, G is said to contain a sequence of *folded* simple domains provided that there exist sets $D_1, D_2, D_3, \cdots, D_i, \cdots$ $(i = 1, 2, 3, \cdots)$ such that (1) for each i, D_i is an element of G, (2) for each i, D_i contains a spanning arc-segment T_i separating D_i into two components C_i and E_i and (3) if ϵ is a positive number and T is the sequential limiting set of $T_1, T_2, T_3, \cdots$, there exist a point A not belonging to T and a number n such that if $i > n$, both C_i and E_i contain points at a distance from A less than ϵ. The principle result of the paper is as follows: In order that a set M be a cyclic semi-locally-connected continuum it is necessary and sufficient that the components of $S - M$ be a collection of mutually exclusive simple domains containing no sequence of folded simple domains. (Received January 29, 1946.)

102. G. W. Whitehead: *On families of continuous vector fields over spheres.*

Let $f(n)$ be the maximum number of everywhere independent continuous fields of tangent vectors that can exist on the n-sphere S^n. It is well known that $f(2n)=0$, $f(2n+1)\geqq 1$, $f(4n+3)\geqq 3$, and $f(8n+7)\geqq 7$. It has been proved independently by B. Eckmann (Comment. Math. Helv. vol. 15 (1942) pp. 1–26) and the author (Ann. of Math. vol. 43 (1942) pp. 132–146) that $f(4n+1)=1$. In this paper it is shown that $f(8n+3)=3$. It follows from this and results of N. E. Steenrod (Ann. of Math. vol. 45 (1944) pp. 294–311) that if $m>k$ and $k=2n$, $4n+1$, or $8n+3$, with $n>0$, then S^m is not a k-sphere bundle over any complex B. (Received December 10, 1945.)

NEW PUBLICATIONS

A collection of papers in memory of Sir William Rowan Hamilton. (Scripta Mathematica Studies, no. 2.) New York, Scripta Mathematica, 1945. 82 pp. $1.00.

Sequential analysis of statistical data: applications. Prepared by Statistical Research Group, Columbia University, for the Applied Mathematics Panel, National Defense Research Committee, Office of Scientific Research and Development. New York, Columbia University Press, 1945. 315 pp. $6.25.

A NOTE ON AXIOMATIC CHARACTERIZATION OF FIELDS

E. ARTIN AND G. WHAPLES

Since publication of our paper, *Axiomatic characterization of fields by the product formula for valuations*,[1] we have found that the fields of class field theory can be characterized by somewhat weaker axioms; we can drop the assumption, in Axiom 1, that $|\alpha|_{\mathfrak{p}}=1$ for all but a finite number of $\mathfrak{p}$, replacing it by the assumption that the product of all valuations converges absolutely to the limit 1 for all α.

Our original proof can be adapted to the new axiom with a few modifications, which we shall describe here. In §2, we keep Axiom 1 for reference and introduce:

AXIOM 1*. *There is a set $\mathfrak{M}$ of prime divisors $\mathfrak{p}$ and a fixed set of valuations $|\ \ |_{\mathfrak{p}}$, one for each $\mathfrak{p}\in\mathfrak{M}$, such that, for every $\alpha\neq 0$ of k, the product $\prod_{\mathfrak{p}}|\alpha|_{\mathfrak{p}}$ converges absolutely to the limit* 1. (*That is, the series $\sum_{\mathfrak{p}}\log|\alpha|_{\mathfrak{p}}$ converges absolutely to* 0.)

We must then omit the statement that there are only a finite number of archimedean primes, since this does not follow immediately from 1*; instead of it, we use the fact that $\sum_{\mathfrak{p}_\infty}\rho(\mathfrak{p}_\infty)$ and $\sum_{\mathfrak{p}_\infty}\lambda(\mathfrak{p}_\infty)$ converge absolutely. These quantities are defined on p. 480; the convergence follows from the fact that the product over all $\mathfrak{p}_\infty$ of $|1+1|_{\mathfrak{p}_\infty}$ must converge absolutely. Also, we must temporarily broaden the definition of "parallelotope" so as to permit a parallelotope to be defined by any valuation vector $\mathfrak{a}$ for which $\prod_{\mathfrak{p}}|\mathfrak{a}|_{\mathfrak{p}}$ converges absolutely (rather than restricting $\mathfrak{a}$ to be an idèle). In the statement of Axiom 2 we must replace "Axiom 1" by "Axiom 1*," Theorem 2, however, is left unchanged, together with Lemmas 4, 5, and 6, which are needed only to prove it; this theorem shows that the fields of class field theory really satisfy Axiom 1, so that at the end of the whole proof we shall find that Axiom 1 is a consequence of Axioms 1* and 2.

In §3, k is assumed to be any field for which Axioms 1* and 2 hold. Lemma 8 holds under assumption of Axiom 1*, for our slightly more general parallelotopes; in its proof we have only to note, in case of archimedean primes, that the product $\prod_{\mathfrak{p}_\infty}4^{\rho(\mathfrak{p}_\infty)}$ converges absolutely. In Lemma 9, property 2 must be replaced by:

2*. $|\alpha|_{\mathfrak{p}_\infty}\leqq B_{\mathfrak{p}_\infty}|y|_{\mathfrak{p}_\infty}$, *with a set of constants $B_{\mathfrak{p}_\infty}$ for which $\prod_{\mathfrak{p}_\infty}B_{\mathfrak{p}_\infty}$ converges absolutely.*

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[1] Bull. Amer. Math. Soc. vol. 51 (1945) pp. 469–492.

To prove existence of these constants, let, at each $\mathfrak{p}_\infty$, $M_{\mathfrak{p}_\infty}$ be the maximum of $|\alpha_i|_{\mathfrak{p}_\infty}$ for $i=1 \cdots l$; then $\prod_{\mathfrak{p}_\infty} M_{\mathfrak{p}_\infty}$ converges to a finite limit. Take $B_{\mathfrak{p}_\infty}=M_{\mathfrak{p}_\infty} l^{\lambda(\mathfrak{p}_\infty)}$; since $\sum_{\mathfrak{p}_\infty}\lambda(\mathfrak{p}_\infty)$ was absolutely convergent, our conclusion follows.

Lemma 10 holds as stated, although the set of $\mathfrak{p}_\infty$ is not now known to be finite. But as soon as we have proved that n is finite, it follows from Theorem 2 that our original Axiom 1 holds, so no further changes are necessary. (The theorems about parallelotopes in §4 hold only for parallelotopes defined by ideal elements.)

It is easy to construct an example of a field which satisfies Axiom 1* but does not satisfy Axiom 1 (nor, of course, Axiom 2). Let $k=R(x, z)$ be the set of all rational functions of x and z over the rational field. Let $k_0=R(x)$, consider k as the set $k_0(z)$ of all rational functions of z with k_0 as constant field, and denote by $\mathfrak{M}_0$ the set of all divisors which are trivial on k_0. We construct $\mathfrak{M}_0$, and define the set of normed valuations, exactly as in the proof of Lemma 6 of our original paper (pp. 477–479). Let $V_0(A)=\prod\|A\|_{p_0}$ where the product is taken over all $p_0\in\mathfrak{M}_0$; by Lemma 6, $V_0(A)=1$ for all $A\in k$.

Now let $x_1=x+z$, $x_2=x+2z, \cdots, x_i=x+iz, \cdots$; let $k_i=R(x_i)$ and for each i construct the sets $\mathfrak{M}_i$ of divisors p_i by repeating exactly the above process with k_0 replaced by k_i. The products $V_i(A)$ are all equal to 1. These sets $\mathfrak{M}_i$ are by no means disjoint; for example one can easily see that the irreducible polynomial z defines the same valuation in each $\mathfrak{M}_i$. However, it is unnecessary to explore these duplications in detail; we shall need only the facts that the valuations $p_{i\infty}$ and $p_{j\infty}$ are inequivalent for $i\neq j$, and are not equivalent to any of the finite p_ν. Namely, $x_i=x+iz=x_j+(i-j)z$ has value 1 at $p_{i\infty}$, but value $q>1$ at all $p_{j\infty}$ with $j\neq i$. And z has value $q>1$ at all $p_{i\infty}$, but has value $\leqq 1$ at all finite p_ν.

To construct our example, let ϵ_ν $(\nu=0, 1, 2, \cdots)$ be an infinite sequence of positive numbers whose sum is finite. Form the product

$$\prod \|A\|_{p_i}^{\epsilon_i}$$

over all $p_i\in\mathfrak{M}_i$, all i, and in this product unite each set of equivalent valuations into a single valuation. The exponents insure the convergence of the infinite products involved in this step. To show that the whole product is absolutely convergent for each $A\in k$, write A in the form $A=g(x, z)/h(x, z)$ where g and h are polynomials with rational coefficients. If N and M are the maximum degrees in x and z, respectively, for both numerator and denominator, then A can be written in the form $g_i(z)/h_i(z)$, where numerator and denominator are polynomials in z with coefficients in k_i, and are of degree at most $N+M$ in z. It follows from this that, for fixed A, the number of factors of $V_i(A)$ which are greater than 1 (or which are less than 1) is bounded, and their size is bounded; and this bound is uniform for all i. Hence the exponents ϵ_i insure absolute convergence. Finally, we note that our product, applied to z, contains an infinity of factors different from 1.

Taking the product over sets $\mathfrak{M}_0$ and $\mathfrak{M}_1$ only gives an example in which Axiom 1 is satisfied but Axiom 2 is not; for the field of constants with respect to $\mathfrak{M}_0 \cup \mathfrak{M}_1$ is the rational field $k_0 \cap k_1$.

To get an example of a field possessing a valuation satisfying Axiom 2, but such that this valuation cannot be contained in any set $\mathfrak{M}$ satisfying Axiom 1, take the p-adic closure of either the rational field or any of the fields $k_0'(z)$ of our original paper, with p any of the divisors of Lemma 6. Because of Theorem 3, such an $\mathfrak{M}$ cannot exist.

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SOME PROPERTIES OF PARTIAL SUMS OF THE HARMONIC SERIES

PAUL ERDÖS AND IVAN NIVEN

It has been proved that $\sum_{k=m}^{n} k^{-1}$ cannot be an integer[1] for any pair of positive integers m and n. More generally, $\sum_{k=0}^{n}(m+kd)^{-1}$ cannot be an integer.[2] We prove two theorems of a similar nature.

THEOREM 1. *There is only a finite number of integers n for which one or more of the elementary symmetric functions of* $1, 1/2, 1/3, \cdots, 1/n$ *is an integer.*

PROOF. Let $\sum_{k,n}$ denote the kth symmetric function of $1, 1/2, 1/3, \cdots, 1/n$. Since each term of $\sum_{k,n}$ is contained $k!$ times in the expansion of $(1+1/2+\cdots+1/n)^k$, we have, for $k>3\log n$ and n sufficiently large,

$$\sum_{k,n} < \frac{(1+1/2+\cdots+1/n)^k}{k!} < \frac{(1+\log n)^k}{k!} < 1,$$

where the second inequality arises from the usual comparison of $\log n$ with the harmonic series, and the third inequality is implied by the hypothesis $k>3\log n$.

Henceforth we take $k<3\log n$. By a theorem of A. E. Ingham[3] there is a prime between x and $x+x^{5/8}$. This implies that there is a prime p between $1+n/(k+1)$ and n/k for $k<3\log n$ and n sufficiently large. Hence $\sum_{k,n}$ contains the term

$$\frac{1}{p}\cdot\frac{1}{2p}\cdots\frac{1}{kp}=\frac{1}{k!p^k}.$$

Now $(k!, p)=1$ since $k<n/(k+1)$, and hence no other term in $\sum_{k,n}$ has a denominator divisible by p^k. So if $\sum_{k,n}=a/b$, we know that $p^k \mid b$ and $p\nmid a$, which proves the theorem.

By a similar but more complicated argument we can prove the same

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[1] Cf. Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925, chap. 8, p. 159, problem 250.

[2] Cf. T. Nagell, *Eine Eigenschaft gewissen Summen*, Skrifter Oslo, no. 13 (1923) pp. 10–15.

[3] *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) p. 256. This result is actually stronger than necessary for our use here. The classical estimates will suffice.

result for the elementary symmetric functions of $1/m$, $1/(m+1)$, $\cdots$, $1/n$, and of $1/m$, $1/(m+d)$, $1/(m+2d)$, $\cdots$, $1/(m+nd)$.

It should be noted that $\sum_{2,3}$ is an integer; we know of no other integral case. Theorem 1 can be proved without the use of the prime number theorem, and this proof could be used to determine the bound on n, above which the result of the theorem holds. For smaller values of n, $\sum_{k,n}$ could be checked, but the proof is complicated and the limits would be large.

THEOREM 2. *No two partial sums of the harmonic series can be equal; that is, it is not possible that*

$$\begin{aligned} 1/m + 1/(m+1) + \cdots + 1/n \\ = 1/x + 1/(x+1) + \cdots + 1/y. \end{aligned} \tag{1}$$

PROOF. We assume that $n<x$. Clearly if (1) has a solution, then any prime divisor of one of the denominators must divide another. Hence by Bertrand's postulate we can be certain that $y<2x-1$, since otherwise a prime $p>n$ would be one of the denominators on the right side of (1).

LEMMA. *Any solution of* (1) *must satisfy* $y<x+x^{1/2}-1$.

To prove this we use a theorem of Sylvester and Schur[4] which states that if $n>k$, then in the set $n, n+1, \cdots, n+k-1$ there is an integer containing a prime divisor greater than k. In our case $x>y-x+1$, so that there is a prime $p>y-x+1$ which divides one and only one (say ap) of the integers $x, x+1, x+2, \cdots, y$. Also p must divide one (say bp) of the set $m, m+1, m+2, \cdots, n$, and certainly not more than one, since $n-m<y-x$. Then $1/ap$ and $1/bp$ are the only terms in equation (1) whose denominators are divisible by p, and since

$$1/bp - 1/ap = (a-b)/abp,$$

we conclude that p must divide $a-b$, whence $a-b\geqq p$ and $a\geqq p+1$. This implies that

$$y \geqq ap \geqq p^2 + p > (y-x+1)^2 + y - x + 1$$

or

$$x - 1 > (y-x+1)^2,$$

which proves the lemma.

Next we obtain estimates for the expressions in (1). First we note that

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[4] Cf. Paul Erdös, J. London Math. Soc. vol. 9 (1934) p. 282.

$$\log\frac{2k+1}{2k-1}=\log\left(1+\frac{1}{2k}\right)-\log\left(1-\frac{1}{2k}\right)$$
$$=\frac{1}{k}+\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}}.$$

Solving for $1/k$, and summing the result for $k=m, m+1, \cdots, n$, we obtain

$$\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{n}$$
$$=\log\frac{2n+1}{2m-1}-\sum_{k=m}^{n}\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}}, \tag{2}$$

and similarly

$$\frac{1}{x}+\frac{1}{x+1}+\cdots+\frac{1}{y}$$
$$=\log\frac{2y+1}{2x-1}-\sum_{k=x}^{y}\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}}. \tag{3}$$

Now (1) and our assumption that $n<x$ imply that for any $j\geqq 1$,

$$\sum_{k}\frac{2}{(2j+1)(2k)^{2j+1}}$$

is greater when summed over $k=m, m+1, \cdots, n$ than over $k=x, x+1, \cdots, y$ and so, comparing the right sides of (2) and (3), we see that

$$(2n+1)/(2m-1)>(2y+1)/(2x-1).$$

Thus, ignoring the sum on the right side of (3), we may write

$$\log\frac{(2n+1)(2x-1)}{(2m-1)(2y+1)}<\sum_{k=m}^{n}\sum_{j=1}^{\infty}\frac{2}{(2j+1)(2k)^{2j+1}}. \tag{4}$$

The infinite sum on the right can be replaced by $4/3$ times the first term, since each term is more than 4 times the next. The numerator of the fraction on the left exceeds the denominator by at least 2, since both are odd, and hence the left side exceeds

$$\log\left(1+\frac{2}{(2m-1)(2y+1)}\right)>\frac{1}{(2m-1)(2y+1)}.$$

Thus we have

$$(5)\qquad \frac{1}{(2m-1)(2y+1)} < \sum_{k=m}^{n} \frac{2\cdot 4/3}{3(2k)^3} < \frac{1}{9m^2} \sum_{k=m}^{n} \frac{1}{k} = \frac{1}{9m^2} \sum_{k=x}^{y} \frac{1}{k}.$$

But the last sum has fewer than $x^{1/2}$ terms (by the lemma) and each term is not greater than $1/x$. And since $(2m-1)(2y+1) < 4my$, inequality (5) implies that

$$\frac{1}{4my} < \frac{1}{9m^2}\cdot\frac{x^{1/2}}{x}$$

or

$$(6)\qquad 9mx^{1/2} < 4y.$$

But also $1/m \leqq 1/m + \cdots + 1/n < x^{1/2}\cdot(1/x) = 1/x^{1/2}$, so that $x^{1/2} < m$, which together with (6) implies that $9x < 4y$, which contradicts the lemma. This completes the proof of Theorem 2.

In conclusion, we observe that $1/2+1/3+1/4 \equiv 1/12 \pmod 1$. Whether the sums in equation (1) are congruent (mod 1) for infinitely many values m, n, x, y is an unsolved problem.

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PRIME IDEALS AND INTEGRAL DEPENDENCE

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Let $\mathfrak{R}$ and $\mathfrak{S}$ be commutative rings such that $\mathfrak{S}$ contains, and has the same identity element as, $\mathfrak{R}$. If $\mathfrak{p}$ and $\mathfrak{P}$ are prime ideals in $\mathfrak{R}$ and $\mathfrak{S}$ respectively such that $\mathfrak{P}\cap\mathfrak{R}=\mathfrak{p}$ then we shall say that $\mathfrak{P}$ lies over, or contracts to, $\mathfrak{p}$. If over every prime ideal in $\mathfrak{R}$ there lies a prime ideal in $\mathfrak{S}$, we shall say that the "lying-over" theorem holds for the pair of rings $\mathfrak{R}$ and $\mathfrak{S}$.

Suppose now that $\mathfrak{q}$ and $\mathfrak{p}$ are prime ideals in $\mathfrak{R}$ such that $\mathfrak{q}\subset\mathfrak{p}$. If for every prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ there exists a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$ and containing $\mathfrak{Q}$, then the "going-up" theorem will be said to hold for $\mathfrak{R}$ and $\mathfrak{S}$. Similarly, if for every prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$ there exists a prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ and contained in $\mathfrak{P}$, then the "going-down" theorem will be said to hold.

Below we are concerned with the case where $\mathfrak{S}$ is integrally dependent on $\mathfrak{R}$. In this case we shall prove the "lying-over" and "going-up" theorems (§1). With certain additional conditions on $\mathfrak{R}$ and $\mathfrak{S}$, also the "going-down" theorem is proved (§2). Counterexamples are given to show that none of these conditions can be omitted (§3).

All of the results of this paper (except Theorem 7) have been proved by Krull[1] when the rings are free from zero-divisors. The present proofs are essentially simpler than Krull's and at the same time do not require that the rings be integral domains.

1. The "lying-over" and "going-up" theorems. Let $\mathfrak{R}$ and $\mathfrak{S}$ be commutative rings with $\mathfrak{R}$ contained in $\mathfrak{S}$ and with a common identity element, and let $\mathfrak{S}$ be integral over $\mathfrak{R}$. We examine first the question of whether the "lying-over" theorem holds for the rings $\mathfrak{R}$ and $\mathfrak{S}$. We remark that if a maximal ideal in $\mathfrak{S}$ necessarily contracts to a maximal ideal in $\mathfrak{R}$, and if $\mathfrak{R}$ has a single maximal ideal $\mathfrak{p}$, then for the prime ideal $\mathfrak{p}$ it is certainly true that there exists a prime ideal in $\mathfrak{S}$ lying over $\mathfrak{p}$; in fact, every maximal ideal of $\mathfrak{S}$ will lie over $\mathfrak{p}$. This remark is the motivation behind the following theorem.

THEOREM 1. *Let $\mathfrak{S}$ be integral over $\mathfrak{R}$, and let the prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lie over the prime ideal $\mathfrak{p}$ in $\mathfrak{R}$, that is, $\mathfrak{P}\cap\mathfrak{R}=\mathfrak{p}$. Then $\mathfrak{p}$ is maximal if and only if $\mathfrak{P}$ is maximal.*

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[1] *Zum Dimensionsbegriff der Idealtheorie*, Math. Zeit. vol. 42 (1937) pp. 745–766. Especially relevant are Theorems 1–6 and the considerations on pp. 756–757.

Proof. Consider the residue-class rings $\mathfrak{S}^* = \mathfrak{S}/\mathfrak{P}$ and $\mathfrak{R}^* = \mathfrak{R}/\mathfrak{p}$. Since $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$ the ring $\mathfrak{S}^*$ may be considered to contain $\mathfrak{R}^*$; $\mathfrak{R}^*$ and $\mathfrak{S}^*$ are integral domains, and $\mathfrak{S}^*$ remains integral over $\mathfrak{R}^*$. Theorem 1 then reduces to the following well known statement:

If $\mathfrak{R}$ and $\mathfrak{S}$ are integral domains and $\mathfrak{S}$ is integral over $\mathfrak{R}$ then $\mathfrak{R}$ is a field if and only if $\mathfrak{S}$ is a field.

To prove this statement we note that if $\mathfrak{R}$ is a field it follows trivially that $\mathfrak{S}$ is a field. Conversely, let $\mathfrak{S}$ be a field and let $a(\neq 0)$ be an element of $\mathfrak{R}$. We wish to prove $1/a \in \mathfrak{R}$. Since $1/a \in \mathfrak{S}$, it is integral over $\mathfrak{R}$ and hence we have an equation of integral dependence:

$$(1/a)^n + c_1(1/a)^{n-1} + \cdots + c_n = 0, \qquad c_i \in \mathfrak{R}.$$

Multiplying this equation by a^{n-1} we obtain

$$1/a = -(c_1 + c_2 a + \cdots + c_n a^{n-1}) \in \mathfrak{R}.$$

This proves that $\mathfrak{R}$ is a field, and completes the proof of Theorem 1.

Theorem 2. *Let $\mathfrak{S}$ be integral over $\mathfrak{R}$. Then for every prime ideal $\mathfrak{p}$ in $\mathfrak{R}$ there exists a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$.*

Proof. If $\mathfrak{R}$ contains only one maximal ideal, and this is $\mathfrak{p}$, then certainly the theorem holds for $\mathfrak{p}$; namely, as noted above, any maximal ideal of $\mathfrak{S}$ (such ideals certainly exist[2]) lies over $\mathfrak{p}$. If $\mathfrak{R}$ and $\mathfrak{S}$ are integral domains then the theorem can be reduced to this trivial case by the device of forming quotient rings. In fact, we form the quotient rings $\mathfrak{R}' = \mathfrak{R}_{\mathfrak{p}}$ (= the set of elements in the quotient field of $\mathfrak{R}$ of the form a/b, $a, b \in \mathfrak{R}$, $b \notin \mathfrak{p}$) and $\mathfrak{S}' = \mathfrak{S}_{\mathfrak{p}}$ (= the set of elements in the quotient field of $\mathfrak{S}$ of the form α/b, $\alpha \in \mathfrak{S}$, $b \in \mathfrak{R}$, $b \notin \mathfrak{p}$). The ring $\mathfrak{S}'$ is integral over $\mathfrak{R}'$; moreover $\mathfrak{R}'$ has a single maximal ideal, namely $\mathfrak{p}' = \mathfrak{R}' \cdot \mathfrak{p}$, and $\mathfrak{p}' \cap \mathfrak{R} = \mathfrak{p}$. If $\mathfrak{P}'$ is a maximal ideal of $\mathfrak{S}'$, then by our initial remark, $\mathfrak{P}'$ lies over $\mathfrak{p}'$. We assert that $\mathfrak{P} = \mathfrak{P}' \cap \mathfrak{S}$ lies over $\mathfrak{p}$. In fact, $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{P}' \cap \mathfrak{S} \cap \mathfrak{R} = \mathfrak{P}' \cap \mathfrak{R} = \mathfrak{P}' \cap \mathfrak{R}' \cap \mathfrak{R} = \mathfrak{p}' \cap \mathfrak{R} = \mathfrak{p}$. This completes the proof if $\mathfrak{R}$ and $\mathfrak{S}$ are integral domains.

If $\mathfrak{R}$ and $\mathfrak{S}$ are not integral domains we cannot form the required quotient rings. Nevertheless, the above argument can be adapted to commutative rings in general. Namely, consider the set W of ideals in $\mathfrak{S}$ which contract to ideals contained in $\mathfrak{p}$; W is not empty, since it certainly contains the zero ideal. Let $\mathfrak{P}$ be a maximal element[2] of W. We assert that $\mathfrak{P}$ is prime and lies over $\mathfrak{p}$.

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[2] We make use here of the following statement, which is an immediate consequence of Zorn's Lemma: *Let W be a nonvoid set of ideals in some ring; assume that if a subset W' of W has the property that of any two of its ideals one contains the other, then the union of all the ideals of W' is a member of W. Then W has a maximal element.*

For let α and β be elements of $\mathfrak{S}$ not in $\mathfrak{P}$, $\alpha\beta \in \mathfrak{P}$. Then $(\mathfrak{P}, \alpha)$ and $(\mathfrak{P}, \beta)$ contain $\mathfrak{P}$ properly, so that $(\mathfrak{P}, \alpha) \cap \mathfrak{R} \not\subseteq \mathfrak{p}$, $(\mathfrak{P}, \beta) \cap \mathfrak{R} \not\subseteq \mathfrak{p}$. Hence there exist elements a and b in $\mathfrak{R}$ but not in $\mathfrak{p}$ such that $a \equiv \sigma\alpha(\mathfrak{P})$, $b \equiv \tau\beta(\mathfrak{P})$, $\sigma, \tau \in \mathfrak{S}$. Then $ab \equiv \sigma\tau\alpha\beta(\mathfrak{P})$, $ab \in \mathfrak{P} \cap \mathfrak{R} \subseteq \mathfrak{p}$; this is a contradiction, and thus $\mathfrak{P}$ is prime. If $\mathfrak{P} \cap \mathfrak{R} \subset \mathfrak{p}$ properly, let $d \in \mathfrak{p}$, $d \notin \mathfrak{P}$. Then $(\mathfrak{P}, d) \supset \mathfrak{P}$ properly, whence $(\mathfrak{P}, d) \cap \mathfrak{R} \not\subseteq \mathfrak{p}$, so that there exists an element $c \in \mathfrak{R}$, $c \notin \mathfrak{p}$, such that $c \equiv \sigma d(\mathfrak{P})$, $\sigma \in \mathfrak{S}$. Since $\mathfrak{S}$ is integral over $\mathfrak{R}$, σ satisfies an equation

$$\sigma^n + a_1\sigma^{n-1} + \cdots + a_n = 0, \qquad a_i \in \mathfrak{R}.$$

From this follows

$$(\sigma d)^n + da_1(\sigma d)^{n-1} + \cdots - d^n a_n = 0.$$

Since $c \equiv \sigma d(\mathfrak{P})$, we have

$$c^n + da_1c^{n-1} + \cdots + d^n a_n \equiv 0 \ (\mathfrak{P}).$$

Since the left side is in $\mathfrak{R}$, this congruence holds mod $\mathfrak{p}$, and since $d \in \mathfrak{p}$, we have $c^n \in \mathfrak{p}$, $c \in \mathfrak{p}$, a contradiction. This completes the proof.

THEOREM 3. *Let $\mathfrak{S}$ be integral over $\mathfrak{R}$. Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{R}$ containing the ideal $\mathfrak{a}$. If $\mathfrak{A}$ is an ideal in $\mathfrak{S}$ such that $\mathfrak{A} \cap \mathfrak{R} = \mathfrak{a}$ then there exists a prime ideal in $\mathfrak{S}$ containing $\mathfrak{A}$ and lying over $\mathfrak{p}$.*

PROOF. This theorem follows directly from Theorem 2 if we take the residue class rings $\mathfrak{S}/\mathfrak{A}$ and $\mathfrak{R}/\mathfrak{a}$.

Conversely, Theorem 2 follows from Theorem 3 by placing $\mathfrak{A} = (0)$. Theorem 3 can, in fact, be proved directly along the same lines as Theorem 2. Thus the two theorems are equivalent; but in Theorem 2 the content takes the form of the "lying-over" theorem, in Theorem 3 it takes the form of the "going-up" theorem.

Theorem 1 implies that if two distinct prime ideals of $\mathfrak{S}$ lie over the same maximal ideal of $\mathfrak{R}$, then neither of these two prime ideals can contain the other. This side of the theorem can be strengthened through the following.

THEOREM 4. *Let $\mathfrak{S}$ be integral over $\mathfrak{R}$, and let the prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lie over $\mathfrak{p}$ in $\mathfrak{R}$. Then no ideal in $\mathfrak{S}$ properly containing $\mathfrak{P}$ can contract to $\mathfrak{p}$ in $\mathfrak{R}$.*

PROOF. If we take the residue-class rings $\mathfrak{S}/\mathfrak{P}$ and $\mathfrak{R}/\mathfrak{p}$, the theorem reduces to the following:

If the integral domain $\mathfrak{S}$ is integral over the ring $\mathfrak{R}$ then any nonzero ideal of $\mathfrak{S}$ contracts to a nonzero ideal of $\mathfrak{R}$.

Suppose, then, that $\alpha \in \mathfrak{S}$, $\alpha \neq 0$; if

$$\alpha^n + c_1\alpha^{n-1} + \cdots + c_n = 0, \qquad c_i \in \mathfrak{R},$$

is an equation of integral dependence of least possible degree for α then $c_n \neq 0$. For $c_n = 0$ would yield

$$\alpha^{n-1} + c_1\alpha^{n-2} + \cdots + c_{n-1} = 0,$$

that is, an equation of integral dependence for α of degree less than n. Since $c_n \in (\alpha) \cap \mathfrak{R}$, $(\alpha) \cap \mathfrak{R} \neq (0)$. This completes the proof.

2. **The "going-down" theorem.** Unlike the "going-up" theorem, the "going-down" theorem requires assumptions on the 0-divisors of $\mathfrak{R}$ and $\mathfrak{S}$. Even in the case of integral domains, however, the "going-down" theorem will not hold without further assumption on $\mathfrak{R}$: the assumption made below is that $\mathfrak{R}$ is integrally closed in its quotient field.

THEOREM 5. *Let $\mathfrak{R}$ be an integral domain integrally closed in its quotient field, $\mathfrak{S}$ a ring integral over $\mathfrak{R}$, with none of its zero-divisors in $\mathfrak{R}$. Then the "going-down" theorem holds for $\mathfrak{R}$ and $\mathfrak{S}$; that is, if $\mathfrak{q}$ and $\mathfrak{p}$ are prime ideals in $\mathfrak{R}$ with $\mathfrak{q} \subset \mathfrak{p}$, then for every prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$ there exists a prime ideal $\mathfrak{Q}$, contained in $\mathfrak{P}$, and lying over $\mathfrak{q}$.*

Before proving this theorem we prove two lemmas.[3]

LEMMA 1. *Let $\mathfrak{S}$ be integral over $\mathfrak{R}$ and let $\mathfrak{q}$ be an ideal in $\mathfrak{R}$. Then the set of elements in $\mathfrak{S}$ satisfying an equation of the form*

$$(1) \qquad \alpha^m + c_1\alpha^{m-1} + \cdots + c_m = 0, \qquad c_i \in \mathfrak{q},$$

is the radical of $\mathfrak{S} \cdot \mathfrak{q}$.

PROOF. If α satisfies the above equation then $\alpha^m \in \mathfrak{S} \cdot \mathfrak{q}$, so that α is in the radical of $\mathfrak{S} \cdot \mathfrak{q}$. Conversely if α is in the radical, then $\alpha^k \in \mathfrak{S} \cdot \mathfrak{q}$ for some k. Thus it is sufficient to prove that every element α in $\mathfrak{S} \cdot \mathfrak{q}$ satisfies an equation of the given form. For elements of the form σq, where $\sigma \in \mathfrak{S}$, $q \in \mathfrak{q}$, this is immediate. For let σ satisfy the equation

$$\sigma^m + d_1\sigma^{n-1} + \cdots + d_m = 0, \qquad d_i \in \mathfrak{R};$$

then

$$(\sigma q)^m + d_1 q(\sigma q)^{m-1} + \cdots + d_m q^m = 0.$$

Since every element of $\mathfrak{S} \cdot \mathfrak{q}$ is the sum of a finite number of terms of the form σq it remains to show that the sum of two elements of $\mathfrak{S}$

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[3] These lemmas are somewhat stronger than is actually necessary for the theorem.

satisfying an equation of the form of (1) also satisfies such an equation.

Let then

$$\alpha^m + c_1\alpha^{m-1} + \cdots + c_m = 0, \qquad c_i \in \mathfrak{q},$$

$$\beta^n + d_1\beta^{n-1} + \cdots + d_n = 0, \qquad d_i \in \mathfrak{q}.$$

It follows from these equations that every power α^r, $r \geqq m$, can be written in the form

$$\alpha^r = e_1\alpha^{m-1} + \cdots + e_m, \qquad e_i \in \mathfrak{q},$$

and every power β^s, $s \geqq n$, in the form

$$\beta^s = f_1\beta^{n-1} + \cdots + f_n, \qquad f_i \in \mathfrak{q}.$$

Let the products $\alpha^i\beta^j$, $0 \leqq i \leqq m-1$, $0 \leqq j \leqq n-1$, arranged in some order, be designated by τ_k, $k=1, \cdots, mn$. Then every product $\alpha^r\beta^s$ with $r+s \geqq m+n-1$ can be written as a linear combination of the τ's with coefficients in $\mathfrak{q}$. Hence if $l=m+n-1$ then

$$(\alpha + \beta)^l\tau_i = \sum_j q_{ij}\tau_j, \qquad q_{ij} \in \mathfrak{q}, i = 1, \cdots, mn.$$

From these equations we obtain

$$\tau_i \cdot \det ((\alpha + \beta)^l\delta_{ij} - q_{ij}) = 0, \qquad i = 1, \cdots, mn.$$

Since one of the τ_i is 1, the determinant is zero, and this is an equation for $\alpha+\beta$ of the form of (1).

Lemma 2. *Let $\mathfrak{R}$ be integrally closed in its total quotient ring*[4] *$\mathfrak{K}$. If $f(x)$ and $g(x)$ are monic polynomials in $\mathfrak{K}[x]$ and $h(x)=f(x)g(x)$ is in $\mathfrak{R}[x]$ then $f(x)$ and $g(x)$ are in $\mathfrak{R}[x]$.*

Proof. Consider the residue-class ring $\mathfrak{K}[x]/f(x)$. Since $f(x)$ is monic, this ring contains a ring isomorphic to $\mathfrak{K}$ and a root θ of $f(x)$. In this way, it is possible to form an extension ring of $\mathfrak{K}$ in which $f(x)$ and $g(x)$ factor into linear factors, say

$$f(x) = \prod_i (x - \alpha_i), \qquad g(x) = \prod_j (x - \beta_j).$$

Since $h(\alpha_i)=0$ and $h(\beta_j)=0$, the α_i and β_j are integral over $\mathfrak{R}$. Hence also the elementary symmetric functions of the α_i and β_j, which are the coefficients of $f(x)$ and $g(x)$, are integral over $\mathfrak{R}$. Since $\mathfrak{R}$ is integrally closed, these coefficients are in $\mathfrak{R}$.

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[4] The total quotient ring of $\mathfrak{R}$ consists of all quotients a/b, where a and b are in $\mathfrak{R}$ and b is not a zero-divisor.

Proof of Theorem 5. Let $\mathfrak{L}$ be the total quotient ring of $\mathfrak{S}$; since no element of $\mathfrak{R}$ is a 0-divisor in $\mathfrak{L}$, $\mathfrak{L}$ contains the quotient field $\mathfrak{K}$ of $\mathfrak{R}$.

Let D be the multiplicatively closed system in $\mathfrak{S}$ consisting of elements of the form $d\delta$, where $d\in\mathfrak{R}$, $d\notin\mathfrak{q}$, $\delta\in\mathfrak{S}$, $\delta\notin\mathfrak{P}$. We consider the set W of ideals in $\mathfrak{S}$ which contain $\mathfrak{S}\cdot\mathfrak{q}$ and do not meet D. We first show that W is not empty; specifically it contains $\mathfrak{S}\cdot\mathfrak{q}$. For suppose $d\delta\in\mathfrak{S}\cdot\mathfrak{q}$. By Lemma 1 there exists an equation $h(x)=0$ of integral dependence for $d\delta$ all of whose coefficients except the leading one are in $\mathfrak{q}$. Let $f(x)=0$ be an equation of least degree which is satisfied by $d\delta$ over $\mathfrak{K}$. Since the leading coefficient of $f(x)$ is not a 0-divisor, we may assume that $f(x)$ is monic. Then $h(x)=f(x)g(x)$, where $g(x)\in\mathfrak{K}[x]$. By Lemma 2, the coefficients of $f(x)$ and $g(x)$ are in $\mathfrak{R}$. Since all the coefficients of $h(x)$ except the first are in $\mathfrak{q}$, it follows by the familiar argument of Gauss' lemma that the same is true for $f(x)$ and $g(x)$.

Let $f(x)=x^n+c_1x^{n-1}+\cdots+c_n$. Clearly $x^n+(c_1/d)x^{n-1}+\cdots+(c_n/d^n)=0$ is the monic equation of least degree satisfied by δ over $\mathfrak{K}$. Just as above for $f(x)$, we have $b_i=c_i/d^i\in\mathfrak{R}$. Since $b_id^i=c_i\in\mathfrak{q}$ and $d^i\notin\mathfrak{q}$ we have $b_i\in\mathfrak{q}$. Hence

$$\delta^n = -b_1\delta^{n-1} - \cdots - b_n \in \mathfrak{S}\cdot\mathfrak{q} \subseteq \mathfrak{P},$$

whence $\delta\in\mathfrak{P}$, a contradiction.

Let, now, $\mathfrak{Q}$ be maximal[2] in W. We have $\mathfrak{Q}\cap\mathfrak{R}\supseteq\mathfrak{S}\cdot\mathfrak{q}\cap\mathfrak{R}\supseteq\mathfrak{q}$, but $\mathfrak{Q}\cap\mathfrak{R}$ cannot contain $\mathfrak{q}$ properly since $\mathfrak{Q}$ does not meet D. Also $\mathfrak{Q}\subset\mathfrak{P}$ since $\mathfrak{Q}$ does not meet D. It remains to prove that $\mathfrak{Q}$ is prime. Suppose, then, that $\gamma\delta\in\mathfrak{Q}$ but $\gamma\notin\mathfrak{Q}$ and $\delta\notin\mathfrak{Q}$. Since $(\mathfrak{Q}, \gamma)$ and $(\mathfrak{Q}, \delta)$ contain $\mathfrak{Q}$ properly, each of them must intersect D, whence also their product intersects D, since D is multiplicatively closed. This is a contradiction since this product is in $\mathfrak{Q}$. This completes the proof.

3. **Counterexamples.** In order better to see to what extent the hypotheses of Theorem 5 can be weakened, we formulate the three hypotheses as follows:

(A) The ring $\mathfrak{R}$ is integrally closed in its total quotient ring.

(B) A non-zero divisor of $\mathfrak{R}$ remains a non-zero divisor in $\mathfrak{S}$.

(C) $\mathfrak{R}$ is an integral domain.

We show by counterexamples that none of these assumptions can be dropped from Theorem 5.

(A) Let $f(x, y)=y^2-x^2-x^3\in K[x, y]$, where K is a groundfield of characteristic 0. Since the curve $f(x, y)=0$ has a singularity at the origin of the xy-plane, the ring $K[\xi, \eta]=K[x, y]/(f(x, y))$ is not integrally closed in its quotient field; in fact, $\tau=\eta/\xi$ satisfies the equation $\tau^2-1-\xi=0$, and hence is integral over $K[\xi, \eta]$. But $\eta/\xi \notin K[\xi, \eta]$, since $\eta/\xi \in K[\xi, \eta]$ leads to $y \in (x, y^2-x^2-x^3)=(x, y^2)$, which is impossible. Consider now the cylinder erected on the curve $f(x, y)=0$, and let $\mathfrak{R}=K[\xi, \eta, \zeta]=K[x, y, z]/(f(x, y))$ be its ring of nonhomogeneous coordinates. On this cylinder consider an irreducible curve which passes through the point (0, 0, 1) and which lies on only one branch of the cylinder in the neighborhood of this point, say the curve whose prime ideal is $\mathfrak{q}=(\zeta\xi+\eta, \zeta^2-1-\xi)$. Let now $\mathfrak{S}=K[\xi, \tau, \zeta]$, where $\tau=\eta/\xi$. The surface $t^2-1-x=0$, whose general point is (ξ, τ, ζ), is a Cremona transform of the original cylinder; this transformation has the effect of separating the two branches of the cylinder. In this separation, the point (0, 0, 1) is split into two points, (0, −1, 1) and (0, 1, 1); that is, the prime ideals in $\mathfrak{S}$ lying over the prime ideal $\mathfrak{p}=(\xi, \eta, \zeta-1)$ are the prime ideals $\mathfrak{P}_1=(\xi, \tau+1, \zeta-1)$ and $\mathfrak{P}_2=(\xi, \tau-1, \zeta-1)$. Now the curve given by $\mathfrak{q}$ transforms into a curve passing through the point $\mathfrak{P}_1$ (0, −1, 1) but not through the point $\mathfrak{P}_2$ (0, 1, 1). In terms of ideals, one has the prime ideals $\mathfrak{q}$, $\mathfrak{p}$ in $\mathfrak{R}$, with $\mathfrak{q}\subset\mathfrak{p}$, and the prime ideal $\mathfrak{P}_2$ in $\mathfrak{S}$ which lies over $\mathfrak{p}$. But no prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ could be contained in $\mathfrak{P}_2$. For suppose $\mathfrak{Q}\subset\mathfrak{P}_2$ and $\mathfrak{Q}\cap\mathfrak{R}=\mathfrak{q}$. We have $\zeta\xi+\eta=(\zeta+\tau)\xi\in\mathfrak{Q}$. Now $\zeta+\tau\in\mathfrak{Q}$, together with $\zeta-1\in\mathfrak{P}_2$ and $\tau-1\in\mathfrak{P}_2$, would imply $1\in\mathfrak{P}_2$, which is impossible; $\xi\in\mathfrak{Q}$ would imply $\eta\in\mathfrak{Q}$ and $\zeta^2-1\in\mathfrak{Q}$, hence $\zeta-1\in\mathfrak{Q}$, whence $\mathfrak{Q}\cap\mathfrak{R}$ would contain $\mathfrak{p}$. Thus $\zeta+\tau\notin\mathfrak{Q}$ and $\xi\notin\mathfrak{Q}$. This is a contradiction, and completes the proof that the "going-down" theorem fails for $\mathfrak{R}$ and $\mathfrak{S}$. Geometrically, then, the reason for the failure lies in the fact that a non-integrally closed ring $\mathfrak{R}$ allows a variety on which two branches meet and on which consequently there may be subvarieties which lie locally on only one branch.[5]

(B) Let $\mathfrak{R}$ be the ring of integers and let $\mathfrak{S}=\mathfrak{R}[x]/(x^2-x, 2x)$, where x is an indeterminate. Since no integer is in $(x^2-x, 2x)$, $\mathfrak{S}$ may be considered to contain $\mathfrak{R}$, and if α is the residue of x, then $\mathfrak{S}=\mathfrak{R}[\alpha]$, $\alpha^2-\alpha=0$, $2\alpha=0$. If $\mathfrak{q}=(0)$, $\mathfrak{p}=(2)$, $\mathfrak{P}=(2, \alpha-1)$, then $\mathfrak{P}$ is maximal and lies over $\mathfrak{p}$. There is no $\mathfrak{Q}$ such that $\mathfrak{Q}\cap\mathfrak{R}=\mathfrak{q}$, $\mathfrak{Q}\subset\mathfrak{P}$. For if there were, we would have $2\alpha=0\in\mathfrak{Q}$, but $2\notin\mathfrak{Q}$, hence $\alpha\in\mathfrak{Q}\subset\mathfrak{P}$; this is impossible since $\alpha-1\in\mathfrak{P}$.

(C) Let $\mathfrak{A}=(x^2, xy, xz, yz-y, z^2-z)$ be an ideal in the polynomial ring $K[x, y, z]$. Since $\mathfrak{A}\cap K=(0)$, the ring $\mathfrak{S}=K[x, y, z]/\mathfrak{A}$ contains a field isomorphic to K, which we identify with K, and if ξ, η, ζ are the maps of x, y, z in $\mathfrak{S}$ then $\mathfrak{S}=K[\xi, \eta, \zeta]$. Let $\mathfrak{R}=K[\xi, \eta]$; $\mathfrak{R}$ is not an integral domain and $\mathfrak{S}$ is integral over $\mathfrak{R}$. In $\mathfrak{S}$ we have the decomposition $\mathfrak{S}\cdot(0)=\mathfrak{P}'\cap\mathfrak{Q}$ where $\mathfrak{Q}$ is the prime ideal $(\xi, \zeta-1)$, and $\mathfrak{P}'$ is the primary ideal (η, ζ), of length 2, belonging to the prime ideal $\mathfrak{P}=(\xi,\eta,\zeta)$. In $\mathfrak{R}$ we have the decomposition $\mathfrak{R}\cdot(0)=\mathfrak{p}'\cap\mathfrak{q}$, where $\mathfrak{q}=\mathfrak{Q}\cap\mathfrak{R}=(\xi)$, and $\mathfrak{p}'=\mathfrak{P}'\cap\mathfrak{R}=(\eta)$ is a primary ideal, of length 2, belonging to $\mathfrak{p}=\mathfrak{P}\cap\mathfrak{R}=(\xi, \eta)$. Since any prime ideal in $\mathfrak{S}$ contains $\mathfrak{S}\cdot(0)$, it must contain one of the prime ideals $\mathfrak{P}$, $\mathfrak{Q}$; hence $\mathfrak{P}$ can contain no prime ideal properly. Since, however, $\mathfrak{P}\cap\mathfrak{R}=\mathfrak{p}\supset\mathfrak{q}$ properly, the "going-down" theorem fails.

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[5] This remark was made to us orally by Professor Zariski. The geometrical reasoning permits us to construct a counterexample simpler than Krull's.

We now show that hypotheses (A) and (B) are satisfied. An element of $\mathfrak{S}$ is a 0-divisor if and only if it is contained in at least one of the ideals $\mathfrak{P}$, $\mathfrak{Q}$. If this 0-divisor is in $\mathfrak{R}$, then it is in at least one of the ideals $\mathfrak{p}$, $\mathfrak{q}$; but then it is a 0-divisor *of* $\mathfrak{R}$. Thus (B) holds.

It remains to prove that $\mathfrak{R}$ is integrally closed in its total quotient ring. Since $\xi^2=\xi\eta=0$, every element of $\mathfrak{R}$ can be written in the form $a+b\xi+c(\eta)\eta$, where a, $b\in K$, $c(\eta)\in K[\eta]$; moreover, a is unique. The element $a+b\xi+c(\eta)\eta$ is a 0-divisor if and only if $a=0$, since then and only then would it be in one of the ideals $\mathfrak{p}$, $\mathfrak{q}$. Thus the total quotient ring of $\mathfrak{R}$ consists of the quotients whose denominators are of the form $a+b\xi+c(\eta)\eta$, $a\neq 0$. If we multiply numerator and denominator by $a-b\xi$, we may suppose without loss of generality that $b=0$. Suppose now that $\alpha=[d(\eta)+e\xi]/f(\eta)$, where $e\in K$, $d(y)$, $f(y)\in K[y]$, $f(0)=a\neq 0$, is integral over $\mathfrak{R}$. Since $e\xi/f(\eta)=e\xi/a\in\mathfrak{R}$, we have that $\alpha-e\xi/a=d(\eta)/f(\eta)$ is integral over $\mathfrak{R}=K[\xi, \eta]$, hence also over $K[\eta]$ since $\xi^2=0$. Now $K[\eta]$ is a simple transcendental extension of K, hence is integrally closed. Hence $d(\eta)/f(\eta)\in K[\eta]$ and $\alpha\in\mathfrak{R}$. This completes the proof.

The geometry behind the above counterexample is as follows. The ideal $\mathfrak{S}\cdot(0)$ corresponds to the reducible variety (in the xyz-space) consisting of the line $x=z-1=0$ and the point $x=y=z=0$; the ideal $\mathfrak{R}\cdot(0)$ corresponds to the line $x=0$ in the xy-plane. Over the point (0, 0) lie the points (0, 0, 0) and (0, 0, 1). Since the point (0, 0, 0) is isolated for the variety corresponding to $\mathfrak{S}\cdot(0)$, but the point (0, 0) is not isolated for the variety corresponding to $\mathfrak{R}\cdot(0)$, it is clear that the "going-down" theorem must fail for $\mathfrak{R}$ and $\mathfrak{S}$. Now any polynomial $f(\xi, \eta)$ for which $f(0, 0)=0$ is a 0-divisor in $\mathfrak{S}$, since the cylinder $f(x, y)=0$ contains part of the variety of $\mathfrak{S}\cdot(0)$. If $f(\xi, \eta)$ is to be a 0-divisor also in $\mathfrak{R}$, then the projection of the variety of $\mathfrak{S}\cdot(0)$ must not be, from an algebro-geometric point of view, simply the line $x=0$, but should also have an imbedded point at (0, 0); that is, $\mathfrak{R}\cdot(0)$ should have the prime ideal corresponding to (0, 0) as an imbedded component. This is accomplished above by making the isolated component of $\mathfrak{S}\cdot(0)$ corresponding to (0, 0, 0) a proper primary ideal.

4. Some additional remarks. We saw above that if $\mathfrak{S}$ is integral over $\mathfrak{R}$ then the "lying-over" theorem holds for $\mathfrak{R}$ and $\mathfrak{S}$. An obvious necessary condition for the "lying-over" theorem to hold for a pair of rings $\mathfrak{R}$, $\mathfrak{S}$ is that $\mathfrak{S}\cdot\mathfrak{p}\cap\mathfrak{R}=\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ in $\mathfrak{R}$. This is known to be also a sufficient condition in the case that $\mathfrak{R}$ and $\mathfrak{S}$ are integral domains. We now show this to be true in general.

THEOREM 6. *Let $\mathfrak{R}$ and $\mathfrak{S}$ be two rings, with $\mathfrak{R}\subset\mathfrak{S}$, and let $\mathfrak{p}$ be a prime ideal in $\mathfrak{R}$. A necessary and sufficient condition that there exist a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$ is that $\mathfrak{S}\cdot\mathfrak{p}\cap\mathfrak{R}=\mathfrak{p}$.*

PROOF. Suppose $\mathfrak{S}\cdot\mathfrak{p}\cap\mathfrak{R}=\mathfrak{p}$. Let $\mathfrak{P}$ be a maximal element[2] in the set of ideals in $\mathfrak{S}$ which contract to $\mathfrak{p}$. That $\mathfrak{P}$ is prime follows just as in Theorem 2.

THEOREM 7. *Let the ring $\mathfrak{R}$ be contained in the field $\mathfrak{K}$, and let $\theta\in\mathfrak{K}$, $\theta\neq 0$. If $\mathfrak{p}$ is a prime ideal in $\mathfrak{R}$ then in either $\mathfrak{R}[\theta]$ or $\mathfrak{R}[\theta^{-1}]$ there lies a prime ideal over $\mathfrak{p}$.*[6]

PROOF. Suppose that no prime ideal in $\mathfrak{R}[\theta]$ lies over $\mathfrak{p}$. Then $\mathfrak{p}\cdot\mathfrak{R}[\theta]\cap\mathfrak{R}\supset\mathfrak{p}$ properly. Hence there exists an element $b\in\mathfrak{R}$, $b\notin\mathfrak{p}$ such that $b=p_0+p_1\theta+\cdots+p_m\theta^m$, $p_i\in\mathfrak{p}$. But then θ^{-1} satisfies an equation with leading coefficient not in $\mathfrak{p}$. Hence $\mathfrak{R}'[\theta^{-1}]$ is integral over $\mathfrak{R}'$, where $\mathfrak{R}'$ is the quotient ring of $\mathfrak{R}$ with respect to $\mathfrak{p}$, and there exists a prime ideal $\mathfrak{P}'$ in $\mathfrak{R}'[\theta^{-1}]$ which lies over the prime ideal $\mathfrak{R}'\cdot\mathfrak{p}$. Then $\mathfrak{P}=\mathfrak{P}'\cap\mathfrak{R}[\theta^{-1}]$ lies over $\mathfrak{p}$.

The following theorem is the basis on which Krull proves the "going-down" theorem in the case of integral domains. We give a somewhat more direct proof.

THEOREM 8. *Let $\mathfrak{R}$ be integrally closed in its quotient field $\mathfrak{K}$, and let $\mathfrak{S}$ be the integral closure of $\mathfrak{R}$ in a finite normal extension $\mathfrak{L}$ of $\mathfrak{K}$. Then any two prime ideals in $\mathfrak{S}$ which lie over the same ideal in $\mathfrak{R}$ are conjugate.*

PROOF. It is clear that $\mathfrak{S}$ is invariant under every automorphism of $\mathfrak{L}/\mathfrak{K}$; two ideals of $\mathfrak{S}$ are said to be conjugate if one is carried into the other by such an automorphism. Let $\mathfrak{P}_1$ be a prime ideal in $\mathfrak{S}$ and let $\mathfrak{P}_1,\cdots,\mathfrak{P}_m$ be a complete set of distinct conjugates of $\mathfrak{P}$ We show that if $\mathfrak{P}$ is a prime ideal in $\mathfrak{S}$ such that $\mathfrak{P}\cap\mathfrak{R}$ then $\mathfrak{P}$ is one of the ideals $\mathfrak{P}_1,\cdots,\mathfrak{P}_m$. For suppose $\mathfrak{P}\neq\mathfrak{P}_i$, $i=1,\cdots,m$. Then by Theorem 4, $\mathfrak{P}\not\subseteq\mathfrak{P}_i$, $i=1,\cdots,m$. Hence there exists an element $\alpha\in\mathfrak{P}$, $\alpha\notin\mathfrak{P}_i$, $i=1,\cdots,m$. But then none of the conjugates of α is in $\mathfrak{P}_i$, $i=1, \cdots, m$; hence neither is any power of their product. Some such power, however, is in $\mathfrak{R}$, hence in $\mathfrak{P}\cap\mathfrak{R}\subset\mathfrak{P}_1$. This is a contradiction and completes the proof.

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[6] This theorem was given by Professor Chevalley in a Princeton lecture.

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NOTE ON AN ASYMMETRIC DIOPHANTINE APPROXIMATION

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1. **Introduction.** In a recent paper B. Segre [1][1] introduced a new type of Diophantine approximation which he called *asymmetric*, since the intervals of approximation are divided into two partial intervals which are in an arbitrarily given ratio. His main result is the following theorem [1, p. 357]:

Theorem 1. *Every irrational θ has an infinity of rational approximations x/y such that*

$$\frac{-1}{y^2(1+4\tau)^{1/2}} < \frac{x}{y} - \theta < \frac{\tau}{y^2(1+4\tau)^{1/2}} \qquad (y>0), \tag{1}$$

where τ is any given non-negative real number.

This theorem is classic for $\tau=0$, cf. [2, p. 139], and for $\tau=1$ it reduces to the fundamental result due to Hurwitz [2, p. 163]. No other particular cases of the theorem seem to be known.

Segre's proof of (1) is geometrical. The purpose of this note is to show that when $\tau\geqq 1$ it is possible to give a very simple arithmetical proof. The method is a generalization of that used by Khintchine [3] for the special case when $\tau=1$.

2. **Proof of Theorem 1.** We suppose that θ is irrational and that $0<\theta<1$. For an arbitrary positive integer n form the Farey series[2] of order n, that is, the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. Let a/b and a'/b' be the two successive terms of this series which satisfy the inequalities $a/b<\theta<a'/b'$. We distinguish two cases.

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[1] Numbers in brackets refer to the references.

[2] See Hardy and Wright [2, p. 23].

Case 1. Suppose that $\tau>0$, and that $b/b'>(\xi+1)/2\tau$, or $b/b'<(\xi-1)/2\tau$, where $\xi=(1+4\tau)^{1/2}$. Then setting $\omega=b/b'$ we see that

$$\frac{1}{\xi}\left(\tau+\frac{1}{\omega^2}\right)-\frac{1}{\omega}=\frac{\tau}{\xi\omega^2}\left(\omega-\frac{\xi+1}{2\tau}\right)\left(\omega-\frac{\xi-1}{2\tau}\right)>0; \tag{2}$$

consequently, since $a'b-ab'=1$,

$$\frac{a'}{b'}-\frac{a}{b}=\frac{1}{b'^2\omega}<\frac{1}{b'^2\xi}\left(\tau+\frac{1}{\omega^2}\right)=\frac{\tau}{b'^2\xi}+\frac{1}{b^2\xi}, \tag{3}$$

which implies that

$$\frac{a}{b}+\frac{1}{b^2\xi}>\frac{a'}{b'}-\frac{\tau}{b'^2\xi}. \tag{4}$$

Hence θ must be interior to one or the other of the intervals

$$\left(\frac{a}{b},\ \frac{a}{b}+\frac{1}{b^2\xi}\right) \quad \text{or} \quad \left(\frac{a'}{b'}-\frac{\tau}{b'^2\xi},\ \frac{a'}{b'}\right). \tag{5}$$

Then, according as θ belongs to the first or to the second interval, we have, respectively, the inequalities

$$-1/b^2\xi<a/b-\theta<0, \quad \text{or} \quad 0<a'/b'-\theta<\tau/b'^2\xi.$$

Thus (1) is true, where for y we take either b or b'. The infinity of solutions is assured since, for irrational θ, b and b' increase as n increases.

If $\omega=b/b'=(\xi\pm1)/2\tau$, then both τ and ξ must be rational. Then the inequality sign in (4) is replaced by an equality sign, and the right and left end points of the intervals in (5) coincide. But θ, being irrational, cannot be equal to this common end point. Hence θ must be interior to one or the other of these intervals, and the proof proceeds as already explained.

Case 2. We now suppose that $(\xi-1)/2\tau<b/b'<(\xi+1)/2\tau$. We consider in turn the two sub-intervals

$$\left(\frac{a}{b},\ \frac{a+a'}{b+b'}\right) \quad \text{and} \quad \left(\frac{a+a'}{b+b'},\ \frac{a'}{b'}\right).$$

For the first sub-interval, write $\omega=b/(b+b')$. Then it is easy to see that $\omega<(\xi-1)/2\tau$, $\tau>0$, and that the inequality (2) is again true. Hence

$$\frac{a+a'}{b+b'}-\frac{a}{b}=\frac{1}{(b+b')^2\omega}<\frac{1}{b^2\xi}+\frac{\tau}{(b+b')^2\xi},$$

and, proceeding as in Case 1, we see that (1) is true where for y we take either b or $b+b'$.

The second sub-interval is handled in exactly the same way. We set $\omega=(b+b')/b'$, then $\omega>(\xi+1)/2\tau$, provided $\tau\geqq 1$. This is the first time we need this restriction on τ. The inequality (1) is again true where y is either b' or $b+b'$.

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A VARIANT OF A RECURSIVELY UNSOLVABLE PROBLEM

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By a string on a, b we mean a row of a's and b's such as *baabbbab*. It may involve only a, or b, or be null. If, for example, g_1, g_2, g_3 represent strings *bab*, *aa*, *b* respectively, string $g_2g_1g_1g_3g_2$ on g_1, g_2, g_3 will represent, in obvious fashion, the string *aababbabbaa* on a, b. By the *correspondence decision problem* we mean the problem of determining for an arbitrary finite set (g_1, g_1'), (g_2, g_2'), $\cdots$, (g_μ, g_μ') of pairs of corresponding non-null strings on a, b whether there is a solution in n, i_1, i_2, $\cdots$, i_n of equation

$$g_{i_1}g_{i_2}\cdots g_{i_n} = g'_{i_1}g'_{i_2}\cdots g'_{i_n}, \qquad n \geqq 1,\ i_j = 1, 2, \cdots, \mu. \tag{1}$$

That is, whether some non-null string on g_1, g_2, $\cdots$, g_μ, and corresponding string on g_1', g_2', $\cdots$, g_μ', represent identical strings on a, b.

In special cases, of course, the question posed by (1) may be answerable. Thus, if, with $\mu=3$, (g_1, g_1'), (g_2, g_2'), (g_3, g_3') are (bb, b), (ab, ba), (b, bb) respectively, $g_1g_2g_2g_3 = bbababb = g_1'g_2'g_2'g_3'$, and (1) has a solution. Again, if each g_i is of greater length than the corresponding g_i', or if each g_i starts with a different letter than the corresponding g_i', (1) has no solution. We proceed to prove, on the other hand, that in its full generality *the correspondence decision problem is recursively unsolvable*,[1] and hence, no doubt, unsolvable in the intuitive sense.

We start with the known recursive unsolvability of the decision problem for the class of normal systems on a, b.[2] A normal system S on

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[1] It suffices here to consider "recursively unsolvable" to mean unsolvable in the sense of Church [1]. Of course the general problem remains recursively unsolvable if we allow null g's and g''s. Numbers in brackets refer to the references cited at the end of the paper.

[2] See [4, §2] for an informal proof. As far as the printed literature is concerned, we must refer to [2] for a formal proof, though there then remains the actual verification, via Gödel representations, that the reduction effected is indeed recursive. This verification, at least for the reduction of S' to S''' [2, p. 51], is immediate if we use the following simpler method of reducing S' to a system S'' in canonical form than that there given by Church. The primitive symbols of our S'' are those of S' and one additional primitive symbol α. The basis of S'' in part consists of the two primitive assertions αI, αJ, and the operation αP, αQ *produce* $\alpha(PQ)$. It will follow that αP is asserted in S'' when and only when P is a combination without free variables. The remainder of the basis of S'' consists of the primitive assertion of S' as primitive assertion, and the thirty-eight operations of S' each modified as follows. For each operational variable P occurring in the operation, αP is introduced as additional premise.

a, b is given by a basis consisting of an initial non-null string A on a, b, and a finite set of operations $\alpha_i P$ *produces* $P\alpha_i'$, $i=1, 2, \cdots, \nu$, where the α's and α''s are given strings on a, b, while the operational variable P represents an arbitrary string on a, b, possibly null. The assertions of S consist of A and all non-null strings obtainable from A by repeated use of the ν operations. The known recursively unsolvable problem is then the problem of determining for arbitrary S, as given by a basis therefore, and arbitrary non-null string B on a, b, whether B is an assertion of S. This unsolvability is undisturbed if the α's and α''s are all non-null,[3] a condition which automatically excludes the possibility of null assertions, and will henceforth be assumed.

Referring to operation $\alpha_i P$ *produces* $P\alpha_i'$ by the subscript i, string B on a, b will be an assertion of S when and only when some finite sequence of operations $i_1, i_2, \cdots, i_n$ leads from A to B. Now operation $\alpha_i P$ *produces* $P\alpha_i'$ can be applied to string C to yield string D when and only when for some string P, possibly null, $C=\alpha_i P$, $P\alpha_i'=D$. Hence B is an assertion of S when and only when the following set of equations has a solution in $n, i_1, i_2, \cdots, i_n$, and the P's.

$$(2)\quad A=\alpha_{i_1}P_1,\ P_1\alpha_{i_1}'=\alpha_{i_2}P_2,\ \cdots,\ P_{n-1}\alpha_{i_{n-1}}'=\alpha_{i_n}P_n,\ P_n\alpha_{i_n}'=B.$$

Here n may be 0, (2) then becoming $A=B$. We proceed to show that (2) is equivalent to a single equation somewhat like (1) subject, however, to certain length conditions.

Given (2), we can eliminate the P's by forming $A\alpha_{i_1}'\alpha_{i_2}'\cdots\alpha_{i_n}'$ and successively substituting for the left members of (2) the right to obtain

$$(3)\qquad A\alpha_{i_1}'\alpha_{i_2}'\cdots\alpha_{i_n}'=\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}B.$$

Likewise, starting with $A\alpha_{i_1}'\alpha_{i_2}'\cdots\alpha_{i_{m-1}}'$, we obtain

$$(4)\qquad A\alpha_{i_1}'\cdots\alpha_{i_{m-1}}'=\alpha_{i_1}\cdots\alpha_{i_m}P_m,$$

whence,

$$(5)\quad \textit{length}\ (A\alpha_{i_1}'\cdots\alpha_{m-1}')\geqq\textit{length}\ (\alpha_{i_1}\cdots\alpha_{i_m}),\quad m=1, 2, \cdots, n,$$

the length of a string being the total number of occurrences of letters therein, here a's and b's. Conversely, let (3) be given, with (5) satisfied. With the length of $\alpha_{i_1}\cdots\alpha_{i_m}$ less than or equal to that of $A\alpha_{i_1}'\cdots\alpha_{i_{m-1}}'$, (3) shows that the former must be identical with an

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[3] It suffices to modify the production starting on page 214 of [3] in accordance with footnote 3 thereof to insure that the final normal system has no g or g' null.

"initial segment" of the latter. Hence, P_m can be determined so that (4) is satisfied, and for $m=1, 2, \cdots, n$. For $m=1$, (4) yields the first equation of (2). By substituting the right side of (4), with $m=j$, for the left, (4) for $m=j+1$ becomes $\alpha_{i_1}\cdots\alpha_{i_j}P_j\alpha_{i_j}=\alpha'_{i_1}\cdots\alpha_{i_j}\alpha_{i_{j+1}}P_{j+1}$, whence $P_j\alpha'_{i_j}=\alpha_{i_{j+1}}P_{j+1}$, $j=1, 2, \cdots, n-1$. Likewise, the last equation of (2) is obtained from (3) via (4) for $m=n$. Hence, (2) has a solution when and only when (3) has a solution subject to (5). That is, B is in normal system S when and only when (3) has a solution in $n, i_1, i_2, \cdots, i_n$ subject to (5). Comparing (3) with (1), we see that to reduce the decision problem for the class of normal systems on a, b to the correspondence decision problem, and thus have the unsolvability of the former lead to the unsolvability of the latter, we must on the one hand eliminate the length condition (5), on the other, the A and B of (3).

We achieve the first aim by reducing normal system S in three stages to a normal system in which (3) implies (5). If $C=x_1x_2\cdots x_n$, the x's a's or b's, let $\overline{C}=x_n\cdots x_2x_1$. For a letter with subscript, superscript, we shall only bar the letter. Now for the normal system S on a, b with initial string A and operations α_iP *produces* $P\alpha'_i$, $i=1, 2, \cdots, \nu$, form the system S', not normal, with initial string $\overline{A}$, and operations $P\bar{\alpha}_i$ *produces* $\bar{\alpha}'_iP$. Clearly, string B on a, b will be an assertion of S when and only when $\overline{B}$ is an assertion of S'. Next form S'' with initial string $\overline{A}h$, and operations $P\bar{\alpha}_ih$ *produces* $\bar{\alpha}'_iPh$. String $\overline{B}$ is then in S' when and only when $\overline{B}h$ is in S''. We finally form a normal system S''', though on the three letters a, b, h, whose assertions are the assertions of S'' and all cyclic permutations thereof, a cyclic permutation of string $x_1\cdots x_jx_{j+1}\cdots x_n$ here meaning any string $x_{j+1}\cdots x_nx_1\cdots x_j$. The initial string of S''' is again $\overline{A}h$. For its first ν operations we take $\bar{\alpha}_ihP$ *produces* $Ph\bar{\alpha}'_i$, premise and conclusion being a cyclic permutation of the premise and conclusion of the corresponding operation of S''. We finally add the operations aP *produces* Pa, bP *produces* Pb, hP *produces* Ph which serve to transform a string on a, b, h into any of its cyclic permutations. System S''' is therefore normal, and, by induction, is easily seen to have the stated property.[4] It follows that B is an assertion of normal system S when and only when $\overline{B}h$ is an assertion of normal system S'''.

Let the operations of S''' be resymbolized β_iP *produces* $P\beta'_i$, $i=1, 2, \cdots, \nu+3$. Though S''' is a normal system on three letters, the discussion of equations (2)–(5) is equally applicable to it. Hence B is an assertion of S when and only when the following equation (6) has a solution subject to (7).

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[4] Cf. [3], final reduction.

$$\bar{A}h\beta'_{i_1}\beta'_{i_2}\cdots\beta'_{i_n} = \beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}\bar{B}h. \tag{6}$$

$$\textit{length}\ (\bar{A}h\beta'_{i_1}\cdots\beta'_{i_{m-1}}) \geqq \textit{length}\ (\beta_{i_1}\cdots\beta_{i_m}),\ m = 1, 2, \cdots, n. \tag{7}$$

Suppose (6) had a solution with (7) not satisfied for a certain m. For that m, the length of $\beta_{i_1}\cdots\beta_{i_m}$ would exceed the length of $\bar{A}h\beta'_{i_1}\cdots\beta'_{i_{m-1}}$, and hence, in virtue of (6), we would have

$$\beta_{i_1}\cdots\beta_{i_m} = \bar{A}h\beta'_{i_1}\cdots\beta'_{i_{m-1}}Q \tag{8}$$

with non-null Q. Recall that (β_i, β'_i) is $(\bar{\alpha}_i h, h\bar{\alpha}'_i)$ for $i \leqq \nu$, (a, a), (b, b), (h, h) for the three remaining i's. With α's and α''s on a, b only, β_i and β'_i are then either both free from h, or have exactly one h apiece. Were the β_{i_m} of (8) a or b, the right side of (8) would have at least one more occurrence of h than the left, which is impossible. In any other case, β_{i_m} ends with h. Non-null Q therefore ends with h, and again the right side of (8) would have at least one more h than the left. Hence, every solution of (6) must satisfy (7). That is, B is an assertion of S when and only when (6) has a solution.

The elimination of $\bar{A}h$ and $\bar{B}h$ from (6) is more easily effected. Corresponding to the $\nu+3$ couples (β_i, β'_i), $i = 1, 2, \cdots, \nu+3$, and $\bar{A}h$, $\bar{B}h$, we introduce $\nu+5$ couples (γ_i, γ'_i) as follows. With x's and y's representing letters, in this case a, b, or h, if β_i and β'_i are $x_1x_2\cdots x_\kappa$ and $y_1y_2\cdots y_\lambda$ respectively, γ_i and γ'_i are to be $x_1kx_2k\cdots x_\kappa k$ and $ky_1ky_2\cdots ky_\lambda$ respectively. If $\bar{A}h$ and $\bar{B}h$ are $y_1y_2\cdots y_\lambda$ and $x_1x_2\cdots x_\kappa$ respectively, $\gamma_{\nu+4}$ and $\gamma'_{\nu+4}$ are to be kk and $kky_1ky_2\cdots ky_\lambda$, $\gamma_{\nu+5}$ and $\gamma'_{\nu+5}$, $x_1kx_2k\cdots x_\kappa kk$ and kk, respectively. It then follows that (6) has a solution in $n, i_1, i_2, \cdots, i_n$, $n \geqq 0$, $i_p = 1, 2, \cdots, \nu+3$, when and only when the equation

$$\gamma'_{j_1}\gamma'_{j_2}\cdots\gamma'_{j_m} = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_m} \tag{9}$$

has a solution in $m, j_1, j_2, \cdots, j_m$, $m \geqq 1$, $j_q = 1, 2, \cdots, \nu+5$. In fact, if $i_1, i_2, \cdots, i_n$ makes both sides of (6) equal to $z_1z_2\cdots z_l$, $(j_1, j_2, \cdots, j_m) = (\nu+4, i_1, i_2, \cdots, i_n, \nu+5)$ makes both sides of (9) equal to $kkz_1kz_2\cdots kz_lkk$. On the other hand, suppose (9) has a solution. Then since γ'_{j_1} and γ_{j_1} must start with the same letter, $j_1 = \nu+4$. For in every other case γ'_{j_1} starts with k, γ_{j_1} with a, b, or h. Similarly, $j_m = \nu+5$, (9) forcing γ'_{j_m} and γ_{j_m} to end with the same letter. If, now, the intermediate j's are all different from $\nu+4$ and $\nu+5$, they directly give a sequence of i's satisfying (6). Otherwise, let j_μ be the first j beyond j_1 that is $\nu+4$ or $\nu+5$. Were $j_\mu = \nu+4$, $\gamma'_{j_1}\gamma'_{j_2}\cdots\gamma'_{j_\mu}$ and $\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_\mu}$ would take the forms $kkx_1kx_2\cdots kx_pkkx_{p+1}k\cdots x_q$ and $kky_1ky_2k\cdots y_rkkk$ respectively, with x's and y's a, b, or h. But then the second occurrence of kk in the left side of (9) would be immediately followed by a, b, or h, in the right side, by k, contradicting (9). Hence $j_\mu=\nu+5$, and $\gamma'_{j_1}\gamma'_{j_2}\cdots\gamma'_{j_\mu}$ and $\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_\mu}$ consequently take the form $kkx_1kx_2\cdots kx_pkk$ and $kky_1ky_2\cdots ky_qkk$. But the left side of (9) through the second occurrence of kk must equal the right side of (9) through its second occurrence of kk. Hence $\gamma'_{j_1}\gamma'_{j_2}\cdots\gamma'_{j_\mu}=\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_\mu}$, and we have a solution of (9) of the type previously seen to lead to a solution of (6). It follows that B is an assertion of S when and only when (9) has a solution.

In the reduction thus effected we have introduced the new letters h and k. But now in (γ_i, γ'_i) replace the letters a, b, h, k by *bab*, *baab*, *baaab*, *baaaab* respectively,[5] and call the resulting pair of strings on a, b, (δ_i, δ'_i). Then (9) is seen to be equivalent to

$$\delta'_{j_1}\delta'_{j_2}\cdots\delta'_{j_m}=\delta_{j_1}\delta_{j_2}\cdots\delta_{j_m}, \tag{10}$$

immediately so in passing from (9) to (10), and conversely. For if, for example, δ'_{j_1} starts with *baab*, δ_{j_1} must also start with *baab*, and likewise for the next group of letters, and so on till (10) is seen to be a translation of (9).

Given normal system S on a, b with basis A, $\alpha_i P$ *produces* $P\alpha'_i$, $i=1, 2, \cdots, \nu$, and string B on a, b, the above gives an effective method for forming the pairs of strings on a, b, (δ_i, δ'_i), $i=1, 2, \cdots, \nu+5$, such that B is an assertion of S when and only when (10) has a solution. But (10), with left and right hand members interchanged, is a case of (1). We have therefore effectively reduced the decision problem for the class of normal systems on a, b to our correspondence decision problem. If, then, the former is unsolvable in the intuitive sense, so must be the latter. Actually, by introducing Gödel representations, we readily verify that the above effective reduction is indeed recursive, the recursive unsolvability of the former problem then leading to the recursive unsolvability of the correspondence decision problem.

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[5] Cf. [4], footnote 5.

TWO BRIEF FORMULATIONS OF BOOLEAN ALGEBRA

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This paper presents two new formulations of Boolean algebra which appear to have some direct interest on their own account, and which further take a place among the most economical versions, while also exhibiting more intuitive clarity than other versions of a similar degree of economy.[1] As is to some extent customary in short expositions of this kind, I am taking for granted without formal presentation: (1) that the system contains more than one element; (2) that it is closed with respect to such operations as appear explicitly in the axioms; (3) that I may employ an identity (or equality) relation without formal statement of its properties; (4) that in addition I may utilize recognized principles of logic (in fact only an ordinary or elementary logic). With a view to comparisons I shall say that what I am presenting are the "transformation axioms." The variables x, y, z, $\cdots$ will represent elements. No other symbolism is required for features of the system except an accent $'$ and juxtaposition for an undefined singulary and binary operation respectively. A symbol for the class of elements is not introduced, because "formation" rules (for example, closure) are left informal. But certain further symbols are used in the formalism (and regarded as taken over from ordinary logic) to enable us to make our statements about the elements and operations (functions) of the system, either as axioms, theorems, or steps in proofs; these further symbols are an identity or equality sign $=$ (the only relation sign employed), and signs serving as connectives between our statements, namely & "and," $\rightarrow$ "only if" or "if . . . then . . . ," and $\leftrightarrows$ "if and only if."[2] The last-named and the equality sign are the only ones appearing in axioms. Parentheses are used in customary ways. In the derivation of theorems indication is made of axioms and theorems used, except that Axioms II and III (associativity and commutativity) may sometimes be brought into play without explicit mention, and likewise certain frequently em-

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[1] Comparisons as to economy are intended to be limited to axiomatizations in mathematical (not metamathematical) language. The vague expression "intuitive clarity" here means that, in some familiar interpretation, the postulates adopted seem readily intelligible and plausible for a mind of limited mathematical experience.

[2] The "&" between statements is, of course, not to be confused with the "and" between elements, expressed by mere juxtaposition when we choose to interpret juxtaposition in the "and" rather than in the "or" manner (two ways in which it is feasible to interpret juxtaposition).

ployed theorems (for example, 1, 2, 4, 5) after they have been proved. For simplicity of argument, no definitions are introduced.

FORMULATION A

Axioms

I. $xy'=zz' \rightleftarrows xy=x$.
II. $(xy)z=x(yz)$.
III. $xy=yx$.

Theorems

1. $xx=x$. From I (since $xx'=xx'$).

2. $xx'=yy'$. From 1 by I.

3. $xy=x \;\&\; yz=y \rightarrow xz=x$.

PROOF. a. $xy=x \rightarrow (xy)z=xz$.
b. $yz=y \rightarrow (xy)z=xy$ by II.
c. $xy=x \;\&\; yz=y \rightarrow xz=xy=x$. From a, b.

4. $x''=x$.

PROOF. a. $x''x'=zz'$. From 2 by III.
b. $x''x=x''$. From a by I.
c. $x'''x'=x'''$. From b.
d. $x''''x''=x''''$. From b.
e. $x''''x=x''''$. From d, b by 3.
f. $x''''x'=zz'$. From e by I.
g. $x'x'''=x'$. From f by III, I.
h. $x'''=x'$. From c, g by III.
i. $xx'''=zz'$. From 2 by h.
j. $xx''=x$. From i by I.
k. $x''=x$. From b, j by III.

5. $x(yy')=yy'$.

PROOF. a. $x(yy')=x(xx')$. From 2.
b. $x(yy')=xx'$. From a by II, 1.
c. $x(yy')=yy'$. From b by 2.

6. $x(x'y)'=x$.

PROOF. a. $x(x'y)=zz'$. From 5 by II, III, 2.
b. $x(x'y)'=x$. Fom a by 4, I.

7. $x(xy)'=xy'$.

PROOF. a. $xy''(xy)'=zz'$. From 2 by 4.
b. $xy'(xy)'=x(xy)'$. From a by I (and II, III).

c. $xy'(xy)'=xy'$. From 6 by 4 (and III).
d. $x(xy)'=xy'$. From b, c.

8. $(x'y')'(x'y)'=x$.

PROOF. a. $x(x'y')'(x'y)'=x$. From 6.
b. $x'(x'y')'(x'y)'=x'y(x'y)'=zz'$. From 7, 4, and 2.
c. $x(x'y')'(x'y)'=(x'y')'(x'y)'$. From b by I (and III, II).
d. $(x'y')'(x'y)'=x$. From c, a.

As it is well known that the axioms above have true interpretations in Boolean algebra, proof that they are necessary will be omitted. To show that they suffice for Boolean algebra, I have now derived from them a well attested set of transformation axioms for this algebra, known as Huntington's Fourth Set [4].[3] The latter is worked out in terms of "not" and "or" operations. As I have left the present operations uninterpreted, I can at this point assign them the "not" and "or" characters to match Huntington's. Then his three transformation postulates appear here as Axioms II, III, and Theorem 8.

Incidentally, it would not be hard to show that each of Axioms I, II, III is independent of the two others.

FORMULATION B

Axioms

I. $xy'=zz' \leftrightarrows xy=x$.
II B. $(xy)z=(yz)x$.

Theorems

1. $xx=x$. From I (as before).

2′. $xy=yx$.

PROOF. $xy=(xy)(xy)=(y(xy))x=((xy)x)y=((yx)x)y=((xx)y)y=(yy)(xx)=yx$. The first and last steps are from 1, the others from II B.

3′. $(xy)z=x(yz)$.

PROOF. $(xy)z=(yz)x=x(yz)$. The first step from II B and the second from 2′.

Theorem 2′ is the previous Axiom III and Theorem 3′ the previous Axiom II. Thus in the presence of Axiom I the previous II and III could be replaced by Axiom II B. This is the only version known to the writer limited to two transformation axioms and employing the "not" and "and" or the "not" and "or" operations. No other choice of operations seems to reach the level of intuitive clarity envisaged in this paper.

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[3] Numbers in brackets refer to the references cited at the end of the paper.

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1. H. M. Sheffer, *A set of five independent postulates for Boolean algebras*, Trans. Amer. Math. Soc. vol. 14 (1913) pp. 481–488. This was the first version in which the transformation postulates numbered only three, then a very radical reduction (the other two were "formation" postulates on number of elements and closure). It also showed for the first time that the number of undefined concepts (operations) other than the class of elements could be reduced to one.

2. B. A. Bernstein, *Simplification of the set of four postulates for Boolean algebras in terms of rejection*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 783–787. In effect this reduces Sheffer's three transformation postulates to two, employing the same operation.

3. ———, *A set of four postulates for Boolean algebras in terms of the "implicative" operation*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 876–884. Another version in two transformation postulates.

4. E. V. Huntington, *New sets of postulates for the algebra of logic*, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 274–304, 557–558, 971. The fourth and fifth sets are limited to three transformation postulates.

5. S. Hoberman and J. C. C. McKinsey, *A set of postulates for Boolean algebra*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 588–592. Has only *one* transformation postulate, but this is metamathematical in character, and equivalent to an infinite bundle of "object-language" axioms of the kind considered in this paper.[4] See my Footnote 1.

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[4] The metamathematical character might be in doubt in the published version, but the senior author has in correspondence suggested what he regards as a more satisfactory wording, which seems without question metamathematical. See also review by Alonzo Church in J. Symbolic Logic vol. 2 (1937) pp. 172–173.

PROJECTIVITIES WITH FIXED POINTS ON EVERY LINE OF THE PLANE

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The system of fixed elements of a projectivity contains with any two points the line connecting them and with any two lines their intersection. It is, therefore, in its structure very much like a subplane of the plane under consideration; and thus one may expect the structure of the projectivity to be dominated by the structure of the system of its fixed elements, provided this system is not "too small." To substantiate this we propose to investigate in this note a class of projectivities which we term quasi-perspectivities. They are characterized by the property that every line carries a fixed point or, equivalently, that every point is on some fixed line.

Every perspectivity is a quasi-perspectivity, and a quasi-perspectivity is not a perspectivity if, and only if, the system of its fixed elements is a projective subplane. Involutions are quasi-perspectivities too, and if the Theorem of Pappus is valid in the plane under consideration, then every quasi-perspectivity is a perspectivity or an involution. But already in the projective plane over the field of real quaternions there exist quasi-perspectivities which are neither perspectivities nor involutions, and we give a complete survey of the quasi-perspectivities in Desarguesian projective planes.

Our results become particularly striking in the case of finite projective planes. If every line in such a plane carries $n+1$ points, then we may show that there do not exist projectivities possessing exactly n fixed points, that a projectivity is a quasi-perspectivity if, and only if, the number of its fixed points is at least $n+1$, and that it is a perspectivity if, and only if, the number of its fixed points is $n+1$ or $n+2$. If a quasi-perspectivity is not a perspectivity, then $n=i^2$ where $i+1$ is the number of fixed points on a fixed line.

The following notations will be used throughout.

We consider a projective plane Π in which the Theorem of Desargues may or may not hold.[1] If P and Q are two different points in Π, then $P+Q$ is the uniquely determined line passing through P and Q; if h and k are two different lines, then hk is the uniquely determined point in which they meet.

A projectivity ϕ is a 1:1 and exhaustive correspondence between the

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[1] For a definition of "projective plane" see, for example, Baer [1, p. 138]. Numbers in brackets refer to the Bibliography at the end of the paper.

points of Π and between the lines of Π such that the point P is on the line h if, and only if, the image point $P\phi$ is on the image line $h\phi$.

The point P is a fixed point of the projectivity ϕ if $P=P\phi$, and fixed lines are defined likewise. The system of all the fixed elements of ϕ (fixed points as well as fixed lines) shall be denoted by $\Phi(\phi)$.

THEOREM 1. *The following properties of a projectivity ϕ imply each other.*

(i) *P, $P\phi$ and $P\phi^2$ are collinear for every point P.*

(ii) *Every point lies on a fixed line.*

(iii) *Every line carries a fixed point.*

(iv) *h, $h\phi$ and $h\phi^2$ are copunctual for every line h.*

PROOF. If (i) is true and P a point, then we distinguish two cases.

Case 1. P is not a fixed point. Then $P\neq P\phi$ and $h=P+P\phi$ is a well determined line containing $P\phi^2$. But $h\phi=P\phi+P\phi^2=h$ shows that the fixed line h passes through the point P.

Case 2. P is a fixed point. There exists a point $Q\neq P$. If Q is a fixed point, then $P+Q$ is the desired fixed line passing through P. Thus we assume that Q is not a fixed point. Then it follows as under Case 1 that the line $q=Q+Q\phi$ is a fixed line. If P is on q, then q is the desired fixed line through P. Hence we assume that P is not on q. There exists a point $R\neq P$ which is not on q. If R is a fixed point, then $P+R$ is a fixed line through P. Thus we assume that R is not a fixed point. Then we show as before that $R+R\phi=r$ is a fixed line. But $r\neq q$, since R is not on q. Thus r and q meet in a fixed point S. This fixed point S is different from P, since P is not on q, but S is on q. Then $P+S$ is a fixed line through P. Hence we have shown in every case that (i) implies (ii).

Assume now the validity of (ii). If P is some point, then there exists a fixed line p, passing through P. Since P is on p, $P\phi^i$ is on $p\phi^i=p$ so that P, $P\phi$ and $P\phi^2$ are collinear as points on the same line p. Thus (i) is a consequence of (ii); and we have shown the equivalence of (i) and (ii).

By duality we have the equivalence of (iv) and (iii).

Assume now the validity of (ii) and consider a line h. If h is a fixed line, then consider a point P not on h. This point P lies by (ii) on a fixed line p which is different from h. Hence p and h are two different fixed lines meeting in a fixed point on h. Assume next that h is not a fixed line. Then h and $h\phi$ meet in a point K and this point K lies by (ii) on a fixed line k. Clearly $K=hk$, since h and k are different, the second line being fixed and the first one not. Hence $K\phi=(h\phi)(k\phi)=(h\phi)k=K$, showing that h carries the fixed point K. Thus we proved that (iii) is a consequence of (ii); and it follows by duality that (ii) is a consequence of (iii), completing the proof.

Every projectivity $\phi \neq 1$ meeting the requirements (i) to (iv) of Theorem 1 shall be termed a *quasi-perspectivity.* It is clear that every perspectivity meets requirement (ii), since a perspectivity possesses a line all of whose points are fixed points,[2] and since every line meets this line in a fixed point. But every involution meets requirement (i) because of $P = P\phi^2$, showing that perspectivities as well as involutions are quasi-perspectivities.

COROLLARY 1. *If ϕ is a quasi-perspectivity, and if the line h is not a fixed line, then h carries one and only one fixed point.*

We infer from Theorem 1, (iii), that h carries fixed points. But h would be a fixed line, if h carried two fixed points.

COROLLARY 2. (a) *The quasi-perspectivity ϕ possesses fixed lines carrying but one fixed point if, and only if, ϕ is an elation.*[3]

(b) *If the quasi-perspectivity ϕ is not an elation, then a necessary and sufficient condition for the line h to be a fixed line is the existence of at least two different fixed points on h.*

PROOF. If ϕ is an elation, the line h its axis and the point H on h its center, then a point is a fixed point if, and only if, it is on h, and a line is a fixed line if, and only if, it passes through H. Clearly the lines through H which are different from h are fixed lines carrying one and only one fixed point, namely H.

Suppose conversely that ϕ is a quasi-perspectivity and that the fixed line h of ϕ carries one and only one fixed point H. If k is a fixed line different from h, then k and h meet in a fixed point on h so that k passes through H as the only fixed point on h. If P is a point not on h, then either P is a fixed point and $P+H$ is a fixed line; or else P is not a fixed point and $P+P\phi$ is by Theorem 1, (i), a fixed line which necessarily passes through H. Thus $P+H$ is always a fixed line. Hence every line through H is a fixed line so that H is a center of ϕ. But projectivities possess an axis whenever they possess a center,[4] and the axis is a fixed line so that it must pass through the center H. Consequently ϕ is an elation, completing the proof of (a). (b) is an immediate consequence of (a) and Corollary 1.

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[2] Namely the axis of the perspectivity; see Veblen-Young [1, p. 72].

[3] An elation is a perspectivity whose center is on its axis; see Veblen-Young [1, p. 72].

[4] See, for example, Baer [1, Corollary 2.3, p. 140].

Preparatory to characterizing the perspectivities among the quasi-perspectivities we introduce the following concept. The set Γ of points and lines is *a closed configuration*[5] if it meets the following requirements:

If $P \neq Q$ are points in Γ, then $P+Q$ is in Γ. If $h \neq k$ are lines in Γ, then hk is in Γ.

We note that the set of fixed elements of a projectivity is always a closed configuration. Another example are the projective subplanes[6] of Π.

LEMMA 1.[7] *The closed configuration Γ which contains both points and lines is not a projective subplane if, and only if, there exist a point H and a line h with the following properties: every point in Γ with the possible exception of H is on h; and every line in Γ with the possible exception of h passes through H.*

PROOF. The sufficiency of this condition is obvious. Assume conversely that Γ is not a projective subplane. Then there exists a line w in Γ carrying at most two points in Γ.

Case 1. No point in Γ is on w. Then w is the only line in Γ, since the existence of a line $v \neq w$ in Γ would imply that the intersection of v and w would be on w and in Γ. There exists at least one point W in Γ. If there would exist a point $V \neq W$ in Γ, then $V+W \neq w$, since neither V nor W is on w; but the line $V+W$ would be in Γ which has been shown to be impossible. Thus Γ consists of the line w and the point W not on w.

Case 2. There is one and only one point W in Γ and on w. If x is a line in Γ, $x \neq w$, then x and w meet in a point which is both in Γ and on w. Hence x passes through W, showing that every line in Γ passes through W.

If $P \neq Q$ are points in Γ, then $P+Q$ is a line in Γ so that $P+Q$ passes through W. This implies the collinearity of all the points in Γ, proving again the validity of our condition.

Case 3. There exist exactly two points in Γ which are on w. Suppose W' and W'' are the two different points in Γ which are on w. If x is any line in Γ, $x \neq w$, then x and w meet in a point in Γ and on w. Hence x passes through one and only one of the two points W' and W''.

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[5] The term "configuration" refers usually only to finite systems, a restriction that we do *not* impose on this concept. M. Hall [1] terms "degenerate projective plane" what we would call a closed configuration which is not a subplane.

[6] These are sometimes referred to as "nets."

[7] This lemma has been stated without proof by M. Hall [1, p. 232].

Our condition is certainly satisfied by Γ if there exists at most one point in Γ which is not on w. Thus let us assume the existence of two different points Q', Q'' in Γ which are not on w. Then $Q'+Q''$ belongs to Γ, is not w, and passes therefore through one and only one of the points W' and W''. We assume without loss in generality that W', Q' and Q'' are collinear.

Suppose now that the point U in Γ is neither on w nor on $Q'+Q''$. Then $Q'+U$ is a line in Γ which cannot pass through W', since otherwise U would be on $Q'+W'=W'+W''$. Thus $Q'+U$ passes through W''; likewise we see that $Q''+U$ passes through W''. If the lines $Q'+U$ and $Q''+U$ were different, then they would meet in $U=W''$ which is impossible, and if the lines $Q'+U$ and $Q''+U$ were equal, then U would be on $Q'+Q''$ which we excluded too. It follows that the only point not on $Q'+Q''$, but in Γ, is W'', and w is readily seen to be the only line in Γ which does not pass through W''. Thus our condition is satisfied, completing the proof of Lemma 1.

THEOREM 2. *The quasi-perspectivity ϕ is a perspectivity if, and only if, the system Φ of fixed elements of ϕ is not a projective subplane.*

PROOF. If ϕ is a perspectivity, then there exists a point H and a line h such that the set of fixed points, with the possible exception of H, is just the set of points on h, and such that the set of fixed lines, with the possible exception of h, is just the pencil of lines through H. It is clear that Φ is not a projective subplane.

Assume conversely that Φ is not a projective subplane. There exist fixed points and fixed lines, since by Theorem 1 every point is on a fixed line and every line passes through a fixed point. We note furthermore that Φ is a closed configuration. Hence we may infer from Lemma 1 the existence of a fixed point H and of a fixed line h with the following properties:

If P is a fixed point, $P\neq H$, then P is on h; and if k is a fixed line, $k\neq h$, then k passes through H.

Suppose that Q is a point on h. There exists a line $q\neq h$ through Q which does not pass H. This line q carries a fixed point which is different from H and which therefore is on h. But q and h meet in Q, proving that Q is a fixed point. Thus every point on h is a fixed point; that every line through H is a fixed line is shown likewise. The point H is the center and the line h is the axis of ϕ, proving that ϕ is a perspectivity.

A closed configuration Γ in the projective plane Π is termed *maximal* if it is different from Π, and if Π is the only closed configuration which contains Γ as a proper subset. One deduces without difficulty from Lemma 1 that the maximal closed configurations which are not projective subplanes consist of a line h, a point H, the points on h and the lines through H.

THEOREM 3. *The systems of fixed elements of quasi-perspectivities are maximal closed configurations.*

PROOF. Suppose that Φ is the system of fixed elements of the quasi-perspectivity ϕ. Then Φ is a closed configuration. If Φ is a proper subset of the closed configuration Γ, then we may assume without loss in generality that Γ contains a point P which is not in Φ. Every line through P carries a fixed point of ϕ which is necessarily different from P. Hence every line through P is in Γ. If h is a fixed line not passing through P, then every point on h is the intersection of h and of a line through P which both belong to Γ. Hence every point on h is in Γ. Every point is on a fixed line (Theorem 1), and thus every point is in Γ with the possible exception of the points on the uniquely determined fixed line through P (Corollary 1). Now it is easily seen that every line and, therefore, every point is in Γ, completing the proof.

It is very easy to construct examples of maximal closed configurations which are not the systems of fixed elements of a suitable projectivity and examples of maximal closed configurations which are the systems of fixed elements of a suitable projectivity, but not of a quasi-perspectivity.

If Γ is a maximal closed configuration, then we may form *the group* $A=A(\Gamma)$ of all the projectivities which leave invariant every element in Γ. It is possible that $A=1$. The following facts are immediate consequences of Theorem 1 and the fact that the systems of fixed elements are closed configurations.

COROLLARY 3. *Suppose that* Γ *is a maximal closed configuration in the projective plane* Π, *that every point in* Π *is on a line in* Γ, *and that every line in* Π *passes through a point in* Γ.

(a) *If* $\phi\neq 1$ *is in* $A(\Gamma)$, *then* ϕ *is a quasi-perspectivity whose system of fixed elements is* Γ.

(b) *If* ϕ *and* γ *are in* $A(\Gamma)$, *and if there exists an element not in* Γ *on which* ϕ *and* γ *have the same effect, then* $\phi=\gamma$.

If the point P is not a point on Γ, and if P^* is the set of all the images of P under projectivities in $A(\Gamma)$, then it follows from the preceding Corollary 3 that the points in P^* are collinear and that $A(\Gamma)$ induces a regular group of permutations in P^*. If in particular $A(\Gamma)$ is a finite group, then all systems P^*, for P not in Γ, contain the same number of elements, this number being the order of the group $A(\Gamma)$.

The following lemma will be convenient for the construction of projectivities whose system of fixed elements is a subplane.

LEMMA 2. *The system Φ of fixed elements of the projectivity ϕ is a projective subplane of the projective plane Π if, and only if, there exist four fixed points of ϕ no three of which are collinear.*[8]

PROOF. The necessity of the condition being obvious, let us assume the existence of four fixed points U, V, W and Z of ϕ such that no three of these four points are collinear. If P is any fixed point of ϕ, then we have to show that three different fixed lines pass through P. If P happens to be one of the four basic points, say $P=U$, then $U+V$, $U+W$ and $U+Z$ are three different fixed lines passing through P. If P is different from the four basic points, then the lines $P+U$, $P+V$, $P+W$ and $P+Z$ are fixed lines passing through P. Thus we have proved our contention, provided three of these lines are different. Assume therefore that only two of these lines are different. Noting the impossibility that three of these lines are equal, we may assume without loss in generality that $P+U=P+V$ and $P+W=P+Z$ so that P is the intersection of the fixed lines $U+V$ and $W+Z$. But the lines $W+U$ and $V+Z$ meet in a fixed point M and the line $M+P$ is the desired third fixed line through P.

Of the four fixed lines $U+V$, $V+W$, $W+Z$ and $Z+U$ no three are copunctual. Hence we infer by duality from the result of the first paragraph that there are on every fixed line of ϕ at least three fixed points; and this completes the proof of the lemma, since fixed lines meet in fixed points and since the lines through fixed points are fixed lines.

LEMMA 3. *If Π is the projective plane over the (not necessarily commutative) field*[9] *G, and if $\phi\neq 1$ is a projectivity of Π, then the following condition is necessary and sufficient for the system Φ of fixed elements of ϕ to be a projective subplane of Π:*

(*) *There exists an automorphism $\alpha\neq 1$ of the field G and a system of homogeneous coordinates of Π such that the point with coordinates (x_0, x_1, x_2) is mapped by ϕ upon the point with coordinates $(x_0^\alpha, x_1^\alpha, x_2^\alpha)$.*

This may be deduced by the customary arguments[10] from Lemma 2.

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[8] M. Hall [1, p. 231] uses this property as a postulate instead of the equivalent requirements that every line carries at least three points and every point is on at least three lines on which requirement our discussion is based; see Baer [1, p. 138].

[9] The word "field" shall always be used in this general sense.

[10] See, for example, Baer [2] or Brauer [1].

LEMMA 4. *The projectivity* ϕ *of the projective plane* Π *over the field* G *which is determined by the automorphism* α *of* G *according to Lemma* 3, (*), *is a quasi-perspectivity if, and only if, there exists a number* $c \neq 0$ *in* G *such that* $c^\alpha = c$ *and*[11] $(x^\alpha - x)^\alpha = c(x^\alpha - x)$ *for every* x *in* G.

PROOF. If the condition is satisfied, then

$$(x_0^{\alpha^2}, x_1^{\alpha^2}, x_2^{\alpha^2}) = (1 + c)(x_0^\alpha, x_1^\alpha, x_2^\alpha) - c(x_0, x_1, x_2);$$

and we infer from Theorem 1, (i), that ϕ is a quasi-perspectivity.

Assume conversely that ϕ is a quasi-perspectivity. Consider the point with coordinates $(1, x, y)$ for x and y in G. Then we infer from Theorem 1, (i), the existence of numbers r, s in G such that $(1, x^{\alpha^2}, y^{\alpha^2}) = r(1, x^\alpha, y^\alpha) + s(1, x, y)$ or $1 = r + s$, $x^{\alpha^2} = rx^\alpha + sx$, $y^{\alpha^2} = ry^\alpha + sy$; this is equivalent to the following statement:

If x and y are numbers in G, then there exists a number s in G such that $x^{\alpha^2} = x^\alpha + s(x - x^\alpha)$ *and* $y^{\alpha^2} = y^\alpha + s(y - y^\alpha)$.

From $\alpha \neq 1$ we infer the existence of an element z in G such that $z^\alpha \neq z$. Then there exists one and only one number $c \neq 0$ in G such that

$$(z^\alpha - z)^\alpha = c(z^\alpha - z).$$

If t is any number in G, then it follows from the result, obtained in the preceding paragraph of this proof, that

$$(t^\alpha - t)^\alpha = c(t^\alpha - t)$$

is valid too. Apply this in particular on $t = z^\alpha$ and obtain

$$c(z^{\alpha^2} - z^\alpha) = (z^{\alpha^2} - z^\alpha)^\alpha = ((z^\alpha - z)^\alpha)^\alpha = (c(z^\alpha - z))^\alpha = c^\alpha(z^{\alpha^2} - z^\alpha)$$

proving $c = c^\alpha$, since $z \neq z^\alpha$ implies $z^\alpha \neq z^{\alpha^2}$, and this completes the proof of Lemma 4.

THEOREM 4. *Suppose that* Π *is the projective plane over the field* G, *and that the projectivity* ϕ *of* Π *is determined by the automorphism* α *of* G *according to Lemma* 3, (*). *Then* ϕ *is a quasi-perspectivity, but not an involution*[12] *if, and only if, there exists a number* c *in* G *which does not belong to the center of* G *such that* $c^{-1} + c$ *is in the center of* G *and such that*

$$x^\alpha = (1 + c)x(1 + c)^{-1} \text{ for every } x \text{ in } G.$$

PROOF. Assume firstly the existence of a number c in G, not in the center of G, such that $c^{-1} + c$ is in the center of G and such that

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[11] Note that α and ϕ are involutions if, and only if, $c = -1$.

[12] It is a consequence of Theorem 2 and Lemma 3 that ϕ cannot be a perspectivity either.

$x^\alpha=(1+c)x(1+c)^{-1}$. Then $c=c^\alpha$ and

$$\begin{aligned} x^{\alpha^2}-cxc^{-1} &= (1+c)^2x(1+c)^{-2}-cxc^{-1} \\ &= [(1+c)^2x-cxc^{-1}(1+c)^2](1+c)^{-2} \\ &= [x+c^2x-cx(c^{-1}+c)](1+c)^{-2} \\ &= [x+c^2x-(c^{-1}+c)cx](1+c)^{-2}=0 \end{aligned}$$

so that $x^{\alpha^2}=cxc^{-1}$. But c is not in the center of G. Hence $\alpha^2\neq 1$, proving that ϕ is not an involution. Furthermore we find that

$$\begin{aligned} (x^\alpha-x)^\alpha-c(x^\alpha-x) &= x^{\alpha^2}-(1+c)x^\alpha+cx \\ &= x^{\alpha^2}-x^{\alpha^2}(1+c)+cx=-x^{\alpha^2}c+cx \\ &= -(cxc^{-1})c+cx=0, \end{aligned}$$

and it follows from Lemma 4 that ϕ is a quasi-perspectivity. This proves the sufficiency of our condition.

Assume conversely that ϕ is a quasi-perspectivity, but not an involution. Then $\alpha^2\neq 1$, since ϕ^2 is determined by α^2, and we infer from Lemma 4 the existence of a number $c\neq 0$ in G such that $c=c^\alpha$ and such that

$$x^{\alpha^2}=(1+c)x^\alpha-cx \text{ for every } x \text{ in } G. \tag{1}$$

From $\alpha^2\neq 1$ we infer $c\neq -1$.

If y is an element in G, then we may apply (1) on $x=y^2$, and we find that

$$\begin{aligned} (1+c)y^{2\alpha}-cy^2 = y^{2\alpha^2} = (y^{\alpha^2})^2 &= [(1+c)y^\alpha-cy]^2 \\ &= (1+c)y^\alpha(1+c)y^\alpha-(1+c)y^\alpha cy-cy(1+c)y^\alpha+cycy \end{aligned}$$

or

$$(1+c)y^\alpha(y^\alpha-(1+c)y^\alpha+cy)=cy(y+cy-(1+c)y^\alpha);$$

thus we have shown that

$$(1+c)y^\alpha c(y-y^\alpha)=cy(1+c)(y-y^\alpha) \text{ for every } y \text{ in } G. \tag{2}$$

If in particular $y\neq y^\alpha$, then we may divide by $y-y^\alpha$ in (2), proving that

$$(1+c)y^\alpha c=cy(1+c) \quad \text{for} \quad y\neq y^\alpha. \tag{3}$$

If c were equal to 1, then the characteristic of the field G would be different from 2, since we have already shown that $c\neq -1$. There exist elements $y\neq y^\alpha$, since $\alpha\neq 1$; we should infer from (3) that $y^\alpha=y$

is a consequence of $y \neq y^\alpha$ which is impossible. Thus we have shown that

$$c \neq 0,\ 1,\ -1. \tag{4}$$

If $w \neq w^\alpha$, then $y = w^\alpha - w$ is an element not equal to 0, satisfying $y^\alpha = (w^\alpha - w)^\alpha = c(w^\alpha - w) = cy \neq y$ by (1) and (4). Hence we may apply (3) on this particular element y, and we obtain, because of (1),

$$\begin{aligned} c(w^\alpha - w)(1 + c) &= cy(1 + c) = (1 + c)y^\alpha c = (1 + c)(w^\alpha - w)^\alpha c \\ &= (1 + c)c(w^\alpha - w)c. \end{aligned}$$

Since $c \neq 0$, this implies $(w^\alpha - w)(1 + c) = (1 + c)(w^\alpha - w)c$ or

$$w^\alpha - w = c(w^\alpha - w)c \text{ for every } w \text{ in } G, \tag{5}$$

since this formula (5) is trivially true, in case $w = w^\alpha$.

If $y \neq y^\alpha$, then we infer from (3) and (5) that

$$\begin{aligned} cy - y^\alpha c &= [cy(1 + c) - cyc] - y^\alpha c = (1 + c)y^\alpha c - cyc - y^\alpha c \\ &= cy^\alpha c - cyc = c(y^\alpha - y)c = y^\alpha - y; \end{aligned}$$

and from this result we deduce that

$$y^\alpha(1 + c) = (1 + c)y \text{ whenever } y \neq y^\alpha. \tag{6}$$

Consider now an element x such that $x = x^\alpha$. We deduce from $\alpha \neq 1$ the existence of an element z in G such that $z \neq z^\alpha$. Then $y = x + z$ is an element in G, satisfying $y \neq y^\alpha$, and it follows from (6) that

$$\begin{aligned} x^\alpha(1 + c) &= (y - z)^\alpha(1 + c) = y^\alpha(1 + c) - z^\alpha(1 + c) \\ &= (1 + c)y - (1 + c)z = (1 + c)(y - z) = (1 + c)x. \end{aligned}$$

Combining this result with (6) and (4), we have shown that

$$x^\alpha = (1 + c)x(1 + c)^{-1} \textit{ for every } x \textit{ in } G. \tag{7}$$

If x is an element in G, then it follows from (1) and (7) that

$$\begin{aligned} 0 &= [x^{\alpha^2} - (1 + c)x^\alpha + cx](1 + c)^2 \\ &= (1 + c)^2x - (1 + c)^2x(1 + c) + cx(1 + c)^2 \\ &= cx(1 + c)^2 - (1 + c)^2xc = cx - xc + cxc^2 - c^2xc \end{aligned}$$

or, since $c \neq 0$ by (4),

$$0 = xc^{-1} - c^{-1}x + xc - cx$$

or

$$x(c + c^{-1}) = (c + c^{-1})x \text{ for every } x \text{ in } G,$$

proving that $c+c^{-1}$ is in the center of G. It follows from (7) and $\alpha\neq 1$ that c cannot be in the center of G, and this completes the proof of the necessity of our condition.

COROLLARY 4. *If* Π *is the projective plane over the field* G, *then the following condition is necessary and sufficient for the existence of a quasi-perspectivity which is neither a perspectivity nor an involution:*

There exists a number c *in* G *which does not belong to the center of* G, *though* $c+c^{-1}$ *is in the center.*

This is an almost immediate consequence of Theorem 2, Lemma 3, and Theorem 4.

COROLLARY 5. *If* Π *is the projective plane over the commutative field* G *then every quasi-perspectivity is a perspectivity or an involution, and the existence of quasi-perspectivities that are not perspectivities is equivalent to the existence of involutorial automorphisms not equal to* 1 *of* G.

This is readily deduced from Corollary 4, Theorem 2 and Lemma 3.

In order to show the existence of quasi-perspectivities that are neither perspectivities nor involutions we consider the projective plane Π over the field G of real quaternions. The number i in G satisfies $i^2=-i$ so that $i+i^{-1}=0$ is an element in the center of G, though i itself is not in the center of G. Thus i is an element c meeting the requirements of Corollary 4, and this shows the existence of quasi-perspectivities of Π which are neither perspectivities nor involutions.

If Π is the projective plane over the (not necessarily commutative) field G, and if Γ is a projective subplane of Π, then Γ is the projective plane over a suitable subfield G_0 of G. It is easy to establish a 1:1 and monotone increasing correspondence mapping the (partially ordered) set of the projective subplanes of Π which contain Γ upon the set of subfields of G which contain G_0. In particular Γ is maximal if, and only if, G_0 is maximal. Lemma 3 establishes a correspondence between the projectivities of Π which leave invariant the elements in Γ and the automorphisms of G which leave invariant the elements in G_0. This shows that this type of Galois theory of projective planes may be reduced completely to the Galois theory of fields. If in particular G is a commutative field, and if G is a normal separable extension of G_0, then G_0 is maximal if, and only if, the degree of G over G_0 is a prime p. In this case both the group of automorphisms of G which leave invariant the elements in G_0 and the group $A(\Gamma)$ of projectivities of Π which leave invariant the elements in Γ are of order p, and it is readily seen that the projectivities in $A(\Gamma)$ are quasi-perspectivities (involutions) if, and only if, $p=2$. This shows in particular that the maximality of $\Phi(\phi)$ does not imply that ϕ is a quasi-perspectivity.

We apply finally the results obtained so far to the special case of finite projective planes. If Π is a finite projective plane, then every line carries the same number $n+1$ of points and every point is on $n+1$ lines; the total number of points (lines) is $1+n+n^2$. This integer n $(\geqq 2)$ shall retain this significance throughout.

We denote by $N=N(\phi)$ *the total number of fixed points*[13] of the projectivity ϕ of the finite projective plane Π.

LEMMA 5. *If ϕ is a projectivity of the finite projective plane* Π, *and if $n \leqq N$, then ϕ is a quasi-perspectivity.*

PROOF. Suppose that no fixed line passes through the point W. Since there exist at least two fixed points, there exists a fixed point $P \neq W$. If W were a fixed point, then $P+W$ would be a fixed line which is impossible. Hence W is not a fixed point.

Suppose next that there exists a fixed point Z on the line $W+W\phi$. Then Z is different from both W and $W\phi$ so that

$$(W + W\phi)\phi = (Z + W)\phi = Z + W\phi = W + W\phi.$$

Thus the presence of a fixed point on $W+W\phi$ would imply that $W+W\phi$ is a fixed line through W. This is impossible. Hence there is no fixed point on the line $W+W\phi$.

If P is any fixed point, then $W+P$ is not a fixed line. Thus it is impossible that $W+P$ carries more than one fixed point. But there exist at least n fixed points each of which may be connected with W by a line which does not carry any further fixed points. Since there pass exactly $n+1$ lines through W, it follows that the line $W+W\phi$ is the one and only line through W which does not carry fixed points, and that all the other lines through W carry one and only one fixed point.

Since no line through W is a fixed line, no line through $W\phi$ is a fixed line. Hence it follows from the previous discussion that $W\phi+W\phi^2$ is the one and only one line through $W\phi$ which does not carry fixed points. But $W+W\phi$ is a fixed point free line passing through $W\phi$. Hence

$$W + W\phi = W\phi + W\phi^2 = (W + W\phi)\phi,$$

proving that $W+W\phi$ is a fixed line passing through W. This is a contradiction which shows that every point is on a fixed line. It follows from Theorem 1 that ϕ is a quasi-perspectivity.

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[13] It will be shown elsewhere that this is the number of fixed lines too.

If ϕ is an *elation*, then the set of its fixed points is just the set of points on some suitable line, proving $N(\phi)=n+1$; if ϕ is a *homology*,[14] then the set of its fixed points consists of all the points on its axis and of the center which is not on its axis so that $N(\phi)=n+2$.

THEOREM 5. *The following properties of the projectivity ϕ of the finite projective plane Π imply each other.*

(i) *ϕ is a quasi-perspectivity, but not a perspectivity.*

(ii) *The system Φ of fixed elements of ϕ is a projective subplane of Π and $n=i^2$ where $i+1$ is the number of fixed points on a fixed line.*

(iii) $n+2<N(\phi)$.

PROOF. If (i) is satisfied by ϕ, then we infer from Theorem 2 that Φ is a projective subplane of the finite plane Π. There exists therefore an integer $i\geqq 2$ such that every fixed line carries $i+1$ fixed points and such that $i+1$ fixed lines pass through any given fixed point. $N(\phi)=1+i+i^2$ is then the total number of fixed points and the total number of fixed lines. Consider a line h which is not a fixed line. It is a consequence of Corollary 1 that h carries one and only one fixed point H, and that every point $P\neq H$ on h is situated on one and only one fixed line. Since every fixed line meets h in one and only one point, and since exactly $i+1$ fixed lines pass through the fixed point H, it follows that the other i^2 fixed lines meet h in the n points different from H, proving that $n=i^2$. Thus we have shown that (ii) is a consequence of (i).

If (ii) is satisfied by ϕ, then

$$n+3=i^2+3\leqq i^2+i+1=N(\phi)$$

may be inferred from $2\leqq i$, showing that (iii) is a consequence of (ii).

If finally (iii) is satisfied by ϕ, then we infer from Lemma 5 that ϕ is a quasi-perspectivity. Since $N(\phi)=n+1$ or $n+2$ for perspectivities, ϕ cannot be a perspectivity, proving that (i) is a consequence of (iii).

COROLLARY 6. *Suppose that ϕ is a projectivity of the finite projective plane Π.*

(a) $N(\phi)\neq n$.

(b) $n+1\leqq N(\phi)$ *if, and only if, ϕ is a quasi-perspectivity.*

(c) $n+1=N(\phi)$ *if, and only if, ϕ is an elation.*

(d) $n+2=N(\phi)$ *if, and only if, ϕ is a homology.*

PROOF. If ϕ is a quasi-perspectivity, then either ϕ is a perspectivity and $N(\phi)=n+1$ or $n+2$, or else it follows from Theorem 5 that $n+2<N(\phi)$. If conversely $n+1\leqq N(\phi)$, then it follows from Lemma 5 that ϕ is a quasi-perspectivity, proving (b). If furthermore $N(\phi)=n+1$ or $n+2$, then it follows from Theorem 5 that ϕ is a perspectivity, and (c) and (d) are immediately inferred from a previous remark. If $n\leqq N(\phi)$, then it follows from Lemma 5 that ϕ is a quasi-perspectivity, and hence it follows from (b) that $n+1\leqq N(\phi)$. Thus $n\leqq N(\phi)$ implies $n+1\leqq N(\phi)$, proving (a).

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[14] See Veblen-Young [1, p. 72].

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INEQUALITIES CONNECTING SOLUTIONS OF CREMONA'S EQUATIONS

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1. **Introduction.** Let a complete and regular linear system $\Sigma_{p,d}$ of plane curves of *dimension* d, the *genus* of the general curve being p, be determined by its *order* x_0, and its multiplicities $x_1, \cdots, x_\rho$ at a set of ρ *general* base points. $x=(x_0; x_1, \cdots, x_\rho)$ is called the *characteristic* of $\Sigma_{p,d}$ and satisfies Cremona's equations:

$$(1)\qquad \begin{aligned} x_0^2 - x_1^2 - x_2^2 - \cdots - x_\rho^2 &\equiv (xx) = d + p - 1, \\ 3x_0 - x_1 - x_2 - \cdots - x_\rho &\equiv (lx) = d - p + 1. \end{aligned}$$

On the other hand, an integer solution x of (1) may or may not determine a linear system. If an x does determine a $\Sigma_{p,d}$, it is said to be *proper*. In this definition is included the usual convention that $(0; -1, 0, \cdots, 0)$ is a proper characteristic of the set of directions at a base point [1].[1]

If a system $\Sigma_{p,d}$ of characteristic x is subjected to a Cremona transformation C with F-points at the base points of Σ, $\Sigma \rightarrow \Sigma'_{p,d}$ whose characteristic x' at the F-points of C^{-1} is given by:

$$(2)\qquad L: \begin{aligned} x_0' &= (cx) \equiv c_0x_0 - c_1x_1 - c_2x_2 - \cdots - c_\rho x_\rho, \\ x_i' &= (\overset{i}{f}x) \equiv \overset{i}{f_0}x_0 - \overset{i}{f_1}x_1 - \overset{i}{f_2}x_2 - \cdots - \overset{i}{f_\rho}x_\rho, \\ &\qquad\qquad i = 1, 2, \cdots, \rho. \end{aligned}$$

Here c is the characteristic of the homaloidal net of C^{-1} and the f^i are the characteristics of the P-curves of this net. Thus proper characteristics c of $p=0$, $d=2$ and proper characteristics f of $p=d=0$ play a central role in the theory and will be prominent in this article. The collection of all transformations L for a given ρ forms a group, G_ρ. G_ρ is generated by transformations L for which c is of type $(2; 1110 \cdots 0)$, and for any $L \in G_\rho$ the forms (xx), (lx) and (xy) are invariant.

In this paper attention is restricted to characteristics of $x_0 > 0$, and $p \geqq 0$ and $d \geqq 0$. We shall designate this as *property* A and obtain inequalities implied by (1) and property A. The inequalities are interesting in themselves and lead to a criterion for distinguishing proper characteristics.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

2. Inequalities involving the characteristics of homaloidal nets.

THEOREM 1. *If x has property* A, *then* $2x_0-x_1-x_2-x_3\geqq 0$. *Moreover, the equals signs hold only for* $p=d=0$; $x=(1;110)$, $(1;101)$ *or* $(1;011)$.

Since $(xx)=d+p-1\geqq -1$, it may be shown that $x_0\geqq x_i$, $i=1,\cdots,p$. Indeed, set $x_i=x_0-a$ in $(xx)\geqq -1$:

$$2ax_0-a^2-x_1^2-\cdots-x_{i-1}^2-x_{i+1}^2-\cdots-x_p^2\geqq -1,$$

or

$$a(2x_0-a)\geqq -1.$$

Since x_0 is a positive integer, a may not be negative. Thus the integers a_1, a_2, a_3 in $x_1=x_0-a_1$, $x_2=x_0-a_2$, $x_3=x_0-a_3$ are non-negative. Substituting these in the quadratic relation yields:

$$-2x_0^2+2x_0(a_1+a_2+a_3)-a_1^2-a_2^2-a_3^2-x_4^2-\cdots-x_p^2\geqq -1.$$

Now $a_1, a_2, a_3, x_4, \cdots, x_p$ cannot all vanish, for this would imply that $-2x_0^2\geqq -1$. Thus:

$$2x_0(a_1+a_2+a_3)-2x_0^2>-1$$

or

$$a_1+a_2+a_3-x_0>-1/2x_0.$$

It follows that $a_1+a_2+a_3-x_0\geqq 0$ and thus that

$$2x_0-(x_0-a_1)-(x_0-a_2)-(x_0-a_3)\geqq 0.$$

If x is a characteristic with property A and $2x_0-x_1-x_2-x_3=0$, then the image of x under

$$\begin{array}{rll} & x_0' = 2x_0-x_1-x_2-x_3, & \\ A_{123}: & x_i' = x_i+(x_0-x_1-x_2-x_3), & i=1,2,3, \\ & x_j' = x_j, & j=4,\cdots,p, \end{array}$$

has $x_0'=0$ and satisfies the same Cremona equations. Thus

$$-x_1'^2-x_2'^2-\cdots-x_p'^2=d+p-1,$$
$$-x_1'-x_2'-\cdots-x_p'=d-p+1.$$

This is possible only for d, $p=0$, 0; 1, 0 and 0, 1. A canvass of the cases reveals that d, $p=0$, 0 and $x'=(0;\ -1\ 0\cdots 0)$ comprise all possibilities. Thus $x=(1;110)$, $(1;101)$, $(1;011)$ are the only values of x for which the equals sign holds.

Since (2; 1110 · · · 0) is the characteristic of a homaloidal net of conics, the form of the inequality clearly suggests the following generalization:

THEOREM 2. *If x has property* A *and c is the characteristic of a homaloidal net, then $(cx) \geqq 0$. Moreover, the equals sign holds only for the characteristics of the principal curves of the homaloidal net.*

Consider first characteristics x of $p+d>0$. In this case, Theorem 1 asserts that any x' obtained from x under A_{ijk} has $x_0' > 0$. Since c is the characteristic of a homaloidal net, c is the image of $(1; 0, 0, \cdots, 0)$ under a sequence of transformations of the form A_{ijk}. Let $x \to x'$ under the sequence that sends $c \to c' = (1; 0, \cdots, 0)$. Since $x_0' > 0$, it follows that $(c'x') > 0$. Thus $(cx) > 0$, for this bilinear relation is invariant under G_ρ.

If $p=0$, $d=0$, a modification of the argument is required since in this case x_0' might vanish under some A_{ijk}. But in this case x is by Theorem 1 a proper characteristic. Thus an improper characteristic x of $p=d=0$ always goes into a characteristic of $x_0' > 0$ under A_{ijk} and the argument above applies. For proper characteristics x of $p=d=0$, it is clear that $(cx) \geqq 0$, else the rational curve would have too many intersections with the homaloidal net. If $(cx) = 0$, x is the characteristic of a rational curve meeting the curves of the net only at the base points, and hence is the characteristic of a principal curve of the net.

3. Inequalities involving characteristics of rational curves.

LEMMA. *If x has property* A *and x^* denotes the same characteristic with x_ρ deleted, then x^* has property* A.

A simple computation yields for p', d' of x^*:

$$d' = d + x_\rho(x_\rho + 1)/2, \qquad p' - 1 = p - 1 + x_\rho(x_\rho - 1)/2.$$

Since $x_\rho(x_\rho+1)/2$ and $x_\rho(x_\rho-1)/2$ are non-negative functions of the integer x_ρ, the conclusion follows.

THEOREM 3. *If x has property* A *and $p+d>0$, and f is a proper characteristic of $p=d=0$ and $(fx)<0$, then $x_0>f_0$.*

Since f is proper, there is [2] an $L \in G_\rho$ such that $\bar{f} = L(f) = (0; 0, \cdots, 0, -1)$. $\bar{x} = L(x)$ has $\bar{x}_0 > 0$ by Theorem 2 and $(fx) = (\bar{f}\bar{x}) < 0$. But $(\bar{f}\bar{x}) = \bar{x}_\rho < 0$. Thus $\bar{x}$ may be written in the form

$$\bar{x} = \bar{x}^* + k\bar{f},$$

where k is a positive integer, and $\bar{x}^*$ is $\bar{x}$ with $\bar{x}_\rho$ deleted. Now consider the image of $\bar{x}$ under L^{-1}.

$$L^{-1}(\bar{x}) = L^{-1}(\bar{x}^* + k\bar{f}) = L^{-1}(\bar{x}^*) + kL^{-1}(\bar{f}),$$

or

$$x = L^{-1}(\bar{x}^*) + kf.$$

Now $\bar{x}^*$ has $\bar{x}_0^* > 0$, and $p' + d' > 0$ by the lemma. Hence by Theorem 2 its image $(\bar{x}^*)'$ has $(\bar{x}_0^*)' > 0$. Since

$$x_0 = (\bar{x}_0^*)' + kf_0,$$

it follows that $x_0 > f_0$.

THEOREM 4. *If x has property* A *and $p+d>0$, and f is a proper characteristic such that $p=d=0$ and $x_0 \leqq f_0$, then $(fx) \geqq 0$.*

Theorem 4 follows from Theorem 3 by formal reasoning and offers a generalization of a property of proper characteristics. For if x is a proper characteristic, $(fx) \geqq 0$ follows from the fact that the curves of the system may not have more than $f_0 x_0$ intersections with the irreducible rational curve associated with f. The significance of the theorem is that all characteristics $f_0 \geqq x_0 > 0$, $p \geqq 0$, $d \geqq 0$, $p+d>0$ must enjoy this same property.

There are examples of characteristics with property A and $p+d>0$ which even have $x_i > 0$, $i=1, \cdots, \rho$, for which there is an f of $f_0 < x_0$ such that $(fx) < 0$. An early example is $(5;\ 3^2 1^6)$ and $(1;\ 1^2 0^6)$.

4. Applications.

THEOREM 5. *Let x be a characteristic of property* A *and $p+d>0$, such that $(fx) \geqq 0$ for all proper f of $p=d=0$ and $f_0 < x_0$; then $x_i \geqq 0$ and, moreover, if x' is the image of x under any $L \in G_\rho$, then $x_0' > 0$ and $x_i' \geqq 0$ for $i=1, \cdots, \rho$.*

Since $\bar{f} = (0;\ 0^{\rho-1} - 1)$ is a proper f of $f_0 = 0 < x_0$ and $(\bar{f}x) \geqq 0$, it follows that $x_i \geqq 0$. By Theorem 4, $(fx) \geqq 0$ for all proper f, $p=d=0$ of $f_0 \geqq x_0$. Then $(fx) \geqq 0$ for all proper f. These characteristics f are simply permuted by any $L \in G_\rho$. Thus if $x' = L(x)$, it follows that $(fx') \geqq 0$ for all proper f. Since these include $(0;\ 0^{\rho-1} - 1)$, it follows as before that $x_i' \geqq 0$. Theorem 2 asserts that $x_0' > 0$.

The following important result is now easily established:

THEOREM 6. *Let c be a solution of (1) for $p=0$, $d=2$, $c_0 > 0$ such that $(fc) \geqq 0$ for all proper f of $p=d=0$ and $f_0 < c_0$, then c is the characteristic of a homaloidal net.*

As before, $c_i \geqq 0$ and it is known [3] that in such a case $c_0 - c_1 - c_2 - c_3 < 0$ if c_1, c_2, c_3 are the greatest of the numbers c_i and $c_0 > 1$. Thus under A_{123}, $c \rightarrow c'$ of $c_0' < c_0$ and by Theorem 5, $c_i' \geqq 0$, $i=1, 2, \cdots, \rho$. Thus this reduction may be continued until $c_0'' = 1$, in which case $c'' = (1; 0, \cdots, 0)$. Under the given hypotheses, c is then the image of $(1; 0, \cdots, 0)$ under some $L \in G_\rho$ and must be proper.

This result has been conjectured much earlier and indeed was proved [4] by the writer, but the proof given on that occasion was quite elusive and unsatisfactory. Fragmentary results indicate that Theorem 5 has other important applications to cases where a generalization of Noether's inequality is possible. It would be desirable to avoid the restriction $p+d>0$. It is possible that Theorem 5 might still be true if one removed this restriction and added at the end of the theorem "or else x' is of the type $(0; 0, \cdots, 0, -1)$."

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ON CERTAIN LIMIT THEOREMS OF THE THEORY OF PROBABILITY

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1. Introduction. In this paper we prove the following four limit theorems:

Let $X_1, X_2, X_3, \cdots$ *be independent identically distributed random*[1] *variables each having mean* 0 *and standard deviation* 1. *Let*

$$s_k = X_1 + X_2 + \cdots + X_k,$$

then:

I.

$$\lim_{n\to\infty} \text{prob. } \{\max(s_1, s_2, \cdots, s_n) < \alpha n^{1/2}\} = \sigma_1(\alpha)$$

where

$$\sigma_1(\alpha) = 0 \qquad (\alpha \leqq 0)$$

and

$$\sigma_1(\alpha) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\alpha \exp\left(-\frac{u^2}{2}\right) du \qquad (\alpha \geqq 0).$$

II.

$$\lim_{n\to\infty} \text{prob. } \{\max(|s_1|, |s_2|, \cdots, |s_n|) < \alpha n^{1/2}\} = \sigma_2(\alpha)$$

where

$$\sigma_2(\alpha) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \exp\{-(2m+1)^2\pi^2/8\alpha^2\} \qquad (\alpha \geqq 0).$$

III.

$$\lim_{n\to\infty} \text{prob. } \left\{\frac{s_1^2 + s_2^2 + \cdots + s_n^2}{n^2} < \alpha\right\} = \sigma_3(\alpha),$$

where

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[1] This condition can be replaced by a weaker one. In fact, it is enough to assume that the X's are such that the central limit theorem is applicable.

$$\sigma_3(\alpha) = \frac{\pi^{-3/2}}{4} \int_0^{\alpha/2} du\, u^{-3/2} \int_0^{\pi/2} (\cos t)^{-1/2} \theta_1' \left(\frac{t}{2}, e^{-1/4u} \right) dt \qquad (\alpha \geqq 0)$$

and

$$\theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/4} \sin (2n+1) z, \qquad \theta_1' = \frac{\partial}{\partial z} \theta_1.$$

IV.

$$\lim_{n \to} \mathrm{pr} \left\{ \frac{| + |s_2| + \cdots + |s_n|}{n^{3/2}} < \alpha \right\} = \sigma_4(\alpha),$$

eal Laplace transform of $\sigma_4(\alpha)$ *is given by the formula*

$$\int_0^{\infty} e^{-z\alpha} d\sigma_4(\alpha) = \sum_{j=1}^{\infty} \kappa_j \exp(-\delta_j z^{2/3}) \qquad (z > 0),$$

and where δ_j *is the jth positive root of the derivative of*

$$P(y) = \frac{(2y)^{1/2}}{3} \left\{ J_{1/3}\left(\frac{2^{3/2}}{3} y^{3/2} \right) + J_{-1/3}\left(\frac{2^{3/2}}{3} y^{3/2} \right) \right\}$$

and

$$\kappa_j = \frac{1 + 3\int_0^{\delta_j} P(y) dy}{3\delta_j P(\delta_j)}.$$

The proofs of all these theorems follow the same pattern. It is first proved that the limiting distribution exists and is *independent of the distribution of the X's*; then the distribution of the X's is chosen conveniently so that the limiting distribution can be calculated explicitly. This simple principle has, to the best of our knowledge, never been used before except in a paper by one of the authors in which IV is proved in all detail.[2]

Theorems I and II generalize and simplify several results of Bachelier.[3] Bachelier's work in spite of being both inspired and important does not always satisfy the modern requirements of rigor. His methods often depend on replacing difference equations by differential equations, a step not always easy to justify.

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[2] M. Kac, *On the average of a certain Wiener functional and a related limit theorem in calculus of probability*, to appear in Trans. Amer. Math. Soc. vol. 59 (1946).

[3] L. Bachelier, *Les lois des grands nombres du calcul de probabilités*, Paris, Gauthier-Villars, 1937. See in particular §§18, 21, 22, 35. This book contains no proofs but it gives references to earlier papers. For the modern and rigorous approach to these questions, based on the differential equation of diffusion, see the third and fourth chapters of A. Khintchine's book *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Berlin, Springer, 1933. Our proofs, however, are entirely elementary and do not depend on the use of parabolic differential equations.

The limiting distribution of III was discovered by Cameron and Martin in their work on Wiener space.[4] Their result is equivalent to the conclusion of III if the X's are assumed to be normally distributed. We use here a different method and make our considerations independent of the use of Wiener space. The Laplace transform of $\sigma_4(\alpha)$ has been recently calculated by Kac.[5] There seems to be very little hope that a reasonably simple expression for $\sigma_4(\alpha)$ itself can be found.

2. **Proofs of I and II.** Let $G_1, G_2, G_3, \cdots$ be independent, normally distributed random variables each having mean and standard deviation 1 and let

$$R_k = G_1 + G_2 + \cdots + G_k.$$

Let furthermore $\epsilon > 0$ and

$$P_n(\alpha) = \text{prob.} \left\{\max(s_1, s_2, \cdots, s_n) < \alpha n^{1/2}\right\}.$$

We first prove that for every integer k we have

$$\begin{aligned}(1)\qquad &\text{prob.}\left\{\max(R_1, \cdots, R_k) < (\alpha - \epsilon)k^{1/2}\right\} - \frac{1}{\epsilon^2 k} \leqq \liminf_{n\to\infty} P_n(\alpha) \\ &\qquad \leqq \limsup_{n\to\infty} P_n(\alpha) \leqq \text{prob.}\left\{\max(R_1, \cdots, R_k) < \alpha k^{1/2}\right\}.\end{aligned}$$

Let

$$n_j = \left[j\frac{n}{k}\right] \qquad (j = 0, 1, 2, \cdots, k),$$

and

$$P_{n,k}(\alpha) = \text{prob.}\left\{\max(s_{n_1}, s_{n_2}, \cdots, s_{n_k}) < \alpha n^{1/2}\right\}.$$

It follows immediately from the multidimensional central limit theorem that

$$(2)\qquad \lim_{n\to\infty} P_{n,k}(\alpha) = \text{prob.}\left\{\max(R_1, \cdots, R_k) < \alpha k^{1/2}\right\}.$$

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[4] R. H. Cameron and W. T. Martin, *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions*, Journal of Mathematics and Physics vol. 23 (1944) pp. 195–209.

[5] Loc. cit. footnote 2. It should be mentioned that the actual computation of the Laplace transform of $\sigma_4(\alpha)$ forms the major part of the proof of IV.

Let

$$E_r = \text{prob. } \{s_r \geqq \alpha n^{1/2}, s_1 < \alpha n^{1/2}, \cdots, s_{r-1} < \alpha n^{1/2}\}.$$

We see immediately that

$$\sum_{r=1}^{n} E_r = 1 - P_n(\alpha) \leqq 1.$$

For $n_i < r \leqq n_{i+1}$ we write

$$E_r = \text{prob. } \{s_r \geqq \alpha n^{1/2}, s_1 < \alpha n^{1/2}, \cdots, s_{r-1} < \alpha n^{1/2}, \left| s_{n_{i+1}} - s_r \right| \geqq \epsilon n^{1/2}\}$$
$$+ \text{prob. } \{s_r \geqq \alpha n^{1/2}, s_1 < \alpha n^{1/2}, \cdots, s_{r-1} < \alpha n^{1/2}, \left| s_{n_{i+1}} - s_r \right| < \epsilon n^{1/2}\}.$$

The first of these probabilities is obviously equal to E_r prob. $\{ \left| s_{n_i+1} - s_r \right| \geqq \epsilon n^{1/2}\}$ and hence by Tchebychef's inequality it is less than

$$E_r / k\epsilon^2.$$

Thus

$$1 - P_n(\alpha) \leqq \frac{1}{k\epsilon^2} + \sum_i \sum_{n_i < r \leqq n_i+1} \text{prob. } \{s_r \geqq \alpha n^{1/2}, s_1 < \alpha n^{1/2},$$
$$\cdots, s_{r-1} < \alpha n^{1/2}, \left| s_{n_{i+1}} - s_r \right| < \epsilon n^{1/2}\}.$$

The double sum is obviously less than the probability that at least one of the sums $s_{n_1}, s_{n_2}, \cdots, s_{n_k}$ is greater than $(\alpha - \epsilon)n^{1/2}$. Hence

$$1 - P_n(\alpha) < \frac{1}{k\epsilon^2} + 1 - P_{n,k}(\alpha - \epsilon),$$

and since $P_n(\alpha) < P_{n,k}(\alpha)$ we obtain.

$$P_{n,k}(\alpha - \epsilon) - 1/k\epsilon^2 < P_n(\alpha) < P_{n,k}(\alpha).$$

Letting $n \to \infty$ and using (2) we obtain (1). Let us now consider the particular case in which

$$\text{(3)} \qquad \text{prob. } \{X_j = 1\} = \text{prob. } \{X_j = -1\} = 1/2.$$

For these random variables the problem becomes the classical problem of the "ruin of the player" and it is well known[6] that I holds. Thus applying (1) to this particular case we get

$$\text{(4)} \qquad \begin{aligned} &\text{prob. } \{\max (R_1, \cdots, R_k) < (\alpha - \epsilon) k^{1/2}\} - 1/\epsilon^2 k \\ &\qquad \leqq \sigma_1(\alpha) \leqq \text{prob. } \{\max (R_1, \cdots, R_k) < \alpha k^{1/2}\}. \end{aligned}$$

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[6] See for instance R. v. Mises, *Wahrscheinlichkeitsrechnung*, Leipzig and Vienna, F. Deuticke, 1931. In particular pp. 499–506.

Replacing α by $\alpha+\epsilon$ we obtain

$$\text{prob. } \{\max (R_1, \cdots, R_k) < \alpha k^{1/2}\} \leqq \sigma_1(\alpha+\epsilon) + 1/\epsilon^2 k$$

and hence, using (1), for the general case

$$\sigma_1(\alpha-\epsilon) - 1/\epsilon^2 k \leqq \liminf_{n\to\infty} P_n(\alpha) \leqq \limsup_{n\to\infty} P_n(\alpha) \leqq \sigma_1(\alpha+\epsilon) + 1/\epsilon^2 k.$$

We complete the proof of I by letting $k\to\infty$ (while keeping ϵ fixed) and by noting that $\sigma_1(\alpha)$ is continuous.

The proof of II proceeds in exactly the same manner. The only difference comes in calculating

$$\lim_{n\to\infty} \text{prob. } \{\max (|s_1|, \cdots, |s_n|) < \alpha n^{1/2}\}$$

for the special random variables (3). The fact that in this case one is led to $\sigma_2(\alpha)$ is again implied by the classical theory.[7]

We should like to call the reader's attention to the fact that (4) provides a convenient and strong estimate for $P_k(\alpha)$ in the case where the X's are normally distributed. A similar estimate can be obtained by writing out in detail the proof of II.

3. **Proofs of III and IV.** Let $n_1, n_2, \cdots, n_k$ have the same meaning as in §2. Consider the difference

$$D_n = \frac{1}{n^2}\sum_{r=1}^{n} s_r^2 - \frac{1}{n^2}\sum_{i=1}^{k} (n_i - n_{i-1}) s_{n_i}^2.$$

We have

$$|D_n| \leqq \frac{1}{n^2}\sum_{i=1}^{k} \sum_{r=n_{i-1}+1}^{n_i} |s_{n_i}^2 - s_r^2|.$$

Using Schwartz's inequality we obtain for $n_{i-1} < r \leqq n_i$

$$\begin{aligned} \text{m.e. } \{|s_{n_i}^2 - s_r^2|\} &\leqq (\text{m.e. } \{(s_{n_i} - s_r)^2\})^{1/2} (\text{m.e. } \{(s_{n_i} + s_r)^2\})^{1/2} \\ &= (n_i - r)^{1/2}(3r + n_i)^{1/2} < 2n^{1/2}(n_i - r)^{1/2}. \end{aligned}$$

Thus

$$\text{m.e. } \left\{ \sum_{r=n_{i-1}+1}^{n_i} |s_{n_i}^2 - s_r^2| \right\} \leqq 2n^{1/2} \sum_{j=1}^{n_i - n_{i-1}} j^{1/2} < 2n^{1/2}\left(\frac{n}{k} + 1\right)^{3/2}$$

and therefore

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[7] Loc. cit. footnote 6, p. 561. This case corresponds to the problem of random walk in the presence of two absorbing barriers.

$$\text{m.e.}\ \{\,|D_n|\,\} < C/k^{1/2},$$

where C is a certain constant.

This estimate of m.e.$\{\,|D_n|\,\}$ implies immediately that for $\epsilon > 0$,

$$\text{prob.}\ \{\,|D_n| \geqq \epsilon\} \leqq C/\epsilon k^{1/2}.$$

We now write

$$\text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha\right\} = \text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha,\ |D_n| \geqq \epsilon\right\}$$
$$+ \text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_\nu^2 < \alpha,\ |D_n| < \epsilon\right\}$$

and notice that

$$\text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha,\ |D_n| < \epsilon\right\}$$
$$\leqq \text{prob.}\left\{\frac{1}{n^2}\sum_1^k (n_i - n_{i-1})s_{n_i}^2 < \alpha + \epsilon\right\}.$$

Thus

$$\begin{aligned} &\text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha\right\} \\ (5)\qquad &\leqq \frac{C}{\epsilon k^{1/2}} + \text{prob.}\left\{\frac{1}{n^2}\sum_1^k (n_i - n_{i-1})s_{n_i}^2 < \alpha + \epsilon\right\}. \end{aligned}$$

In exactly the same manner we obtain

$$\begin{aligned} &\text{prob.}\left\{\frac{1}{n^2}\sum_1^k (n_i - n_{i-1})s_{n_i}^2 < \alpha - \epsilon\right\} \\ (6)\qquad &\leqq \frac{C}{\epsilon k^{1/2}} + \text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha\right\}. \end{aligned}$$

Combining (5) and (6) we can write

$$\begin{aligned} &\text{prob.}\left\{\frac{1}{n^2}\sum_1^k (n_i - n_{i-1})s_{n_i}^2 < \alpha - \epsilon\right\} - \frac{C}{\epsilon k^{1/2}} \\ (7)\qquad &\leqq \text{prob.}\left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha\right\} \\ &\leqq \text{prob.}\left\{\frac{1}{n^2}\sum_1^k (n_i - n_{i-1})s_{n_i}^2 < \alpha + \epsilon\right\} + \frac{C}{\epsilon k^{1/2}}\cdot \end{aligned}$$

Let us now find the limit, as $n \to \infty$, of

$$\text{prob.} \left\{ \frac{1}{n^2} \sum_{1}^{k} (n_i - n_{i-1}) s_{n_i}^2 < \beta \right\}. \tag{8}$$

Denote by $\phi(\xi)$ the characteristic function of the distribution function of X_j.

The characteristic function of the joint distribution of

$$\frac{n_1^{1/2}}{n} s_{n_1}, \frac{(n_2 - n_1)^{1/2}}{n} s_{n_2}, \cdots, \frac{(n_k - n_{k-1})^{1/2}}{n} s_{n_k}, \tag{9}$$

that is, the mathematical expectation of

$$\exp \left\{ \frac{i}{n} \sum_{1}^{k} \xi_j (n_j - n_{j-1})^{1/2} s_{n_j} \right\}$$

is easily found to be

$$\prod_{j=1}^{k} \phi^{n_j - n_{j-1}} \left(\frac{1}{n} \sum_{i=j}^{k} \xi_i (n_i - n_{i-1})^{1/2} \right). \tag{10}$$

If we let $n \to \infty$ (keeping k fixed) we find easily that (10) approaches (uniformly in every bounded region of the k-dimensional space)

$$\exp \left\{ -\frac{1}{2k^2} \sum_{j=1}^{k} \left(\sum_{i=j}^{k} \xi_i \right)^2 \right\}$$

which can be recognized as the characteristic function of the joint distribution of

$$R_1/k, R_2/k, \cdots, R_k/k. \tag{11}$$

(We recall that $R_j = G_1 + \cdots + G_j$, where $G_1, G_2, \cdots, G_k$ are independent, normally distributed random variables each having mean 0 and standard deviation 1).

From the multidimensional continuity theorem for Fourier-Radon transforms, it follows that the joint distribution of the random variables (9) approaches the joint distribution of the random variables (11). Thus the probability (8) approaches

$$\text{prob.} \left\{ \frac{1}{k^2} \sum_{j=1}^{k} R_j^2 < \beta \right\} = \rho_k(\beta). \tag{12}$$

If in (7) we let $n \to \infty$ we obtain the inequality

$$\begin{aligned}\rho_k(\alpha - \epsilon) - \frac{C}{\epsilon k^{1/2}} &\leqq \liminf_{n\to\infty} \text{prob.} \left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha\right\} \\ &\leqq \limsup_{n\to\infty} \text{prob.} \left\{\frac{1}{n^2}\sum_1^n s_r^2 < \alpha\right\} \\ &\leqq \rho_k(\alpha + \epsilon) + \frac{C}{\epsilon k^{1/2}},\end{aligned}$$

which is analogous to the inequality (1) of §1.

We complete the proof of III by showing that

$$\rho_k(\beta) \to \sigma_3(\beta)$$

for all β. It was shown by Cameron and Martin[8] that the characteristic function of $\sigma_3(\beta)$ is

$$(\sec (2i\xi)^{1/2})^{1/2}$$

where $\sec(2i\xi)^{1/2}$ is defined by the familiar infinite product

$$\prod_{n=0}^{\infty}\left(1 - \frac{2i\xi}{((2n+1)\pi/2)^2}\right)^{-1}$$

and $z^{1/2}$ is defined in the plane with negative real axis removed and satisfying the requirement that $z^{1/2}$ is positive for real and positive z.

It is then sufficient to show that the characteristic function

$$\int_0^\infty e^{i\xi\beta}d\rho_k(\beta)$$

approaches $(\sec(2i\xi)^{1/2})^{1/2}$ uniformly in every finite ξ-interval. We have

$$\begin{aligned}\int_0^\infty e^{i\xi\beta}d\rho_k(\beta) &= (2\pi)^{-k/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k}\left(\sum_{r=1}^{j} x_r\right)^2\right\} \\ &\qquad\cdot\exp\left\{-\frac{1}{2}\sum_1^k x_r^2\right\} dx_1\cdots dx_k \\ &= (2\pi)^{-k/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{\frac{i\xi}{k^2}\sum_1^k y_j^2\right\} \\ &\qquad\cdot\exp\left\{-\frac{y_1^2}{2} - \frac{1}{2}\sum_2^k (y_j - y_{j-1})^2\right\} dy_1\cdots dy_k.\end{aligned}$$

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[8] Loc. cit. footnote 4. Actually, the characteristic function considered by Cameron and Martin is $(\sec(i\xi)^{1/2})^{1/2}$ but their case corresponds to the normalization m.e. $\{G_i^2\} = 1/2$ instead of m.e. $\{G_i^2\} = 1$.

If in the last integral we introduce the coordinate system corresponding to the "principal axes" of the quadratic form

$$y_1^2 + \sum_2^k (y_j - y_{j-1})^2 \tag{13}$$

we notice readily that the integral will be reduced to the form

$$(2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left\{ \frac{i\xi}{k^2} \sum_1^k z_j^2 \right\} \exp \left\{ -\frac{1}{2} \sum_1^k \lambda_j z_j^2 \right\} dz_1 \cdots dz_k,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the eigenvalues of the matrix of the quadratic form (13). In the last integral the variables are separated and we obtain immediately

$$\int_0^{\infty} e^{i\xi\beta} d\rho_k(\beta) = \prod_{j=1}^k \left(\lambda_j - \frac{2i\xi}{k^2} \right)^{-1/2}. \tag{14}$$

The branch of

$$(\lambda_j - 2i\xi/k^2)^{-1/2} \tag{15}$$

is determined by removing from the complex ξ-plane the part of the imaginary axis between $-\infty$ and $-k^2\lambda_j/2$ and requiring that (15) be positive for $\xi = yi$, $y > -k^2\lambda_j/2$.

One could now calculate explicitly the eigenvalues λ_j by elementary (but rather tedious) means and pass to the limit as $k \to \infty$. We prefer a less elementary method which has the advantage that it is applicable to many cases where the explicit calculation of eigenvalues is impossible.

We first notice that putting $\xi = 0$ in (14) we obtain $\lambda_1 \lambda_2 \cdots \lambda_k = 1$ and hence

$$\int_0^{\infty} e^{i\xi\beta} d\rho_k(\beta) = \prod_{j=1}^k \left(1 - \frac{2i\xi}{k^2 \lambda_j} \right)^{-1/2}.$$

Denoting by $((a_{rs}))$ the inverse matrix of the matrix of the quadratic form (13) we can verify directly that

$$a_{rs} = \min(r, s).$$

Noticing that $1/\lambda_1, \cdots, 1/\lambda_k$ are the eigenvalues of $((a_{rs}))$ and using Hilbert's approach to Fredholm's theory[9] we obtain that the quantities $1/k^2\lambda_j$ approach the eigenvalues of the integral equation

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[9] See D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Berlin and Leipzig, Teubner, 1912. In particular see Theorem 2 on p. 14.

$$\int_0^1 \min(s, t) f(t) dt = \lambda f(s). \tag{16}$$

Writing (16) in the form

$$\int_0^s t f(t) dt + s \int_s^1 f(t) dt = \lambda f(s)$$

and differentiating twice with respect to s we obtain the differential equation

$$\lambda f''(s) + f(s) = 0$$

subject to the boundary conditions $f(0) = f'(1) = 0$. Thus the eigenvalues are the numbers

$$((2n+1)\pi/2)^{-2} \qquad (n = 0, 1, 2, \cdots)$$

and it follows that

$$\begin{aligned} \lim_{k\to\infty} \int_0^\infty e^{i\xi\beta} dp_k(\beta) &= \prod_{n=0}^{\infty} \left(1 - \frac{2i\xi}{((2n+1)\pi/2)^2}\right)^{-1/2} \\ &= (\sec (2i\xi)^{1/2})^{1/2}, \end{aligned} \tag{17}$$

where the determination of $(\sec(2i\xi)^{1/2})^{1/2}$ was described above.[10] This completes the proof of III.

The proof of IV proceeds as follows. Denoting by $Q_n(\alpha)$ the probability

$$\text{prob.} \left\{ n^{-3/2} \sum_1^n | s_r | < \alpha \right\}$$

we are led to the inequality

$$\begin{aligned} &\text{prob.} \left\{ k^{-3/2} \sum_1^k | R_j | < \alpha - \epsilon \right\} - \frac{C_1}{\epsilon k^{1/2}} \leqq \liminf_{n\to\infty} Q_n(\alpha) \\ &\leqq \limsup_{n->\infty} Q_n(\alpha) \leqq \text{prob.} \left\{ k^{-3/2} \sum_1^k | R_j | < \alpha + \epsilon \right\} + \frac{C_1}{\epsilon k^{1/2}}. \end{aligned} \tag{18}$$

We then chose particular random variables whose distribution is given by the formula

$$\text{prob.} \{ X_j < \alpha \} = \frac{1}{2^{1/2}} \int_{-\infty}^{\alpha} \exp(-2^{1/2} | u |) du.$$

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[10] That convergence in (17) is uniform in every finite ξ-interval also follows from Hilbert's considerations quoted above.

For these random variables it was proved (loc. cit. footnote 2) that

$$\lim_{n\to\infty} Q_n(\alpha) = \sigma_4(\alpha).^{11}$$

We complete the proof for the general case in the same way as in the proof of I.

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[11] Inasmuch as we have not proved that σ_4 is continuous this statement should be qualified by adding that it holds at each continuity point of σ_4.

Added in proof: In the meantime Dr. Erdös succeeded in proving that σ_4 is everywhere continuous. The proof is quite involved and will not be reproduced here.

CONCERNING THE SEPARABILITY OF CERTAIN LOCALLY CONNECTED METRIC SPACES

F. BURTON JONES

If a connected metric space S is locally separable, then S is separable.[1] If a connected, *locally connected*, metric space S is locally *peripherally* separable, then S is separable.[2] Furthermore if a connected, locally connected, *complete* metric space S satisfies certain "flatness" conditions, it is known to be separable.[3] These "flatness" conditions are rather strong and involve both im kleinen and im grossen properties, which makes application rather awkward in some cases. If, however, this space S contains no skew curve[4] of type 1, then S has a certain amount of "flatness," but not quite enough to imply separability as can be seen from the following example. Let S consist of the points of the 2-sphere, distance being redefined as follows: (1) if the points X and Y of S lie on the same great circle through the poles, then $d(X, Y)$ is the ordinary distance on the sphere but (2) if the points lie on different great circles through the poles, then $d(X, Y)$ is the sum of the ordinary distances from each point to the same pole, using the pole which gives the smaller sum. The space S is a connected, locally connected, complete metric space which contains no skew curve of type 1 but S is not separable. Furthermore, S contains no cut point. However, if this last condition is strengthened slightly, separability follows as is seen in the following theorem.

THEOREM 1. *Let S denote a locally connected, complete metric space such that no pair of points cuts S. If S contains no skew curve of type* 1, *then S is separable.*

PROOF. Suppose, on the contrary, that S is not separable. Let T_0

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[1] Paul Alexandroff, *Über die Metrization der im kleinen kompakten topologischen Räume*, Math. Ann. vol. 92 (1924) pp. 294–301. Also W. Sierpinski, *Sur les espaces métriques localement separables*, Fund. Math. vol. 21 (1933) pp. 107–113.

[2] F. B. Jones, *A theorem concerning locally peripherally separable spaces*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 437–439.

[3] F. B. Jones, *Concerning certain topologically flat spaces*, Trans. Amer. Math Soc. vol. 42 (1937) pp. 53–93, Theorem 31. Also F. B. Jones, Bull. Amer. Math. Soc. Abstract 47-1-93.

[4] Kuratowski in his paper, *Sur le problème des courbes gauches en Topologie*, Fund. Math. vol. 15 (1930) pp. 271–283, defined two "skew curves." One of type 1 is topologically equivalent to the sum of three simple triods each two of which intersect precisely at their end points.

denote a simple triod in S and let $M_1=T_0$. Let T_1 denote a simple triod in S having only its end points in M_1, and let $M_2=T_0+T_1$. Let T_2 denote a simple triod in S having only its end points in M_2, and let $M_3=T_0+T_1+T_2$. This process may be continued, so that if z is an ordinal less than Ω_1, then (a) $M_z=\overline{\Sigma T_\eta}$, $0\leqq\eta<z$, (b) M_z is a separable (hence proper) subcontinuum of S, and (c) T_z is a simple triod having only its end points in M_z. Hence $T_1, T_2, T_3, \cdots, T_z, \cdots$ is an uncountable sequence α of simple triods such that no two of them have a point in common which is not an end point of one of them. For each $z<\Omega_1$, let d_z denote the smallest of the distances: $d(A, \text{arc } BOC)$, $d(B, \text{arc } AOC)$, and $d(C, \text{arc } AOB)$, where A, B, and C are the end points and O is the emanation point of T_z. Let H_1 denote the set of all simple triods T such that for some z, T is T_z of α and each end point of T lies together with a point of M_1 in a connected domain of diameter less than $d_z/5$.

Suppose that H_1 is uncountable. There exists a positive number ϵ such that for uncountably many different ordinals $z<\Omega_1$, T_z belongs to H_1 and $1.1\epsilon>d_z/5>\epsilon$. But since M_1 is separable, there exist three distinct points X_1, X_2, and X_3 and three ordinals $\alpha<\beta<\gamma<\Omega_1$ such that (1) T_α, T_β, and T_γ each belong to H_1, (2) for each ξ, $\xi=\alpha, \beta, \gamma$, $1.1\epsilon>d_\xi/5>\epsilon$, and (3) for each ξ, $\xi=\alpha, \beta, \gamma$, and each i, $i=1, 2, 3$, there exists a connected domain $D_{\xi i}$ which contains X_i and an end point of T_ξ, and whose diameter is less than $d_\xi/5$. Now for each ξ, $\xi=\alpha, \beta, \gamma$, let O_ξ denote the emanation point of T_ξ. From (2), (3), the definition of d_z, and the triangle axiom on the distance function, it follows that the connected domains, $D_1=\Sigma D_{\xi 1}$, $D_2=\Sigma D_{\xi 2}$, and $D_3=\Sigma D_{\xi 3}$, are mutually exclusive and neither $\overline{D}_1$, $\overline{D}_2$, nor $\overline{D}_3$ contains either O_α, O_β, or O_γ. Because of the restricted way in which the triods may intersect, no point outside of $D_1+D_2+D_3$ lies in more than one of the triods T_α, T_β, and T_γ. But $T_\alpha\cdot D_1$ may be joined to $T_\beta\cdot D_1$ by an arc in D_1; $T_\alpha\cdot D_1$ may be joined to $T_\gamma\cdot D_1$ by an arc in D_1; $T_\alpha\cdot D_2$ may be joined to $T_\beta\cdot D_2$ by an arc in D_2; and so on; and in the sum of these arcs together with $T_\alpha+T_\beta+T_\gamma$ there exists a skew curve of type 1. So the assumption that H_1 is uncountable leads to a contradiction. Hence H_1 is countable.

Let z_2 denote the smallest ordinal such that if $z\geqq z_2$, T_z of α does not belong to H_1. Evidently $z_2<\Omega_1$. Let H_2 denote the set of all triods T such that for some z, T is T_z of α and each end point of T lies together with a point of M_{z_2} in a connected domain of diameter less than $d_z/5$. The collection H_2 is countable. Let z_3 denote the smallest ordinal such that if $z\geqq z_3$, T_z of α does not belong to H_2. Evidently $z_3<\Omega_1$. Let H_3 denote the set of all triods T such that for some z, T is T_z of α and each end point of T lies together with a point of M_{z_3} in a connected domain of diameter less than $d_z/5$. The collection H_3 is countable. Continue this process, so that for each natural number n, H_n is defined and countable. There exists an ordinal number $z<\Omega_1$ such that for each n, $z>z_n$. Let $\bar{z}$ denote the first such ordinal. Clearly, $T_{\bar{z}}$ of α does not belong to H_n for any n. Let D_1, D_2, and D_3 denote three mutually exclusive connected domains covering respectively the end points of $T_{\bar{z}}$ such that each has a diameter less than $d_{\bar{z}}$. Since $M_1\subset M_2\subset M_3\subset \cdots \subset M_{\bar{z}}\subset \cdots$ and $T_{\bar{z}}$ has its end points in $M_{\bar{z}}$, there exists an integer i such that each of the domains, D_1, D_2, and D_3, intersects M_{z_i}. Hence $T_{\bar{z}}$ belongs to H_i. This is a contradiction.

THEOREM 2. *Let S denote a locally connected, complete metric space such that no pair of points cuts S. If S does not contain uncountably many skew curves of type* 1, *then S is separable.*

Theorem 2 may be established by the argument for Theorem 1, taking for T_0 the closure of the set consisting of all points X such that X belongs to a skew curve of type 1 lying in S. The connectedness of M_z was not used in the argument.

Comment. This result (Theorem 1) cannot be extended to complete Moore spaces.[5] For a locally connected complete Moore space exists which is not cut by any pair of its points and which contains no skew curve of type 1 but which nevertheless is not separable.[6] Furthermore, a separable such space exists which is not completely (perfectly) separable and hence is not metric.[7] The relation between Moore and metric spaces (in this connection) is shown in Theorem 3.

THEOREM 3. *Let M denote a locally connected, complete Moore space such that* (1) *no pair of points cuts M and* (2) *M contains no skew curve of type* 1. *In order that M be metric it is necessary and sufficient that M be completely* (*perfectly*) *separable.*

PROOF. Since any metric, complete Moore space is a complete metric space[8] and any separable metric space is completely separable, the necessity of the condition follows at once from Theorem 1. Since a

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[5] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloqiuum Publications, vol 13, 1932. Hereinafter this book will be referred to as *Foundations.* A complete Moore space is a space satisfying Axioms 0 and 1 of *Foundations.*

[6] R. L. Moore, *Concerning separability*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 56–58, Example 1.

[7] Ibid. Example 2.

[8] J. H. Roberts, *A property related to completeness*, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 835–838.

Moore space is a regular Hausdorff space, the sufficiency of the condition is well known.[9]

THEOREM 4. *Every metric space satisfying Axioms* 0–4 *of R. L. Moore's Foundations is completely* (*perpectly*) *separable.*

PROOF. Let S be a metric space satisfying Axioms 0–4 of *Foundations*. No finite set of points separates S.[10] Furthermore, with the help of Theorem 7 of Chapter III of *Foundations* it can be shown that S contains no skew curve of type 1. It follows from the preceding theorem that S is completely separable.

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[9] P. Urysohn, *Zum Metrisationsproblem*, Math. Ann. vol. 94 (1925) pp. 309–315, and A. Tychonoff, *Über einen Metrisationssatz von P. Urysohn*, Math. Am. vol. 95 (1926) pp. 139–142. See *Foundations*, p. 464.

[10] F. B. Jones, *Certain consequences of the Jordan curve theorem*, Amer. J. Math. vol. 63 (1941) pp. 531–544, Theorem 25.

MOEBIUS TRANSFORMATIONS AND CONTINUED FRACTIONS

H. SCHWERDTFEGER

R. E. Lane[1] has shown how the criterion for convergence of a periodic continued fraction can be based upon simple properties of the linear fractional transformation (Moebius transformation)

$$Z = A(z) = (az + b)/(cz + d). \tag{1}$$

The method may be further adapted to the pure "transformation point of view" by: (a) interpretation of the two lemmas of Lane's paper as reductions of (1) to its similarity normal forms; (b) proving and expressing the final theorem in terms of the theory of the Moebius transformation only.

This will be done in the present note which can be read without reference to Lane's paper.

The Moebius transformation[2] (1) is defined by its matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of complex elements with nonvanishing determinant δ, or by any matrix λA where $\lambda \neq 0$. Let x_1, x_2 be its fixed points: $A(x_\nu) = x_\nu$; then these are also the fixed points of the iterated transformations $z_n = A^n(z)$ $(n = 1, 2, \cdots)$.

Let T be the matrix of another Moebius transformation $z' = T(z)$ by which the new variable z' is introduced. Then $T(x_\nu)$ are the fixed points of the transformation[3]

$$Z' = TAT^{-1}(z'). \tag{2}$$

Suppose $x_1 \neq x_2$; then $T(x_1) \neq T(x_2)$ and the transformation T may be chosen so that $T(x_1) = 0$, $T(x_2) = \infty$, namely, $z' = (z - x_1)/(z - x_2)$, and (2) becomes

$$Z' = kz' \tag{3}$$

where $k = (a - cx_1)/(a - cx_2)$ is a complex constant, uniquely defined

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[1] R. E. Lane, *The convergence and the values of periodic continued fractions*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 246–250.

[2] Cf. C. Carathéodory, *Conformal representation*, Cambridge, 1932, chap. 1, §26.

[3] The matrix of the product (that is, composition) of two Moebius transformations, first A, then B, is the product BA of the two matrices.

by A and the same for all similar transformations SAS^{-1}. The transformation A is said to be *elliptic* if $|k|=1$ $(k\neq 1)$, *hyperbolic* if $k>0$, in all other cases *loxodromic*; it is the identity if and only if $k=1$.

If $x_1=x_2=x$, a transformation T can be found so that $T(x)=\infty$; then TAT^{-1} becomes a translation $Z'=z'+b'$. By suitable choice of T one can reach the normal form

$$Z' = z' + 1 \tag{4}$$

of A which, in this case, is said to be *parabolic*.

For a given $z\neq x_\nu$ $(\nu=1,\ 2)$ we consider the recurring sequence $z_n=A(z_{n-1})= \cdots =A^n(z)$. If it is convergent its limit is a fixed point x_1 of A. If $z_n\to x_1$ for all z of a neighborhood of x_1, we call x_1 an *attractive* fixed point of A. At the same time $T(x_1)$ is an attractive fixed point of TAT^{-1}. If $x_1\neq x_2$ it follows from (3) that x_1 is attractive if and only if $|k|<1$, and then $z_n\to x_1$ for all $z\neq x_2$; therefore the fixed point x_2 is said to be *repulsive*.

Evidently ∞ is an attractive fixed point of the translation (4); thus the only fixed point x of a parabolic transformation A is always attractive.[4]

If A is elliptic, $|k|=1$, the sequence z_n is bounded and divergent whatever the initial point $z\neq x_\nu$; the points z_n all lie on the circumference of the same invariant circle of A. The fixed points x_1, x_2 are symmetric with respect to this circle; they are called *indifferent* fixed points of A.

Now let

$$f = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots$$

be a given continued fraction where all $a_\mu\neq 0$. Its approximants f_q may be represented in the following way: Let

$$A_1 = \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix}, \cdots .$$

Then one has

$$f_0 = A_1(\infty) = 0, \qquad f_1 = a_1/b_1 = A_1(0) = A_1A_2(\infty), \cdots ,$$
$$f_{q-1} = A_1A_2 \cdots A_{q-1}(0) = A_1A_2 \cdots A_q(\infty).$$

By definition let

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[4] In the book of G. Julia, *Principes géométriques d'analyse*, I, Paris, 1930, it is stated on p. 30 that the fixed point of a parabolic transformation is indifferent. This depends, however, on the inadequate definition of "indifferent" given there on p. 23.

$$f_{-1} = \infty.$$

Now let f be periodic with a period of length m so that for all $n=0, 1, 2, \cdots$

$$a_q = a_r, \qquad b_q = b_r \quad \text{if} \quad q = nm + r$$

where r may be the least non-negative remainder (mod m) of q. We introduce the "period matrix"

$$A = \lambda A_1 A_2 \cdots A_m \qquad (\lambda \neq 0).$$

Then

$$f_{q-1} = A^n A_1 \cdots A_r(\infty) = A^n(f_{r-1})$$

for $r=0, 1, \cdots, m-1$, whence the main result follows immediately:

The periodic continued fraction f is convergent if and only if the following three conditions are satisfied:

1. *The Moebius transformation with the period matrix A is not elliptic.*
2. *The attractive fixed point x_1 of A is finite.*
3. *For all $r=0, \cdots, m-1$ one has $f_{r-1} \neq x_2$ where x_2 is the repulsive fixed point of A.*

Then $f=x_1$.

Remark. For $r=0$ the condition 3 implies that also x_2 must be finite. There is no restriction for f_{r-1} if A is parabolic. The condition 1 implies Lane's inequality (1, 4). In fact $f_{m-1}=A(\infty)$ is the pole of the transformation A^{-1} which has equal distance from both fixed points x_1, x_2 if and only if A is elliptic. In fact from (3) one has $f_{m-1}-x_1 = k(f_{m-1}-x_2)$ whence moreover it follows at once that in the non-elliptic case the pole of A^{-1} is nearer to the attractive fixed point of A.

For the application of the theorem it may be useful to have the necessary and sufficient condition for a Moebius transformation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to be elliptic; it is

$$\Re(\bar{\tau} \cdot (\tau^2 - 4\delta)^{1/2}) = 0$$

where $\tau = a+d$, $\delta = ad-bc$, and τ the conjugate complex of τ.

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ON THE THEOREM OF FEJÉR-RIESZ

A. ZYGMUND

1. Statement of results. Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1}$$

be a function regular for $|z| \leqq 1$. The well known inequality of Fejér and Riesz asserts that

$$\int_D |f(z)|\,|dz| \leqq \frac{1}{2}\int_C |f(z)|\,|dz|, \tag{2}$$

where C is the circumference $|z|=1$, and D any of its diameters.[1]

For $f(z)=F'(z)$, the inequality (2) takes the form

$$\int_D |F'(z)|\,|dz| \leqq \frac{1}{2}\int_C |F'(z)|\,|dz|, \tag{3}$$

which shows that the total variation of $F(z)$ along D does not exceed half of the total variation of F along C. In this form the inequality remains valid for harmonic functions. Let $z=\rho e^{i\theta}$. If $U(z)=U(\rho, \theta)$ is harmonic for $|z| \leqq 1$, the total variation of F along D does not exceed half of the total variation of F along C.[2] In symbols,

$$\int_D |U_\rho|\,d\rho \leqq \frac{1}{2}\int_C |U_\theta|\,d\theta. \tag{4}$$

Let $V(z)=V(\rho, \theta)$ be the harmonic function conjugate to U. In (4) we may replace U_ρ by $\rho^{-1}V_\theta$. Writing $U_\theta=u$, $V_\theta=v$, we obtain an equivalent form of the inequality (4), namely

$$\int_D \left|\frac{v(z)}{z}\right|\,|dz| \leqq \frac{1}{2}\int_C |u(z)|\,|dz|. \tag{5}$$

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[1] L. Fejér and F. Riesz, *Ueber eine funktionentheoretische Ungleichung*, Math. Zeit. vol. 11 (1921) pp. 305–314.

[2] The inequality (4), with 1/2 on the right replaced by an undetermined constant A, was first proved by B. N. Prasad, *On the summability of power series and the bounded variation of power series*, Proc. London Math. Soc. vol. 35 (1933) pp. 407–424. That $C=1/2$ was shown in F. Riesz, *Eine Ungleichung für harmonische Funktionen*, Monatshefte für Mathematik und Physik vol. 43 (1936) pp. 401–406, and A. Zygmund, *Some points in the theory of trigonometric and power series*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 586–617, especially p. 599.

It is valid for any pair of functions

$$u(z) = u(\rho, \theta) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)\rho^\nu, \tag{6}$$

$$v(z) = v(\rho, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)\rho^\nu, \tag{7}$$

harmonic and conjugate in $|z| \leqq 1$, provided that $v(0)=0$.

The purpose of this note is to prove the following complement to (4).

THEOREM 1. *Let*

$$U(z) = \frac{A_0}{2} + \sum_{\nu=1}^{\infty} (A_\nu \cos \nu\theta + B_\nu \sin \nu\theta)\rho^\nu \qquad (z = \rho e^{i\theta}) \tag{8}$$

be a function harmonic for $|z| \leqq 1$, *and let* $U_n(z)$ *be the nth partial sum of the series* (8). *For every n, the total variation of* $U_n(z)$ *over D does not exceed* $2/\pi$ *times the total variation of* $U(z)$ *over C. The constant* $2/\pi$ *here cannot be replaced by any smaller number.*

We shall prove this result in the following equivalent form.

THEOREM 1′. *Let* $u(z)$ *and* $v(z)$, *given by* (6) *and* (7), *be harmonic for* $|z| \leqq 1$ *and conjugate (in particular,* $v(0)=0$). *Let* $v_n(z)$ *be the nth partial sum of the series* (7). *Then*

$$\int_D \left|\frac{v_n(z)}{z}\right| |dz| \leqq \frac{2}{\pi}\int_C |u(z)|\,|dz| \tag{9}$$

for all n, the factor $2/\pi$ *on the right being the best possible.*

Taking $u(z)=zf(z)$, so that $v(z)=-izf(z)$, we deduce from (9) the following corollary.

THEOREM 2. *Let* $f(z)$ *be a function regular for* $|z| \leqq 1$ *and let* $s_n(z)$ *be the nth partial sum of the series* (1). *Then for all n*

$$\int_D |s_n(z)|\,|dz| \leqq \frac{2}{\pi}\int_C |f(z)|\,|dz|. \tag{10}$$

In all these results the assumption that the functions are harmonic (or regular) inside *and on C* can obviously be relaxed, and more general results may be obtained from the special ones by routine passages to the limit.[3] Thus, for example, in Theorem 2 we may assume that $f(z)$ is regular for $|z|<1$ and continuous for $|z|\leqq 1$. The partial sums $s_n(z)$ for such an f may be unbounded, and it is of interest to note that in the passage from (2) to (10) the increase of the coefficient on the right (from $1/2$ to $2/\pi=0.64\cdots$) is not significant.

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[3] See also §3, below.

That $2/\pi$ is the best constant in Theorem 1′ (and so in Theorem 1) is easily seen from the example

$$u(z) = 1/2 + \rho R\cos\theta + \rho^2R^2\cos 2\theta + \cdots \qquad (z=\rho e^{i\theta})$$

where R is a fixed positive number less than 1. The integral on the right of (9) is then π. Taking $n=1$ and the segment $(-i, i)$ for D we find for the integral on the left the value $2R$. Since R may be as close to 1 as we wish, the conclusion follows.

As we shall see later, for each $n>1$ the constant $2/\pi$ on the right can be replaced by a constant $C_n<2/\pi$, and clearly $2/\pi=C_1\geqq C_2 \geqq\cdots\geqq C_n\geqq\cdots$. One might expect that $C_n\to 1/2$, in accordance with (5). It is however not so, and the difference $C_n-1/2$ stays above a positive number. This fact has close connection with the Gibbs' phenomenon.

2. **Proof of Theorem 1′.** We shall use the formula

$$v_n(\rho,\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi} u(1,\theta+t)\rho\Delta_n(\rho,t)dt, \tag{11}$$

where

$$\Delta_n(\rho,t) = \sum_{\nu=1}^{n} \rho^{\nu-1}\sin\nu t. \tag{12}$$

Taking, as we may, for D the segment $(-1, +1)$ of the real axis, we get

$$\int_D\left|\frac{v_n(z)}{z}\right||dz| \leqq \frac{1}{\pi}\int_{-\pi}^{\pi}|u(1,t)|\left\{\int_{-1}^{+1}|\Delta_n(\rho,t)|\,d\rho\right\}dt,$$

and our problem reduces to showing that

$$\int_{-1}^{+1}|\Delta_n(\rho,t)|\,d\rho\leqq 2, \qquad \text{for } n=1,2,\cdots, -\pi\leqq t\leqq\pi. \tag{13}$$

That the integral on the left here is bounded as a function of n would be a simple matter to show. Complications arise when we want to show that the integral in question does not exceed 2.

We shall need the following lemma.

LEMMA. *Let* $0 \leqq t \leqq \pi$. *For every* μ *positive and odd, the values of the trigonometric polynomials*

$$\sin t/1 + \sin 3t/3 + \cdots + \sin \mu t/\mu,$$
$$\sin t/1 + \sin 3t/3 + \cdots + \sin (\mu - 2)t/(\mu - 2) + \sin \mu t/2\mu$$

are contained between 0 *and* 1.

Taking temporarily this lemma for granted, we shall proceed with the proof of the inequality (13). Let us consider the auxiliary function

$$(14) \qquad \Delta_n^*(\rho, t) = \sum_{\nu=1}^{n}{}' \rho^{\nu-1} \sin \nu t = \sum_{\nu=1}^{n-1} \rho^{\nu-1} \sin \nu t + \frac{\rho^{n-1} \sin nt}{2} \cdot$$

We easily find that

$$\Delta_n^*(\rho, t) = \frac{\dfrac{\sin t}{2}\left[2(1 - \rho^n \cos nt) - \rho^{n-1} \dfrac{\sin nt}{\sin t}(1 - \rho^2)\right]}{1 - 2\rho \cos t + \rho^2} \cdot$$

We note that, for $0 \leqq \rho \leqq 1$,

$$\begin{aligned} 2(1 - \rho^n \cos nt) - \rho^{n-1} \frac{\sin nt}{\sin t}(1 - \rho^2) &\geqq 2(1 - \rho^n) - n\rho^{n-1}(1 - \rho^2) \\ &= (1 - \rho)[2(1 + \rho + \cdots + \rho^{n-1}) - \rho^{n-1}n(1 + \rho)] \\ &\geqq (1 - \rho)(2n\rho^{n-1} - 2n\rho^{n-1}) = 0. \end{aligned}$$

Thus Δ_n^* (unlike Δ_n) is non-negative for $0 \leqq t \leqq \pi$, and so nonpositive in the interval $(-\pi, 0)$. This remark applies, in particular, to the trigonometric polynomial

$$\Delta_n^*(1, t) = \sum_{\nu=1}^{n}{}' \sin \nu t.$$

Let us now fix t, $0 \leqq t \leqq \pi$. In the proof of (13) we shall consider various special cases.

Case (1). $\Delta_n(\rho, t) \geqq 0$, $\Delta_n(\rho, t+\pi) \leqq 0$ for $0 \leqq \rho \leqq 1$.

Thus

$$\begin{aligned} \int_{-1}^{+1} |\Delta_n(\rho, t)| \, d\rho &= \int_0^1 [\Delta_n(\rho, t) - \Delta_n(\rho, t + \pi)] d\rho \\ &= \int_0^1 \left\{ \sum_{\nu \leqq n} \rho^{\nu-1} \sin \nu t - \sum_{\nu \leqq n} (-1)^\nu \rho^{\nu-1} \sin \nu t \right\} d\rho \\ &= 2 \sum_{\nu \leqq n, \nu \text{ odd}} \frac{\sin \nu t}{\nu} \leqq 2, \end{aligned}$$

by the lemma.

Case (ii). $\Delta_n(\rho_0, t) < 0$, for some ρ_0, $0 \leqq \rho_0 \leqq 1$. We may assume that $0 < \rho_0 < 1$.

This cannot happen for $n=1$. Thus $n \geqq 2$. We shall show that in this case, for all ρ, $0 \leqq \rho \leqq 1$:

(a) $\Delta_n(\rho, t+\pi) < 0$.

(b) $|\Delta_n(\rho, t)|$ is majorized both by $\Delta_{n-1}(\rho, t)$ and $\Delta_{n+1}(\rho, t)$ (in particular, the latter quantities must be non-negative).

For suppose that

$$\Delta_n(\rho, t) = \sin t + \rho_0 \sin 2t + \cdots + \rho_0^{n-1} \sin nt < 0.$$

We know that for all ρ, $0 \leqq \rho \leqq 1$,

$$\begin{aligned} \Delta_n^*(\rho, t) &= \sin t + \rho \sin 2t + \cdots \\ &\quad + \rho^{n-2} \sin (n-1)t + 2^{-1}\rho^{n-1} \sin nt \geqq 0, \\ \Delta_{n+1}^*(\rho, t) &= \sin t + \rho \sin 2t + \cdots \\ &\quad + \rho^{n-1} \sin nt + 2^{-1}\rho^{n} \sin (n+1)t \geqq 0. \end{aligned}$$

A comparison of these inequalities shows that

$$\sin nt < 0, \qquad \sin (n+1)t > 0. \tag{15}$$

Thus, since $\Delta_n = \Delta_n^* + 2^{-1}\rho^{n-1} \sin nt$, we have

$$\begin{aligned} 2^{-1}\rho^{n-1} \sin nt \leqq \Delta_n(\rho, t) &\leqq \Delta_n^*(\rho, t), \\ |\Delta_n(\rho, t)| &\leqq |\Delta_n^*(\rho, t)| + |2^{-1}\rho^{n-1} \sin nt| \\ &= \Delta_n^*(\rho, t) - 2^{-1}\rho^{n-1} \sin nt = \Delta_{n-1}(\rho, t). \end{aligned} \tag{16}$$

Similarly, the formula $\Delta_n = \Delta_{n+1}^* - 2^{-1}\rho^n \sin(n+1)t$ and (15) imply

$$\begin{aligned} -2^{-1}\rho^{n} \sin (n+1)t \leqq \Delta_n(\rho, t) &\leqq \Delta_{n+1}^*(\rho, t), \\ |\Delta_n(\rho, t)| &\leqq |\Delta_{n+1}^*(\rho, t)| + |2^{-1}\rho^{n} \sin (n+1)t| \\ &= \Delta_{n+1}^*(\rho, t) + 2^{-1}\rho^{n} \sin (n+1)t = \Delta_{n+1}(\rho, t). \end{aligned} \tag{17}$$

Thus assertion (b) is proved. To prove (a) we replace t by $t+\pi$ in the equations

$$\begin{aligned} \Delta_n(\rho, t) &= \Delta_n^*(\rho, t) + 2^{-1}\rho^{n-1} \sin nt, \\ \Delta_n(\rho, t) &= \Delta_{n+1}^*(\rho, t) - 2^{-1}\rho^{n} \sin (n+1)t. \end{aligned}$$

If n is even, the first of these resulting equations, coupled with the inequalities $\Delta_n^*(\rho, t+\pi) \leqq 0$, $\sin nt < 0$, shows that $\Delta_n(\rho, t+\pi) < 0$. If n is odd, we similarly argue with the second equation. Thus (a) is proved.

Using (a) and (b), we see that in case (ii)

$$\int_{-1}^{+1} |\Delta_n(\rho, t)| \, d\rho \leqq \int_0^1 \{\Delta_{n\pm 1}(\rho, t) - \Delta_n(\rho, t + \pi)\} d\rho$$

$$= \int_0^1 \left\{ \sum_{\nu=1}^{n\pm 1} \rho^{\nu-1} \sin \nu t - \sum_{\nu=1}^{n} (-1)^\nu \rho^{\nu-1} \sin \nu t \right\} d\rho.$$

If n is even, we take the sign $+$ in $n \pm 1$, and the last integral becomes

$$2\left\{ \sum_{\nu \leqq n-1, \nu \text{ odd}} \frac{\sin \nu t}{\nu} + \frac{1}{2} \frac{\sin (n+1)t}{n+1} \right\} \leqq 2,$$

by the lemma. If n is odd ($n \geqq 3$), we take the sign $-$ in $n \pm 1$, and the integral in question takes the form

$$2\left(\sum_{\nu \leqq n-2, \nu \text{ odd}} \frac{\sin \nu t}{\nu} + \frac{1}{2} \frac{\sin nt}{n} \right) \leqq 2,$$

again by the lemma. Thus (13) holds in case (ii).

Case (iii). $\Delta_n(\rho_0, t+\pi) > 0$ for some ρ_0, $0 \leqq \rho_0 \leqq 1$.

To prove that (13) holds in this case, we observe that, since $\Delta_n(\rho, t)$ is odd in t,

$$\Delta_n(\rho_0, t + \pi) = -\Delta_n(\rho_0, \pi - t) = -\Delta_n(\rho_0, t'),$$

where $t' = \pi - t$. Thus $0 \leqq t' \leqq \pi$, $\Delta_n(\rho_0, t')$ is negative, and we are in case (ii). It follows that (13) holds if we replace there t by t'. But

$$\begin{aligned} \int_{-1}^{+1} |\Delta_n(\rho, t)| \, d\rho &= \int_{-1}^{+1} |\Delta_n(\rho, t - \pi)| \, d\rho \\ &= \int_{-1}^{+1} |\Delta_n(\rho, t')| \, d\rho \leqq 2, \end{aligned} \tag{18}$$

and (13) holds again.

Cases (i), (ii), (iii) exhaust all possibilities, since the simultaneous occurrence of the inequalities $\Delta_n(\rho_0, t) < 0$, $\Delta_n(\rho_1, t+\pi) > 0$ is excluded by case (ii). Thus for the completion of the proof of Theorem 1′ we need only prove the lemma.

In estimating the polynomial

$$\begin{aligned} &\frac{\sin t}{1} + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{\mu} \\ &= \int_0^t (\cos u + \cos 3u + \cdots + \cos \mu u) du \\ &= \int_0^t \frac{\sin (\mu + 1)u}{2 \sin u} du, \end{aligned} \tag{19}$$

we may assume that $0 \leqq t \leqq \pi/2$, since replacing t by $\pi - t$ does not change the value of the polynomial. It is clear that the maximum of the last integral is attained for $t = \pi/(\mu+1)$, and is equal to

$$(20)\qquad \int_0^{\pi/(\mu+1)} \frac{\sin(\mu+1)u}{2\sin u}\,du = \int_0^{\pi} \frac{\sin u}{2(\mu+1)\sin(u/\mu+1)}\,du.$$

For fixed u, $0 \leqq u \leqq \pi$, and $\mu \geqq 1$, the minimum of the last denominator is attained when $\mu = 1$. For this particular value of μ the last integral becomes $2^{-1}\int_0^\pi \cos(u/2)du = 1$, and one-half of the lemma is proved.

Similarly, assuming, as we may, that $\mu \geqq 3$, we get

$$\sin t + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{2\mu} = \int_0^t \frac{\sin \mu u}{2\tan u}\,du \leqq \int_0^{\pi/\mu} \frac{\sin \mu u}{2\tan u}\,du$$
$$\leqq \int_0^{\pi/\mu} \frac{\sin \mu u}{2\sin u}\,du \leqq 1$$

by the result just obtained. This completes the proof of the lemma.[4]

3. **Additional remarks.** (i) Let C_n be the least number such that

$$\int_D \left| \frac{v_n(z)}{z} \right| |dz| \leqq C_n \int_C |u(z)|\,|dz|$$

for all functions $u(z)$ harmonic in $|z| \leqq 1$. We know that $C_1 = 2/\pi$, and the argument just completed clearly shows that $C_n < 2/\pi$ for $n = 2, 3, \cdots$. The numbers $C_1 \geqq C_2 \geqq C_3 \geqq \cdots$ tend to a limit $C^* \geqq 1/2$. Combining the estimates in cases (i), (ii), (iii) with the inequalities of the lemma, we find that

$$(21)\qquad \begin{aligned} C^* &\leqq \lim_{m\to\infty} \frac{1}{\pi}\int_0^{\pi/m} \frac{\sin mu}{\sin u}\,du = \lim_{m\to\infty} \frac{1}{\pi}\int_0^{\pi/m} \frac{\sin mu}{u}\,du \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{\sin u}{u}\,du. \end{aligned}$$

On the other hand, let R be any positive number less than 1 and let

$$u(z) = u_R(z) = \pi^{-1}(1/2 + R\rho\cos\theta + R^2\rho^2\cos 2\theta + \cdots), \quad z = \rho e^{i\theta},$$

so that $\int_C |u(z)|\,|dz| = 1$. If t, $0 < t < \pi$, is the angle of D with the positive real axis, then

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[4] The author is indebted to a referee for pointing out a slip in the initial proof of the lemma.

$$\begin{aligned}\int_D \left|\frac{v_n(z)}{z}\right| |dz| &= \int_{-1}^{+1} |\Delta_n(\rho, t)| \, d\rho \\ &\geqq \int_0^{+1} \{\Delta_n(\rho, t) - \Delta_n(\rho, t+\pi)\} d\rho \\ &= 2\left\{R\frac{\sin t}{1} + R^3\frac{\sin 3t}{3} + \cdots + R^\mu \frac{\sin \mu t}{\mu}\right\},\end{aligned}$$

where μ is the largest odd integer not greater than n. Taking R as close to 1 as we wish, we see that the last sum comes arbitrarily close to

$$2\left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{\mu}\right), \tag{22}$$

which is a partial sum of the Fourier series of the function $(\pi/2)$ sign t $(-\pi < t < \pi)$. This function has a jump at $t=0$ so that the partial sums (22) must display Gibbs' phenomenon. This also follows from (19) and (20), which show that

$$C^* \geqq \frac{1}{\pi}\int_0^\pi \frac{\sin u}{u} du.$$

Comparing this with (21) we see that

$$C^* = \frac{1}{\pi}\int_0^\pi \frac{\sin u}{u} du = .589490 \cdots .$$

(ii) Let $U(\rho, \theta)$ be the Poisson integral of a function $F(t)$, $0 \leqq t < 2\pi$, of bounded variation and not constant. Thus

$$U(\rho, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\rho^2}{1 - 2\rho\cos(t-\theta) + \rho^2} F(t) dt.$$

Since the total variation of $U(z)$ on the circle $|z| = \rho < 1$ tends to the total variation of $F(t)$ over $(0, 2\pi)$, Theorem 1 gives

$$\int_D \left|\frac{d}{d\rho} U_n(z)\right| d\rho \leqq \frac{2}{\pi}\int_0^{2\pi} |dF(t)|. \tag{23}$$

This, of course, may be obtained directly, by applying to the formula

$$\frac{d}{d\rho} U_n(\rho, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Delta_n(\rho, \theta - t) dF(t)$$

the estimates for $\int_{-1}^{+1} |\Delta_n(\rho, t)| d\rho$. This direct approach shows that if, for simplicity, we take for D the segment $(-1, +1)$, then the sign of equality occurs in (23) if and only if (a) $n=1$, (b) $F(t)$ is $Cx(t)$, where C is any constant, and $x(t)$ equals $+1$ inside $(-\pi/2, +\pi/2)$ and equals -1 inside the intervals $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$. In other words, $U(\rho, \theta)$ must be

$$C\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{\cos(2\nu-1)\theta}{2\nu-1}\rho^{2\nu-1}.$$

UNIVERSITY OF PENNSYLVANIA

A NOTE ON THE RIEMANN ZETA-FUNCTION

FU TRAING WANG

Let $\rho_\nu = \beta_\nu + i\gamma_\nu$ be the zeros of the Riemann zeta-function $\zeta(1/2+z)$ whose real part $\beta_\nu \geqq 0$. Then we have the following formula which is an improvement on Paley-Wiener's [1, p. 78][1]

$$\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2}\,dt = 2\pi \sum_{\nu=1}^{\infty} \frac{\beta_\nu}{|\rho_\nu|^2} + \int_0^{\pi/2} R\{e^{-i\theta} \log \zeta(1/2+e^{i\theta})\}\,d\theta + O\left(\frac{\log T}{T}\right).$$

In order to prove this formula let ρ_ν $(\nu = 1, 2, \cdots, n)$ be the n zeros of $\zeta(1/2+z)$ for which $0 < \gamma_\nu < T$ and $0 \leqq \beta_\nu < 1/2$. We require the following lemma:

LEMMA. *Let K be the unit semicircle with center $z=0$ lying in the right half-plane $R(z) > 0$ and let C be the broken line consisting of three segments L_1 $(0 \leqq x \leqq T,\ y = T)$, L_2 $(0 \leqq x \leqq T,\ y = -T)$ and L_3 $(x = T,\ -T \leqq y \leqq T)$. Then*

$$\begin{aligned}(1)\qquad \frac{1}{\pi}\int_1^T \frac{\log|\zeta(1/2+it)|}{t^2}\,dt &= 2\sum_{\nu=1}^{n} \frac{\beta_\nu}{|\rho_\nu|^2} \\ &\quad + \frac{1}{2\pi i}\int_K \frac{\log \zeta(1/2+z)}{z^2}\,dz - \frac{1}{2\pi i}\int_C \frac{\log\zeta(1/2+z)}{z^2}\,dz.\end{aligned}$$

This is a form of Carleman's theorem which can be proved by a method of proof analogous to that of Littlewood's theorem (Titchmarsh [3, pp. 130–134]).

Let Γ be a contour describing C, K and the corresponding part of the imaginary axis, and let ρ_ν be a point interior to Γ, and $\log(z-\rho_\nu)$ be taken as its principal value. We write C_1 as a contour describing Γ in positive direction to the point $i\gamma_\nu$, then along the segment $y = \gamma_\nu$, $0 < x < \beta_\nu - r$, and describing a small circle with center $z = \rho_\nu$, radius r, then going back along the negative side of this segment to $i\gamma_\nu$, and then along Γ to the starting point.

By Cauchy's theorem we get

$$\int_{C_1} \frac{\log(z-\rho_\nu)}{z^2}\,dz = 0.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Hence

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\log(z-\rho_\nu)}{z^2}dz=-\int_0^{\beta_\nu}\frac{dx}{(x+i\gamma_\nu)^2}$$

where the integral round the small circle with center $z=\rho_\nu$, radius r, tends to zero as $r\to 0$. This formula is also true for $\beta_\nu=0$.

Put $\zeta(1/2+z)=\phi(z)\prod_{\nu=1}^{n}(z-\rho_\nu)\prod_{\nu=1}^{n}(z-\bar{\rho}_\nu)$ where $\phi(z)$ is regular and has no zero in and on Γ. Then we get

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\log\zeta(1/2+z)}{z^2}dz=\sum_{\nu=1}^{n}\left(\frac{1}{\rho_\nu}-\frac{1}{i\gamma_\nu}\right)+\sum_{\nu=1}^{n}\left(\frac{1}{\bar{\rho}_\nu}+\frac{1}{i\gamma_\nu}\right)$$
$$=2\sum_{\nu=1}^{n}\frac{\beta_\nu}{|\rho_\nu|^2}.$$

From this the lemma follows.

Now we have

$$\int_C\frac{\log\zeta(1/2+z)}{z^2}dz=-\int_{L_1}+\int_{L_2}+\int_{L_3}. \tag{2}$$

On account of

$$\log\zeta(1/2+x+iT)=O(1)\qquad\text{for } x\geqq 1$$

we have

$$\int_{L_1}=\int_0^1\frac{\log\zeta(1/2+x+iT)}{(x+iT)^2}dx+O\left(\frac{1}{T}\right). \tag{3}$$

Since (Titchmarsh [2, p. 5])

$$\arg\zeta(1/2+x+iT)=O(\log T)\qquad\text{for } 0\leqq x\leqq 1$$

and (Titchmarsh [2, p. 59])

$$\log|\zeta(1/2+x+iT)|$$
$$=\frac{1}{2}\sum_{|\gamma-T|<1}\log\{(x-\beta)^2+(T-\gamma)^2\}+O(\log T),$$

then

$$\int_0^1\frac{\log\zeta(1/2+x+iT)}{(x+iT)^2}dx=O\left(\frac{\log T}{T^2}\right). \tag{4}$$

From (3) and (4) we get

$$\int_{L_1}=O\left(\frac{\log T}{T}\right). \tag{5}$$

Similarly

$$\int_{L_2} = O\left(\frac{\log T}{T}\right). \tag{6}$$

Since $\log \zeta(1/2+T+iy) = O(2^{-T})$, we get

$$\int_{L_3} = O(T2^{-T}). \tag{7}$$

By (1), (2), (5), (6) and (7) we have

$$\begin{aligned}\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2}\,dt &= 2\pi\sum_{\nu=1}^{n} \frac{\beta_\nu}{|\rho_\nu|^2} \\ &\quad + \frac{1}{2i}\int_K \frac{\log \zeta(1/2+z)}{z^2}\,dz + O\left(\frac{\log T}{T}\right).\end{aligned} \tag{8}$$

But (Ingham [4, p. 70])

$$\sum_{\nu=n+1}^{\infty} \frac{\beta_\nu}{|\rho_\nu|^2} = O\left(\sum_{\gamma>T} \frac{1}{\gamma^2}\right) = O\left(\frac{\log T}{T}\right). \tag{9}$$

The formula follows from (8) and (9).

Finally, if we make $T\to\infty$ then

$$\int_1^\infty \frac{\log |\zeta(1/2+it)|}{t^2}\,dt = \int_0^{\pi/2} R\{e^{-i\theta} \log \zeta(1/2+e^{i\theta})\}\,d\theta$$

gives a necessary and sufficient condition for the truth of the Riemann hypothesis.

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NOTE ON A THEOREM OF MURRAY

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1. **Introduction.** In a recent paper[1] [1][2] Murray has shown that in any reflexive separable Banach space $\mathfrak{B}$ every closed subspace $\mathfrak{M}$ admits what he calls a quasi-complement, that is, a second closed subspace $\mathfrak{N}$ such that $\mathfrak{M}\cap\mathfrak{N}=0$ and such that $\mathfrak{M}\dotplus\mathfrak{N}$, the smallest subspace containing both $\mathfrak{M}$ and $\mathfrak{N}$, is dense in $\mathfrak{B}$. It is the purpose of this note to give a simpler proof of the following somewhat more general theorem.

THEOREM. *Let $\mathfrak{B}$ be a separable normed linear space (not necessarily reflexive or even complete) and let $\mathfrak{M}$ be a closed subspace of $\mathfrak{B}$. Then there exists a second closed subspace $\mathfrak{N}$ such that $\mathfrak{M}\cap\mathfrak{N}=0$ and $\mathfrak{M}+\mathfrak{N}$ is dense in $\mathfrak{B}$.*

In proving this theorem it is convenient to make use of the notion of closed subspace of a linear system discussed at length in Chapter III of [2]. We repeat the necessary definitions here. A linear system X_L is an abstract linear space X together with a linear subspace L of the space X^* of all linear[3] functionals defined on X. If $l(x)=0$ for all l in L implies that $x=0$ (that is, if L is total) we say that X_L is a regular linear system. If M is a subspace of X $[L]$ we denote by M' the set of all l in L $[x$ in $X]$ such that $l(x)=0$ for all x in X $[l$ in $L]$. It is clear that $M\subseteq N$ implies $N'\subseteq M'$ and that $M''\supseteq M$. Since $M'''=(M'')'\subseteq M'$ and since $M'''=(M')''\supseteq M'$ it follows that $M'=M'''$ and hence that $M=M''$ if and only if M is of the form N'. A subspace having either and hence both of these properties is said to be closed. We observe that the operation $'$ sets up a one-to-one inclusion inverting correspondence between the closed subspaces of X and L respectively.

2. **Two lemmas.** The proof of the theorem is based essentially on the following lemma.

LEMMA 1. *Let X_L be a regular linear system such that both X and L are $\aleph_0$ dimensional, that is, have $\aleph_0$ independent generators. Then if M*

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[1] The author is indebted to Professor Murray for letting him read this paper in manuscript form and for several helpful discussions during the preparation of the present note.

[2] Numbers in brackets refer to the bibliography.

[3] By linear we mean additive and homogeneous.

is any closed subspace of X_L *there exists a second closed subspace* N *of* X_L *such that* $M \dotplus N = X$ *and* $M' \dotplus N' = L$.

The proof of Lemma 1 is an easy consequence of a second lemma which is proved in its present form on page 171 of [2] and in other forms elsewhere but which we prove again here for completeness.

LEMMA 2. *Let* X_L *be as in Lemma* 1. *Then there exist sequences of elements* $x_1, x_2, \cdots$ *and* $l_1, l_2, \cdots$ *of* X *and* L *respectively such that* $x_1 \dotplus x_2 \dotplus \cdots = X$, $l_1 \dotplus l_2 \dotplus \cdots = L$ *and* $l_i(x_j) = \delta_i^j$ *for* $i, j = 1, 2, \cdots$.

PROOF. Let $y_1, y_2, \cdots$ and $m_1, m_2, \cdots$ generate X and L respectively. We define $x_1, x_2, \cdots$ and $l_1, l_2, \cdots$ by induction. Let $l_1 = m_1$ and let $x_1 = y_{n_1}/m_1(y_{n_1})$ where n_1 is the first integer such that $m_1(y_{n_1}) \neq 0$. Suppose that $x_1, x_2, \cdots, x_k$ and $l_1, l_2, \cdots, l_k$ have been defined. If k is odd let n_0 be the first integer such that $y_{n_0} \notin x_1 \dotplus x_2 \dotplus \cdots \dotplus x_k$ and let $x_{k+1} = y_{n_0} - (l_k(y_{n_0})x_k + \cdots + l_1(y_{n_0})x_1)$. Then let $\bar{n}$ be the first integer such that $m_{\bar{n}}(x_{k+1}) \neq 0$ and let $l_{k+1} = (m_{\bar{n}} - (m_{\bar{n}}(x_k)l_k + \cdots + m_{\bar{n}}(x_1)l_1))/m_{\bar{n}}(x_{k+1})$. If k is even let n_0 be the first integer such that $m_{n_0} \notin l_1 \dotplus l_2 \dotplus \cdots \dotplus l_k$ and let $l_{k+1} = m_{n_0} - (m_{n_0}(x_k)l_k + \cdots + m_{n_0}(x_1)l_1)$. Then let $\bar{n}$ be the first integer such that $l_{k+1}(y_{\bar{n}}) \neq 0$ and let $x_{k+1} = (y_{\bar{n}} - (l_k(y_{\bar{n}})x_k + \cdots + l_1(y_{\bar{n}})x_1))/l_{k+1}(y_{\bar{n}})$. It follows at once by induction that $l_i(x_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$ and it is clear that $X = x_1 \dotplus x_2 \dotplus \cdots$ and $L = l_1 \dotplus l_2 \dotplus \cdots$.

PROOF OF LEMMA 1. For definiteness we shall assume that M and M' are infinite-dimensional. The only difference in the contrary case is that certain infinite sequences must be replaced by finite ones. That Lemma 2 is true when X and L are finite-dimensional is obvious. Applying Lemma 2 to M and the linear functionals on M defined by the members of L we may infer the existence of sequences of elements $x_1, x_2, x_3, \cdots$ and $m_1, m_2, m_3, \cdots$ of M and L respectively such that $x_1 \dotplus x_2 \dotplus \cdots = M$, $M' \dotplus m_1 \dotplus m_2 \dotplus \cdots = L$ and $m_i(x_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$. Similarly by applying Lemma 2 to M' and the linear functionals on M' defined by members of X and remembering that $M'' = M$ we may infer the existence of sequences of elements $f_1, f_2, f_3, \cdots$ and $z_1, z_2, z_3, \cdots$ of M' and X respectively such that $f_1 \dotplus f_2 \dotplus \cdots = M'$, $M \dotplus z_1 \dotplus z_2 \dotplus \cdots = X$ and $f_i(z_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$. Now for each i and $j = 1, 2, \cdots$ let

$$y_j = z_j - (m_1(z_j)x_1 + m_2(z_j)x_2 + \cdots + m_j(z_j)x_j),$$

$$l_i = m_i - (m_i(z_1)f_1 + m_i(z_2)f_2 + \cdots + m_i(z_{i-1})f_{i-1}),$$

where f_0 and z_0 are to be taken as zero. Then keeping in mind the fact that $f_i(z_j) = m_i(x_j) = \delta_i{}^j$ and $f_i(x_j) = 0$ for $i, j = 1, 2, \cdots$ it is easy to verify that $l_i(y_j) = 0$ and $f_i(y_j) = l_i(x_j) = \delta_i{}^j$. The statement of the lemma now follows at once on setting $N = y_1 \dotplus y_2 \dotplus \cdots$. In fact since $x_1 \dotplus x_2 \dotplus \cdots \dotplus y_1 \dotplus y_2 \dotplus \cdots = x_1 \dotplus x_2 \dotplus \cdots \dotplus z_1 \dotplus z_2 \dotplus \cdots = M \dotplus z_1 \dotplus z_2 \dotplus \cdots = X$ and $l_1 \dotplus l_2 \dotplus \cdots \dotplus f_1 \dotplus f_2 \dotplus \cdots = l_1 \dotplus l_2 \dotplus \cdots \dotplus m_1 \dotplus m_2 \dotplus \cdots = M' \dotplus m_1 \dotplus m_2 \dotplus \cdots = L$, it is easy to see that $N' = l_1 \dotplus l_2 \dotplus \cdots$ and $N'' = y_1 \dotplus y_2 \dotplus \cdots$. Thus $N'' = N$ so that N is closed and $M \dotplus N = X$ while $M' \dotplus N' = L$.

3. **Proof of the theorem.** Since $\mathfrak{B}$ is separable it is clear that $\mathfrak{B} | \mathfrak{M}$ is also.[4] It follows then from Théorème 4 on page 124 of [3] that there exists a countable total set of members of the conjugate of $\mathfrak{B} | \mathfrak{M}$ and a countable total set of members of the conjugate of $\mathfrak{B}$. Now every member of the conjugate of $\mathfrak{B} | \mathfrak{M}$ has associated with it in an obvious fashion a member of the conjugate of $\mathfrak{B}$ which vanishes throughout $\mathfrak{M}$. Thus the first countable total set defines a countable set of elements of the conjugate of $\mathfrak{B}$ the intersections of the null spaces of which is $\mathfrak{M}$. Denote the linear span of these two countable subsets of the conjugate of $\mathfrak{B}$ by L. Since $\mathfrak{M}$ and $\mathfrak{B}$ are separable there exists a dense countable set in $\mathfrak{B}$ a subset of which is a dense set in $\mathfrak{M}$. Let X be the linear span of this countable set and let $M = \mathfrak{M} \cap X$. It is obvious that the X, L and M so defined satisfy the hypotheses of Lemma 1 and that the closures of M and X are $\mathfrak{M}$ and $\mathfrak{B}$ respectively. That M is closed as a subspace of the linear system X_L follows from the fact that $\mathfrak{M}$ is an intersection of null spaces of members of L. Let N be the closed subspace of X_L whose existence is guaranteed by Lemma 1. We define $\mathfrak{N}$ as the closure (in $\mathfrak{B}$) of N. Since $\mathfrak{M} \dotplus \mathfrak{N} \supseteq M \dotplus N = X$ it is clear that $\mathfrak{M} \dotplus \mathfrak{N}$ is dense in $\mathfrak{B}$. Suppose that $x \in \mathfrak{M} \cap \mathfrak{N}$. Since $\mathfrak{M}$ is the closure of M every element in M' vanishes throughout $\mathfrak{M}$. Similarly every element in N' vanishes throughout $\mathfrak{N}$. Thus every element in $M' \dotplus N' = L$ vanishes on x. But L is total. Therefore $x = 0$. Thus $\mathfrak{M} \cap \mathfrak{N} = 0$ and the theorem is proved.

4. **Remarks.** Murray's paper closes with a proof that in reflexive spaces quasi-complements which are not at the same time complements[5] are very non-unique in the sense that every such both properly contains and is properly contained in other quasi-complements. This theorem and its proof may be extended to the nonreflexive case (but not the incomplete one) by considering the closed subspaces of the linear systems $\mathfrak{B}_A$ and $\mathfrak{A}_B$ where $\mathfrak{A}$ is the conjugate of $\mathfrak{B}$ rather than those of the Banach spaces $\mathfrak{A}$ and $\mathfrak{B}$. The needed fact that the linear

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[4] Here $\mathfrak{B} | \mathfrak{M}$ denotes the quotient or difference space of $\mathfrak{B}$ mod $\mathfrak{M}$.

[5] Complements of course are not unique either.

union of a closed subspace with a finite-dimensional one is again closed follows from Theorem III-1 of [2]. The trouble when $\mathfrak{B}$ is not complete is that $\mathfrak{M}'$ and $\mathfrak{N}'$ may be complementary even when $\mathfrak{M}$ and $\mathfrak{N}$ are not. That this can indeed happen is shown by the first example at the bottom of page 173 of [2].

Using this same device one may carry over Murray's theory of the connection between quasi-complements and closed projections almost word for word to the nonreflexive and noncomplete case.

Making further use of the methods and theorems of Chapter III of [2] one can show that if $\mathfrak{B}$ is separable and $\mathfrak{M}$ is a closed subspace of $\mathfrak{B}$ such that neither $\mathfrak{M}$ nor $\mathfrak{B}|\mathfrak{M}$ is finite-dimensional then the quasi-complement of $\mathfrak{M}$ can always be selected so as not to be a complement and that whenever $\mathfrak{B}$ is complete and $\mathfrak{M}$ and $\mathfrak{N}$ are quasi-complementary and not complementary then $\mathfrak{B}$ has infinitely many linearly independent elements mod $\mathfrak{M}+\mathfrak{N}$.

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ON PROXIMATE ORDERS OF INTEGRAL FUNCTIONS

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Let $f(z)$ be an integral function of finite order ρ and let $M(r) = \max_{|z|=r}|f(z)|$. It is possible to find[1] a positive continuous function $\rho(r)$ having the following properties.

(1) $\rho(r)$ is differentiable for $r > r_0$ except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist;

$$\lim_{r\to\infty} \rho(r) = \rho; \tag{2}$$

$$\lim_{r\to\infty} r\rho'(r) \log r = 0; \tag{3}$$

$$\limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho(r)}} = 1. \tag{4}$$

Such a function is called a Lindelöf's proximate order for the integral function $f(z)$. The proof given by Valiron for the existence of proximate orders is based on some rather deep results due to Blumenthal. The object of this note is to give a particularly simple proof of the existence of proximate orders. The proof given here makes no use of the special properties of $M(r)$ and is therefore of wider scope.

Let $\sigma(r) = \log \log M(r)/\log r$. Either (A) $\sigma(r) > \rho$ for a sequence of values of r tending to infinity, or (B) $\sigma(r) \leqq \rho$ for all large r.

In case (A) we define $\phi(r) = \max_{x \geqq r}\{\sigma(x)\}$. Since $\sigma(r)$ is continuous, $\limsup_{r\to\infty} \sigma(r) = \rho$ and $\sigma(r) > \rho$ for a sequence of values of r tending to infinity. Therefore $\phi(r)$ exists. $\phi(r)$ is a nonincreasing function of r.

Let $r_1 > e^{e^e}$ and $\phi(r_1) = \sigma(r_1)$. Such values will exist for a sequence of values of r tending to infinity.

Let $\rho(r_1) = \phi(r_1)$. Let t_1 be the smallest integer not less than $1 + r_1$ such that $\phi(r_1) > \phi(t_1)$, and let $\rho(r) = \rho(r_1) = \phi(r_1)$ for $r_1 < r \leqq t_1$.

Define u_1 as follows:

$$u_1 > t_1,$$

$$\rho(r) = \rho(r_1) - \log\log\log r + \log\log\log t_1 \qquad \text{for } t_1 \leqq r \leqq u_1,$$

$$\rho(r) = \phi(r) \qquad \text{for } r = u_1,$$

but $\rho(r) > \phi(r)$ for $t_1 \leqq r < u_1$. Let r_2 be the smallest value of r for which $r_2 \geqq u_1$ and

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[1] G. Valiron, *Lectures on the general theory of integral functions*, Cambridge, 1923, pp. 64–67.

$$\phi(r_2) = \sigma(r_2).$$

If $r_2 > u_1$, then let $\rho(r) = \phi(r)$ for $u_1 \leqq r \leqq r_2$. Since $\phi(r)$ is constant for $u_1 \leqq r \leqq r_2$, therefore $\rho(r)$ is constant for $u_1 \leqq r \leqq r_2$. We repeat the argument and obtain that $\rho(r)$ is differentiable in adjacent intervals. Further

$$\rho'(r) = 0 \quad \text{or} \quad \frac{-1}{r \log r \log\log r},$$

and $\rho(r) \geqq \phi(r) \geqq \sigma(r)$, for all $r \geqq r_1$. Further $\rho(r) = \sigma(r)$ for an infinity of values of $r = r_1, r_2, \cdots$, $\rho(r)$ is nonincreasing and $\lim_{r\to\infty} \phi(r) = \rho$. Hence

$$\limsup_{r\to\infty} \rho(r) = \lim_{r\to\infty} \rho(r) = \rho,$$

and since $\log M(r) = r^{\sigma(r)} = r^{\rho(r)}$ for an infinity of r, $\log M(r) < r^{\rho(r)}$ for the remaining r, therefore $\limsup_{r\to\infty} \log M(r)/r^{\rho(r)} = 1$.

Let $\sigma(r) \leqq \rho$ for all large r (case (B)). Here there are two possibilities:

$$\text{(B.1)} \qquad \sigma(r) = \rho$$

for at least a sequence of values of r tending to infinity;

$$\text{(B.2)} \qquad \sigma(r) < \rho$$

for all large values of r.

In case (B.1) we take $\rho(r) = \rho$ for all values of r.

In case (B.2) let $\xi(r) = \max_{X \leqq x \leqq r} \{\sigma(x)\}$ where $X > e^{e^e}$ is such that $\sigma(x) < \rho$ whenever $x \geqq X$. $\xi(r)$ is nondecreasing. Take a suitably large value of $r_1 > X$ and let

$$\rho(r_1) = \rho,$$

$$\rho(r) = \rho + \log\log\log r - \log\log\log r_1 \qquad \text{for } s_1 \leqq r \leqq r_1,$$

where $s_1 < r_1$ is such that $\xi(s_1) = \rho(s_1)$. If $\xi(s_1) \neq \sigma(s_1)$, then we take $\rho(r) = \xi(r)$ up to the nearest point $t_1 < s_1$ at which $\xi(t_1) = \sigma(t_1)$. $\rho(r)$ is then constant for $t_1 \leqq r \leqq s_1$.

If $\xi(s_1) = \sigma(s_1)$, then let $t_1 = s_1$.

Choose $r_2 > r_1$ suitably large and let $\rho(r_2) = \rho$,

$$\rho(r) = \rho + \log\log\log r - \log\log\log r_2 \qquad \text{for } s_2 \leqq r \leqq r_2$$

where s_2 $(< r_2)$ is such that[2] $\xi(s_2) = \rho(s_2)$. If $\xi(s_2) \neq \sigma(s_2)$, then let

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[2] s_2 is given by the largest positive root of $\xi(s_2) = \rho - \log\log\log r_2 + \log\log\log s_2$.

$\rho(r)=\xi(r)$ for $t_2\leqq r\leqq s_2$ where t_2 $(<s_2)$ is the point nearest to s_2 at which $\xi(t_2)=\sigma(t_2)$. If $\xi(s_2)=\sigma(s_2)$, then let $t_2=s_2$. For $r<t_2$ let $\rho(r)=\rho(t_2)+\log\log\log t_2-\log\log\log r$ for $u_1\leqq r\leqq t_2$ where u_1 $(<t_2)$ is the point of intersection of $y=\rho$ with $y=\rho(t_2)+\log\log\log t_2-\log\log\log r$.

Let $\rho(r)=\rho$ for $r_1\leqq r\leqq u_1$. It is always possible to choose r_2 so large that $r_1<u_1$. We repeat the procedure and note that

$$\rho(r)\geqq\xi(r)\geqq\sigma(r)$$

and $\rho(r)=\sigma(r)$ for $r=t_1, t_2, t_3, \cdots$. Hence $\lim_{r\to\infty}\rho(r)=\rho$, and

$$\limsup_{r\to\infty}\frac{\log M(r)}{r^{\rho(r)}}=1.$$

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A NOTE ON THE SPECTRAL THEOREM

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1. **Introduction.** Although the connections between the spectral resolution of a self-adjoint transformation in Hilbert space, the moment problem, and Riesz' integral representation [1][1] for linear functionals on the space C are known (cf. Stone [2], Murray [3], Widder [4], Lengyel [5]), the following elementary derivation of the spectral theorem from the Riesz theorem exhibits the connections in, perhaps, the simplest light. We consider only *bounded* self-adjoint transformations H; one can treat an unbounded H by considering $(I+H^2)^{-1}$, which is bounded and self-adjoint [3, p. 95]. Note that the derivation does not involve the separability of the Hilbert space $\mathfrak{H}$.

2. **Six lemmas.** Let H be a self-adjoint transformation with the bounds a, b—that is, $a\|f\|^2\leqq(Hf, f)\leqq b\|f\|^2$ for all $f\in H$, and $\|H\|=\max(|a|, |b|)$. Denote by C the space of continuous real-valued functions defined on the closed interval (a, b), with $\|f(x)\|=\max|f(x)|$ $(a\leqq x\leqq b)$. Let $p(x)=\sum_0^n c_jx^j$ be any polynomial with real coefficients, and let $p(H)$ be the corresponding transformation $p(H)=\sum_0^n c_jH^j$, where $H^0=I$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

LEMMA 1. $\|p(H)\| \leqq \|p(x)\| = \max_{x \in (a,b)} |p(x)|$.

Murray [3, p. 82] gives a proof of the lemma; we remark only that the general case can be reduced to the case of a self-adjoint H in n-dimensional Euclidean space. The inequality is almost obvious when H is represented by a finite diagonal matrix; and the lemma is essentially equivalent to the statement that every n^2-symmetric matrix can be reduced to diagonal form by unitary transformations.

Given any two elements $f, g \in \mathfrak{H}$, consider the expression $(p(H)f, g)$, taking on real or complex values, as an operator defined over the linear set of polynomials included in C. The linearity of the operator is obvious; its continuity follows from Lemma 1:

$$|(p(H)f, g)| \leqq \|p(H)\| \cdot \|f\| \cdot \|g\| \leqq \{\|f\| \cdot \|g\|\} \|p(x)\|.$$

Since the polynomials are dense in C (Weierstrass approximation theorem), the operator can be extended uniquely to a linear functional defined over all C, without increase of norm. The Riesz theorem [1] then yields immediately:

LEMMA 2. *There exists a function of bounded variation* $\rho(\lambda; f, g)$ $(a \leqq \lambda \leqq b)$ *such that*

$$(p(H)f, g) = \int_a^b p(\lambda) d\rho(\lambda; f, g),$$

the uniqueness of ρ *being assured by the normalization conditions*: $\rho(a; f, g) = 0$ *and* $\rho(\lambda; f, g) = \rho(\lambda + 0; f, g)$, $a < \lambda < b$. $\rho(\lambda; f, f)$ *is real, and* $\mathrm{Var}_{(a,b)} \rho(\lambda; f, f) \leqq \|f\|^2$.

The structure of the $\rho(\lambda; f, g)$ is implied by the following elementary properties of the Stieltjes integral (cf. Widder [4]).

LEMMA 3. *Let* $f(x)$, $g(x)$ *be continuous and* $\gamma(x)$ *be a normalized function of bounded variation on* (a, b). *Then* $G(x) = \int_a^x g(t) d\gamma(t)$ *is a normalized function of bounded variation, and* $\int_a^b f(t) dG(t) = \int_a^b f(t) g(t) d\gamma(t)$. *If* $\int_a^b t^n d\gamma(t) = 0$ $(n = 0, 1, 2, \cdots)$, *then* $\gamma(t) \equiv 0$.

LEMMA 4. (1) $\rho(b; f, g) = (f, g)$, $0 \leqq \rho(\lambda; f, f) \leqq \|f\|^2$;
(2) $\rho(\lambda; f_1 + f_2, g) = \rho(\lambda; f_1, g) + \rho(\lambda; f_2, g)$;
(3) $\rho(\lambda; cf, g) = c\rho(\lambda; f, g)$;
(4) $\rho(\lambda; f, g) = \bar{\rho}(\lambda; g, f)$.

The equality in (1) arises on setting $p(\lambda) = 1$ in Lemma 2. But then $\rho(\lambda; f, f)$ is nondecreasing, whence the inequality. The proof of (2) is typical of the remaining statements:

$$\begin{aligned}\int_a^b \lambda^n d\rho(\lambda; f_1 + f_2, g) &= (H^n(f_1 + f_2), g) = (H^n f_1, g) + (H^n f_2, g)\\ &= \int_a^b \lambda^n d\rho(\lambda; f_1, g) + \int_a^b \lambda^n d\rho(\lambda; f_2, g)\\ &= \int_a^b \lambda^n d\{\rho(\lambda; f_1, g) + \rho(\lambda; f_2, g)\}.\end{aligned}$$

Set $\gamma(\lambda) = \rho(\lambda; f_1 + f_2, g) - \{\rho(\lambda; f_1, g) + \rho(\lambda; f_2, g)\}$ and apply Lemma 3.

Lemma 4 and the representation theorem [2, p. 63] for bounded, bilinear symmetric functionals defined over a Hilbert (or unitary) space yield the following theorem:

LEMMA 5. *There exists a set of bounded self-adjoint transformations* $F(\lambda)$ $(a \leqq \lambda \leqq b)$ *such that* $(F(\lambda)f, g) = \rho(\lambda; f, g)$. $F(a) = 0$, *and* $F(b) = I$.

That the $F(\lambda)$ are projections is a consequence of the following result:

LEMMA 6. $F(\mu)F(\lambda) = F(\nu)$, *where* $\nu = \min(\mu, \lambda)$. *In particular,* $F^2(\lambda) = F(\lambda)$.

For, let m and n be arbitrary $(m, n = 0, 1, 2, \cdots)$. Then

$$\begin{aligned}\int_a^b \lambda^n d\left\{\int_a^\lambda \mu^m d(F(\mu)f, g)\right\} &= \int_a^b \lambda^n \cdot \lambda^m d(F(\lambda)f, g) = (H^{n+m}f, g)\\ &= (H^n f, H^m g) = \int_a^b \lambda^n d(F(\lambda)f, H^m g)\\ &= \int_a^b \lambda^n d(H^m F(\lambda)f, g)\\ &= \int_a^b \lambda^n d\left\{\int_a^b \mu^m d(F(\mu)F(\lambda)f, g)\right\}.\end{aligned}$$

Application of Lemma 3 yields

$$\int_a^b \mu^m d(F(\mu)F(\lambda)f, g) = \int_a^\lambda \mu^m d(F(\mu)f, g) = \int_a^b \mu^m d(F(\nu)f, g);$$

and another application, $F(\mu)F(\lambda) = F(\nu)$, where $\nu = \min(\mu, \lambda)$.

3. **The spectral theorem.** In order to put our results into standard form we modify the definition of the family of projections by setting

$E(\lambda)=0$ $(\lambda<a)$, $E(\lambda)=F(\lambda+0)$ $(a\leqq\lambda<b)$, $E(\lambda)=I$ $(\lambda\geqq b)$. The $E(\lambda)$ thus obtained form a *finite resolution of the identity*: They are all projections; $E(\lambda)E(\mu)=E(\nu)$, where $\nu=\min(\mu, \lambda)$; $E(\lambda+0)=E(\lambda)$ $(-\infty<\lambda<\infty)$; $(E(\lambda)f, g)$ is of bounded variation; $E(\lambda)=0$ $(\lambda<a)$ and $E(\lambda)=I$ $(\lambda\geqq b)$.

To allow for the change in definition at the point $\lambda=a$ we modify the limits of integration from (a, b) to $(a-\epsilon, b)$, ϵ being an arbitrary positive number. Our final result is then the following:

SPECTRAL THEOREM. *Let H be a bounded self-adjoint transformation with bounds a, b. Then there exists a finite resolution of the identity $E(\lambda)$ such that*

$$(p(H)f, g) = \int_{a-\epsilon}^{b} p(\lambda)d(E(\lambda)f, g),$$

ϵ being an arbitrary positive number.

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AREOLAR MONOGENIC FUNCTIONS

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1. **Introduction.** There have been several modifications of the definition of the derivative of a complex function of a complex variable which lead to theories of non-analytic functions. These generalizations were initiated by Riemann (1851) and Picard (1892) and followed with others by Pompeiu, Kasner and Cioranescu. The general derivatives of Riemann and Cioranescu depend on direction and have an infinite set of values at a given point, hence Kasner gave to the class of non-analytic functions considered the name polygenic functions to distinguish them from classical analytic, monogenic functions.[1]

The conditions for classical monogenity have been much reduced by Looman-Menchoff [7, pp. 9–16; 9, pp. 198–201].[2] We shall similarly reduce the restrictions for the existence of the Cioranescu single-valued areolar derivative and show that under those reduced conditions the real and imaginary parts of the areolar monogenic function are biharmonic. Finally the class of areolar monogenic functions so determined will be simply characterized in terms of the Pompeiu derivative.

2. **The Cioranescu and Pompeiu derivatives.** Let $f(z)=f(x, y)=u(x, y)+iv(x, y)$ be defined in a domain D of the complex variable $z=x+iy$. Construct a rectangle in D at a point z of D whose vertices in positive order are z, z_1, z', z_2. If z is taken as the pole of a polar coordinate system (ρ, ϕ) then $z_1-z=\rho_1e^{i\phi}$, $z_2-z=\rho_2e^{i(\phi+\pi/2)}$ and $z'-z=(\rho_1^2+\rho_2^2)^{1/2}e^{i(\phi+\alpha)}$ where $\alpha=\tan^{-1}\rho_2/\rho_1$. We now form the quotient

$$\Delta^2f(z) = \frac{f(z') - f(z_1) - f(z_2) + f(z)}{(z_1 - z)(z_2 - z)} \tag{2.1}$$

and consider the limit of $\Delta^2f(z)$ as ρ_1 and ρ_2 approach zero with ϕ held

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[1] See E. R. Hedrick, *Non-analytic functions of a complex variable*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 75–96 for an extensive bibliography. The author is indebted to the referee for the following observation. "Calugareano studied the second derivative of a polygenic function for only one rectilinear path of approach; Nicolesco studied it for any two different rectilinear paths of approach and Cioranescu considered the limit for two mutually perpendicular, rectilinear paths. Kasner and DeCicco have studied the geometry of the second derivative for a general curvilinear path of approach."

[2] Numbers in brackets refer to the references cited at the end of the paper.

constant. If $u(x, y)$ and $v(x, y)$ have continuous second order partial derivatives at z, the limit exists and we have

$$\begin{aligned}\underset{z_1\to z,\, z_2\to z}{L} \Delta^2 f(z) &= D^2 f(z)\\ &= (f_{xx} - f_{yy} - 2if_{xy})/4 - (f_{xx} - f_{yy} + 2if_{xy})e^{-4i\phi}/4\\ &= [(u_{xx} - u_{yy} + 2v_{xy}) + i(v_{xx} - v_{yy} - 2u_{xy})]/4\\ &\quad - [(u_{xx} - u_{yy} - 2v_{xy}) + i(v_{xx} - v_{yy} + 2u_{xy})]e^{-4i\phi}/4.\end{aligned} \tag{2.2}$$

The necessary and sufficient conditions that $D^2f(z)$ shall be independent of ϕ are that

$$u_{xy} = -(v_{xx} - v_{yy})/2, \qquad v_{xy} = (u_{xx} - u_{yy})/2. \tag{2.3}$$

If (2.3) are satisfied then $D^2f(z) = v_{xy} - iu_{xy}$ is the Cioranescu derivative.

If now $u(x, y)$ and $v(x, y)$ are assumed to have fourth order partial derivatives at z then from (2.3) we have $\nabla^4 u(x, y) = \nabla^4 v(x, y)$ where $\nabla^4 = \nabla^2(\nabla^2)$ and ∇^2 is the Laplacian operator.

We shall make the following definition.

Definition. A complex function $f(z) = u(x, y) + iv(x, y)$ defined in a domain D will be said to be areolar monogenic at a point (x, y) of D if $u(x, y)$ and $v(x, y)$ have continuous second order partial derivatives at (x, y) which satisfy (2.3). We shall call $f(z)$ an areolar monogenic function in D if these conditions are satisfied at every point of D.

The Pompeiu areolar derivative is defined as

$$\frac{Df(z)}{Dw} = \underset{\delta=0}{L} \frac{1}{2iw} \int_s f(z)dz \tag{2.4}$$

where s is any simple rectifiable closed curve of diameter δ which with its interior σ is in D and w is the area of σ. If $u(x, y)$ and $v(x, y)$ are continuous with their first and second order partial derivatives we have

$$\frac{Df}{Dw} = [(u_x - v_y) + i(v_x + u_y)]/2 \tag{2.5}$$

and

$$\frac{D^2f}{Dw^2} = [-v_{xy} + (u_{xx} - u_{yy})/2 + i\{u_{xy} + (v_{xx} - v_{yy})/2\}]/2. \tag{2.6}$$

As remarked by Cioranescu [2, p. 29] we see by (2.6) that the class of areolar monogenic functions is the class of solutions of the differential equation $D^2f/Dw^2 = 0$.

3. **Green's lemma.** Most discussions of monogeneity lead to the evaluation of $\int_{C(R)} f(z)dz$ where the integral is a line integral around a closed curve in D which in our case we shall assume is a rectangle with sides parallel to the coordinate axes. This integral is then transformed by Green's lemma into integrals over the area of the rectangle. A reduction of conditions under which

$$(3.1)\qquad \int_{C(R)} f(z)dz = -\iint_R (u_y + v_x)dxdy + i\iint_R (u_x - v_y)dxdy$$

then leads to reduced conditions for monogeneity. We shall give three conditions for (3.1) to hold for all rectangles $C(R)$ in D.

(3.2) *Condition* C. If $u(x, y)$, $v(x, y)$, u_x, u_y, v_x and v_y are continuous in D, then (3.1) holds for all rectangles in D.

(3.3) *Condition* A. If $F(R) = \int_{C(R)} udy$, $G(R) = -\int_{C(R)} udx$ with similar conditions for $v(x, y)$ and F and G are absolutely continuous functions of point sets in D, then (3.1) holds for all rectangles $C(R)$ in D.

This is condition (A) of Evans [4, p. 32].

(3.4) *Condition* M. If u_x, u_y, v_x and v_y exist almost everywhere in D and are summable and, moreover, at each point of D except for at most a finite or denumerably infinite set, the functions $u(x, y)$ and $v(x, y)$ have finite Dini derivatives with respect to x and y, then (3.1) holds for all rectangles $C(R)$ in D.

This condition among others was given by Menchoff [5, p. 29].

4. **The principal theorem.** We are now in a position to prove the following theorem.

THEOREM 1. *If $u(x, y)$ and $v(x, y)$ satisfy conditions* C, A *or* M *in D and the equations*

$$(4.1)\qquad u_x - v_y = U(x, y), \qquad u_y + v_x = V(x, y)$$

hold almost everywhere (everywhere under condition C*) in D, where $U(x, y)$ and $V(x, y)$ are conjugate harmonic in D, then $f(z) = u(x, y) + iv(x, y)$ is areolar monogenic in D.*

PROOF. Under the conditions C, A or M the differential equations (4.1) are equivalent to the integral equations

$$(4.2)\qquad \begin{aligned} \int_{C(R)} vdx + udy &= \iint_R U(x, y)dxdy, \\ \int_{C(R)} vdy - udx &= \iint_R V(x, y)dxdy. \end{aligned}$$

These are the integral equations considered by Evans [4]. Let

$$(4.3)\qquad \begin{aligned} \mu(x, y) &= \mu(M) = -\frac{1}{2\pi}\iint_D \log\frac{1}{MP}\,U(P)d\sigma_P,\\ \nu(x, y) &= \nu(M) = -\frac{1}{2\pi}\iint_D \log\frac{1}{MP}\,V(P)d\sigma_P. \end{aligned}$$

Since $U(x, y)$ and $V(x, y)$ are harmonic in D they satisfy a Hölder condition [6, p. 153] and the first and second partial derivatives of $\mu(M)$ and $\nu(M)$ exist at M and

$$(4.4)\qquad \nabla^2\mu(x, y) = U(x, y), \qquad \nabla^2\nu(x, y) = V(x, y).$$

The general solution of (4.2) is

$$(4.5)\qquad u(x, y) = \frac{\partial\mu}{\partial x} + \frac{\partial\nu}{\partial y} + \frac{\partial\psi}{\partial y}, \qquad v(x, y) = \frac{\partial\nu}{\partial x} - \frac{\partial\mu}{\partial y} + \frac{\partial\psi}{\partial x},$$

where $\psi(x, y)$ is (in D) an arbitrary harmonic function. Now by the theorem of Evans [4, p. 33], (4.5) is a solution of (4.1) almost everywhere in D when $U(x, y)$ and $V(x, y)$ are only bounded and measurable in the Lebesgue sense. In our case with $U(x, y)$ and $V(x, y)$ harmonic in D, (4.5) is a solution of (4.1) everywhere in D and any solution of (4.1) satisfying C, A, or M can be given the form (4.5). We have from (4.5) and (4.4)

$$(4.6)\quad u_x - v_y = \nabla^2\mu(x, y) = U(x, y), \quad u_y + v_x = \nabla^2\nu(x, y) = V(x, y)$$

at all points (x, y) of D and therefore $\mu(x, y)$ and $\nu(x, y)$ satisfy

$$(4.7)\quad \nabla^4\mu(x, y) = \nabla^2U(x, y) = 0, \qquad \nabla^4\nu(x, y) = \nabla^2V(x, y) = 0$$

at all points (x, y) of D. Solutions $\mu(x, y)$ and $\nu(x, y)$ of (4.7) are analytic in D and since $\mu(x, y)$, $\nu(x, y)$ and $\psi(x, y)$ are all solutions of $\nabla^4w=0$ their first partial derivatives are also biharmonic and therefore analytic in D. Therefore $u(x, y)$ and $v(x, y)$ by (4.5) are biharmonic and analytic in D. Now from (4.1) we derive that

$$\begin{aligned} 2u_{xy} + (v_{xx} - v_{yy}) &= U_y + V_x = 0,\\ 2v_{xy} - (u_{xx} - u_{yy}) &= - U_x + V_y = 0 \end{aligned}$$

and therefore $f(z)$ is areolar monogenic in D.

COROLLARY. *If $u(x, y)$ and $v(x, y)$ satisfy the conditions* C, A *or* M *in D and the Pompeiu derivative exists for all rectangles $C(R)$ in D and $Df/DR=[U(x, y)+iV(x, y)]/2=F(z)/2$ almost everywhere in D and $F(z)$ is monogenic in the ordinary sense, then $D^2f(z)/DR^2=0$ at all points of D and $f(z)$ is areolar monogenic.*

The following theorem is an analog of the first theorem of Menchoff [7, p. 9] relative to ordinary monogeneity.

THEOREM 2. *The function $f(z) = u(x, y) + iv(x, y)$ is areolar monogenic in a domain D if $u(x, y)$ and $v(x, y)$ satisfy condition* C *in D and if the second partial derivatives of $u(x, y)$ and $v(x, y)$ exist, are finite and satisfy the conditions* (2.3) *everywhere in D except for a point set E which consists of at most a finite or denumerable infinity of points.*

PROOF. Let $U(x, y) = u_x - v_y$, $V(x, y) = u_y + v_x$. Then by (2.3) we have on $D - E$

$$
\begin{aligned}
U_x - V_y &= -2v_{xy} + u_{xx} - u_{yy} = 0, \\
U_y + V_x &= 2u_{xy} + v_{xx} - v_{yy} = 0.
\end{aligned}
\tag{4.8}
$$

By the theorem of Menchoff [7, p. 9] $U(x, y)$ and $V(x, y)$ are conjugate harmonic functions in D and $F(z) = U(x, y) + iV(x, y)$ is a monogenic function in the ordinary sense. Therefore, by Theorem 1, $f(z)$ is areolar monogenic.

COROLLARY. *If $u(x, y)$ and $v(x, y)$ satisfy condition* C *in D and the second Pompeiu derivative exists, is finite and equal to zero at all points of D except for at most a finite or denumerably infinite set, then $f(z)$ is areolar monogenic in D.*

5. **An analog of Morera's theorem.** We have the following analog of the theorem of Morera.

THEOREM 3. *A necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that it be continuous, and for all circles $C(x_0, y_0; r)$ with center (x_0, y_0) and radius r in D*

$$
\int_C f(z)dz = \pi r^2 i[U(x_0, y_0) + iV(x_0, y_0)]
\tag{5.1}
$$

where $U(x, y)$ and $V(x, y)$ are given conjugate harmonic functions in D.

PROOF. *Necessity.* If $f(z)$ is areolar monogenic in D we let

$$
u_x - v_y = U(x, y), \qquad u_y + v_x = V(x, y)
\tag{5.2}
$$

and, by (5.1) and (2.3), $U(x, y)$ and $V(x, y)$ are conjugate harmonic in D. Therefore we have

$$
\begin{aligned}
\int_C f(z)dz &= \int_C udx - vdy + i\int_C vdx + udy \\
&= i\iint_A [U(x, y) + iV(x, y)]dxdy.
\end{aligned}
\tag{5.3}
$$

Now by the mean value property of harmonic functions we have from (5.2) that (5.1) is satisfied for all circles $C(x_0, y_0; r)$ in D.

Sufficiency. From (5.1) by the mean value property of harmonic functions we have (5.3) and therefore

$$\begin{aligned} \int_C v dx + u dy &= \iint_A U(x, y) dx dy, \\ \int_C v dy - u dx &= \iint_A V(x, y) dx dy. \end{aligned} \tag{5.4}$$

Now by an argument similar to that for Theorem 1, $f(z)$ is areolar monogenic.

COROLLARY. *The class of ordinary monogenic functions is a subclass of areolar monogenic functions* [1, pp. 264–265].

In this case $U(x, y)+iV(x, y)\equiv 0$ in D and by (5.1), $\int_C f(z)dz=0$ for all circles $C(x_0, y_0; r)$ in D. The solution of (5.4) with the right-hand member zero is given by (4.6) with $\mu\equiv\nu\equiv 0$. Therefore $u(x, y)$ and $v(x, y)$ are conjugate harmonic in D and $f(z)$ is monogenic in the classical sense.

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APPROXIMATION IN THE SENSE OF LEAST pTH POWERS WITH A SINGLE AUXILIARY CONDITION OF INTERPOLATION

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Introduction. Let $w=g(z)$ map the interior of the analytic Jordan curve C conformally into the interior of the circle $|w|=1$. We shall say that the function $f(z)$, analytic interior to C, is of class E_p there if $\int_{C_r}|f(z)|^p|dz|$ is bounded for $r<1$, where C_r is the curve $|g(z)|=r$. A function is of class H_p if it is analytic for $|z|<1$, and of class E_p there. We are taking $p>0$.

Of the functions $f(z)$, analytic interior to C, of class E_p there, with $f(\alpha)=A$ ($z=\alpha$ a point interior to C, A an arbitrary constant), let $F_0(z)$ be the one[1] which minimizes the integral $\int_C|f(z)|^p|dz|$. Let $P_n(z)$ be the minimizing polynomial of degree n with $P_n(\alpha)=A$, for this integral. We shall prove that the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, *converges maximally* to $F_0(z)$ on the closed set Γ, consisting of C and its interior, and then derive some extensions. We use the term *maximal convergence* in the sense of J. L. Walsh [3, p. 80].[2]

We denote by ρ the maximum value of R such that the minimizing function can be extended so as to be analytic and single-valued interior to Γ_R, as used by Walsh [3, p. 80].

1. Inequalities: unit circle.

We shall start with the results for the unit circle.

THEOREM 1.1. *Of the functions $f(z)$ of class H_p ($p>0$) interior to C: $|z|=1$, with $f(\alpha)=A$, $|\alpha|<1$, the one which minimizes the integral $\int_C|f(z)|^p|dz|$ is given by $F_0(z)=A[(|\alpha|^2-1)/(\bar{\alpha}z-1)]^{2/p}$, with the branch for which $F_0(\alpha)=A$.*

For, it is true that $\int_C|f(z)|^p|dz|\geqq 2\pi|f(0)|^p$, so that for the case $\alpha=0$, the minimizing function is $F_0(z)=A$. If, in the general case, we map the interior of the unit circle conformally into itself, with $z=\alpha$ corresponding to the origin, the desired result is obtained.

The main new tool exhibited by the paper is the following theorem.

THEOREM 1.2. *Let $f_n(z)$ be a sequence of functions of class H_p with $f_n(\alpha)=A$ and $\int_C|f_n(z)|^p|dz|\leqq\int_C|F_0(z)|^p|dz|+\epsilon_n$, where $\epsilon_n\to 0$ as*

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[1] The minimizing function $F_0(z)$ is unique.

[2] Numbers in brackets refer to the references cited at the end of the paper.

$n \to \infty$. *If D is any closed region lying entirely interior to C, there is a constant M, depending only on D, such that*

$$|f_n(z) - F_0(z)| \leqq M\epsilon_n^{1/2}, \qquad \textit{for } z \textit{ in } D.$$

This implies continuous convergence of $f_n(z)$ to $F_0(z)$ interior to C, and gives a measure of the degree of convergence.

We start the proof with the case $\alpha = 0$, $A = 1$. Then

$$\int_C |f_n(z)|^p |dz| \leqq 2\pi + \epsilon_n, \qquad f_n(0) = 1.$$

With the further restriction that $f_n(z) \neq 0$ for $|z| < 1$, the functions $f_n(z)^{p/2}$ are of class H_2 and have the expansions $1 + \sum_{i=1}^{i=\infty} a_{i,n} z^i$, with $\sum_{i=1}^{i=\infty} |a_{i,n}|^2$ convergent, from which, by Parseval's Theorem

$$\int_C |f_n(z)^{p/2}|^2 |dz| = 2\pi\left[1 + \sum_{i=1}^{\infty} |a_{i,n}|^2\right] \leqq 2\pi + \epsilon_n,$$

so that we get the inequality in

$$\int_C |f_n(z)^{p/2} - 1|^2 |dz| = 2\pi \sum_{i=1}^{\infty} |a_{i,n}|^2 \leqq \epsilon_n.$$

By a lemma of J. L. Walsh [**3**, p. 101] this implies, for some constant M',

$$|f_n(z)^{p/2} - 1| \leqq M'\epsilon_n^{1/2}, \qquad z \text{ in } D,$$

where M' depends only on D, and we can write

$$f_n(z) - 1 = [1 + \eta_n(z)]^{2/p} - 1, \qquad |\eta_n(z)| \leqq M'\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

For n sufficiently large, the binomial expansion may be used, and there is a constant M'' such that, for some integer N,

$$|f_n(z) - 1| \leqq M'' \cdot \max_{z \text{ in } D} |\eta_n(z)|, \qquad n > N.$$

This gives the desired inequality for n sufficiently large, and a suitable choice of the constant makes it valid for all n.

If now the functions $f_n(z)$ do vanish interior to C, let $B_n(z)$ be the Blaschke product for the zeros of $f_n(z)$ interior to C, normalized so that $B_n(0) > 0$. We write

$$(1) \qquad f_n(z) = B_n(z) \cdot \psi_n(z)$$

where $\psi_n(z) \neq 0$ interior to C, and $\psi_n(z)$ is of class H_p for each n. Since

$|B_n(z)|$ has boundary values equal to unity a.e. on C,

$$\int_C |f_n(z)|^p |dz| = \int_C |\psi_n(z)|^p |dz| \leqq 2\pi + \epsilon_n,$$

from which, setting $\Psi_n(z) = \psi_n(z)/\psi_n(0)$,

$$\int_C |\Psi_n(z)|^p |dz| \leqq (2\pi + \epsilon_n)/[\psi_n(0)]^p. \tag{2}$$

From (1) we have $\psi_n(0) \geqq 1$ and hence from (2), $\int_C |\Psi_n(z)|^p |dz| \leqq 2\pi + \epsilon_n$. The first part of the proof now applies to $\Psi_n(z)$ and gives, for some constant M',

$$|\Psi_n(z)^{p/2} - 1| \leqq M' \epsilon_n^{1/2}, \qquad z \text{ in } D. \tag{3}$$

Since the left member of (2) is not less than 2π, we have

$$[\psi_n(0)]^{p/2} \leqq (1 + \epsilon_n/2\pi)^{1/2} \tag{4}$$

which, with (3), means that for some M''

$$|\psi_n(z)^{p/2} - 1| \leqq M'' \cdot \epsilon_n^{1/2}, \qquad z \text{ in } D,$$

and, as in the first part of the proof,

$$|\psi_n(z) - 1| \leqq M \epsilon_n^{1/2}, \qquad z \text{ in } D. \tag{5}$$

We now desire a measure of the degree of convergence of $B_n(z)$. We have a.e. on C,

$$|B_n(z) - 1|^2 = |B_n(z)|^2 + 1 - 2\Re[B_n(z)],$$

so that, since $B_n(0)$ is real, and the Cauchy integral formula applies to $B_n(z)$ on C,

$$\int_C |B_n(z) - 1|^2 |dz| = 4\pi[1 - B_n(0)].$$

By (1) and (4), $B_n(0) \geqq (1 + \epsilon_n/2\pi)^{-1/p}$, so that for some constant M',

$$\int_C |B_n(z) - 1|^2 |dz| \leqq M' \epsilon_n, \qquad n = 0, 1, 2, \cdots,$$

and by the lemma used previously, we have for some M,

$$|B_n(z) - 1| \leqq M \epsilon_n^{1/2}, \qquad z \text{ in } D. \tag{6}$$

In (5) and (6), M is not necessarily the same, but the two inequalities imply the existence of a constant M, depending only on the region D, such that $|f_n(z)-1| \leqq M\epsilon_n^{1/2}$, z in D.

It is clear that this inequality holds in the case $f_n(0)=A$, where the convergence is to the function identically equal to A.

Let us now return to the original problem, where

$$f_n(\alpha) = A, \qquad F_0(z) = A[(|\alpha|^2 - 1)/(\bar{\alpha}z - 1)]^{2/p}.$$

Here we have

$$\int_C |f_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + \epsilon_n. \tag{7}$$

If we make the transformation $w=(z-\alpha)/(1-\bar{\alpha}z)$, (7) becomes

$$\int_{C'} |F_n(w)|^p |dw| \leqq 2\pi |A|^p + \epsilon_n/(1 - |\alpha|^2), \qquad C': |w| = 1,$$

where

$$F_n(w) = (1 + \bar{\alpha}w)^{-2/p}\cdot f_n[(w + \alpha)/(1 + \alpha w)]$$

and since $F_n(0)=A$, this is precisely the situation already treated. The closed region D in the z-plane corresponds in the w-plane to a closed region D' which lies completely interior to C'. Hence, for some constant M',

$$|F_n(w) - A| \leqq M'\epsilon_n^{1/2}, \qquad w \text{ in } D',$$

and we have finally for some constant M,

$$|f_n(z) - F_0(z)| \leqq M\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

Continuous convergence interior to C for the case $f_n(0)=1$ was proved by Keldysch and Lavrentieff [1, p. 35], but no attempt was made to determine the degree of convergence.

2. **Maximal convergence: unit circle.** We now estimate the ϵ_n for minimizing polynomials.

THEOREM 2.1. *Let $P_n(z)$ be the polynomial of degree n with $P_n(\alpha)=A$ for which $\int_C|P_n(z)|^p|dz|$ is a minimum. Then, given any number R such that $1<R<\rho$, there exists a constant M, not depending on n, such that in the inequality*

$$\int_C |P_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + \epsilon_n, \qquad C: |z| = 1,$$

we may take $\epsilon_n \leqq M/R^{2n}$.

Let $\pi_n(z)$ be the polynomial of degree n, of best approximation to $F_0(z)$ on Γ in the sense of least maximum modulus, with the condition $\pi_n(\alpha)=A$. The $\pi_n(z)$ converge maximally to $F_0(z)$ on Γ [3, §11.2], so that for any R as above there is a constant M such that

$$|\pi_n(z)-F_0(z)| \leqq M/R^n, \qquad z \text{ on } \Gamma.$$

This implies first of all that for n sufficiently large, the $\pi_n(z)$ do not vanish interior to C. Let us now write

$$\pi_n(z) = F_0(z) + R_n(z)$$

from which

$$\pi_n(z)^{p/2} = F_0(z)^{p/2}[1 + R_n(z)/F_0(z)]^{p/2}, \qquad |R_n(z)| \leqq M/R^n, \qquad z \text{ in } \Gamma.$$

Since $F_0(z)$ is bounded from zero on Γ, the expansion of the binomial is valid for n sufficiently large, and there is a constant M' and an integer N such that

$$|\pi_n(z)^{p/2}-F_0(z)^{p/2}| \leqq M'\cdot \max_{z \text{ in } \Gamma} |R_n(z)|, \qquad z \text{ in } \Gamma, \quad n > N,$$

or

$$|\pi_n(z)^{p/2}-F_0(z)^{p/2}| \leqq M''/R^n, \qquad z \text{ in } \Gamma, \quad n > N. \tag{8}$$

The function $F_0(z)^{p/2}-\pi_n(z)^{p/2}$ is analytic on Γ for each n which is large enough, and vanishes for $z=\alpha$. Hence it is orthogonal to $(1-\bar{\alpha}z)^{-1}$ on C, by which we get

$$\int_C [F_0(z)^{p/2}-\pi_n(z)^{p/2}]\cdot\bar{F}_0(z)^{p/2}|dz| = 0. \tag{9}$$

Thus,

$$\begin{aligned}\int_C |F_0(z)^{p/2}-\pi_n(z)^{p/2}|^2|dz| &= \int_C [F_0(z)^{p/2}-\pi_n(z)^{p/2}][\bar{F}_0(z)^{p/2}-\bar{\pi}_n(z)^{p/2}]|dz| \\ &= -\int_C F_0(z)^{p/2}\bar{\pi}_n(z)^{p/2}|dz| + \int_C |\pi_n(z)|^p|dz|.\end{aligned} \tag{10}$$

But by (9), $\int_C F_0(z)^{p/2}\bar{\pi}_n(z)^{p/2}|dz| = \int_C |F_0(z)|^p|dz|$, so that (10) becomes

$$\int_C |\pi_n(z)|^p|dz| = \int_C |F_0(z)|^p|dz| + \int_C |F_0(z)^{p/2}-\pi_n(z)^{p/2}|^2|dz|.$$

Applying (8) we have for some M''' and n large enough

$$\int_C |\pi_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + M'''/R^{2n}.$$

But since $P_n(z)$ is the minimizing polynomial of degree n for $P_n(\alpha) = A$, this inequality is also true for $P_n(z)$, and all n, and the theorem is proved.

THEOREM 2.2. *The sequence of minimizing polynomials* $P_n(z)$ *converges maximally to* $F_0(z)$ *on* Γ.

It must be shown that given R, with $1 < R < \rho$, there exists M such that $|P_n(z) - F_0(z)| \leqq M/R^n$, z in Γ. Choose an R_1 such that $R < R_1 < \rho$. We can then find a closed region D, lying completely interior to C, such that $D_{R_1/R}$ contains Γ in its interior [3, §2.2, Theorem 2]. By Theorem 2.1 we have

$$\int_C |P_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + M'/R_1^{2n}.$$

But then by Theorem 1.2,

$$|P_n(z) - F_0(z)| \leqq M''/R_1^n, \qquad z \text{ in } D,$$

which gives [3, §4.7, Theorem 8, corollary]

$$|P_n(z) - F_0(z)| \leqq M'''/R^n, \qquad z \text{ in } D_{R_1/R}.$$

This inequality, being valid on $D_{R_1/R}$, is valid on its subset Γ, and the theorem is proved.

3. **Maximal convergence: analytic Jordan curve.** Let C be an arbitrary analytic Jordan curve of the z-plane, and $z = \alpha$ a point interior to C. Let $w = g(z)$, $z = h(w)$ map the interior of C conformally into the interior of C': $|w| = 1$ so that $z = \alpha$ corresponds to $w = 0$. Let Γ consist of C and its interior.

THEOREM 3.1. *Among the functions* $f(z)$ *of class* $E_p (p > 0)$ *interior to* C, *with* $f(\alpha) = A$, *the minimum of* $\int_C |f(z)|^p |dz|$ *occurs for the function* $F_0(z) = A[g'(z)/g'(\alpha)]^{1/p}$.

For, with the transformation $w = g(z)$, we have

$$\int_C |f(z)|^p |dz| = \int_{C'} |f[h(w)] \cdot h'(w)^{1/p}|^p |dw|.$$

The function $f[h(w)] \cdot h'(w)^{1/p}$ is of class H_p interior to C' and takes on the value $A\cdot h'(0)^{1/p}$ at $w=0$. Hence the minimum of the integral on the right occurs when $F_0[h(w)]\cdot h'(w)^{1/p}=A\cdot h'(0)^{1/p}$, which in the z-plane gives $F_0(z)$ as above.

This result in a restricted form is due to Julia,[3] and in more general form to Keldysch and Lavrentieff.

The previous methods carry over to the present situation. Thus, suppose we have the inequality of Theorem 1.2, where the $f_n(z)$ are of class E_p interior to C, $f_n(\alpha)=A$, and $\epsilon_n\to 0$ as $n\to\infty$. In the w-plane this becomes

$$\int_{C'} | f_n[h(w)]\cdot h'(w)^{1/p}|^p | dw | \leqq \int_{C'} | F_0[h(w)]\cdot h'(w)^{1/p}|^p | dw | + \epsilon_n.$$

The hypotheses of Theorem 1.2 are satisfied, and a closed region D entirely interior to C corresponds to a closed region D' entirely interior to C'; then there is a constant M such that

$$| f_n[h(w)]\cdot h'(w)^{1/p} - F_0[h(w)]\cdot h'(w)^{1/p} | \leqq M\epsilon_n^{1/2}, \qquad w \text{ in } D'.$$

But $h'(w)^{1/p}$ is bounded from zero on Γ, so that in the z-plane this becomes

$$| f_n(z) - F_0(z) | \leqq M'\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

THEOREM 3.2. *Theorem* 1.2 *is valid if all statements in it now refer to an analytic Jordan curve.*

In estimating the ϵ_n for the minimizing polynomials $P_n(z)$, we use polynomials $\pi_n(z)$ of best approximation to $F_0(z)$ in the sense of least maximum modulus on Γ, subject to $\pi_n(\alpha)=A$. The proofs carry over, and we have the general result as follows.

THEOREM 3.3. *Let Γ be a closed region bounded by the analytic Jordan curve C, and let $z=\alpha$ be a point interior to C. If $P_n(z)$ is the minimizing polynomial of degree n, in the sense of least pth powers ($p>0$), to the function $f(z)=0$, on C, with $P_n(\alpha)=A$, then the sequence $P_n(z)$ converges maximally on Γ to the minimizing function $F_0(z)=A[g'(z)/g'(\alpha)]^{1/p}$.*

4. Approximation to $(z-\alpha)^{-1}$: analytic Jordan curve. The results already obtained extend to polynomials of best approximation to certain rational functions.

THEOREM 4.1. *Let Γ be a closed region bounded by the analytic Jordan curve C, and $z=\alpha$ a point interior to C. The function of class E_p interior*

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[3] Julia, G. *Leçons sur la représentation conforme des aires simplement connexes*, Paris, 1931.

to C which minimizes the integral

$$\int_C |(z-\alpha)^{-1} - f(z)|^p \, |dz|$$

is given by

$$F_0(z) = \frac{1}{z-\alpha} - \frac{g'(z)^{1/p}}{g'(\alpha)^{1/p-1}\cdot g(z)}$$

where $w = g(z)$ *is the mapping function of* §3.

It is seen that $F_0(z)$ is analytic at $z=\alpha$ if properly defined there.

THEOREM 4.2. *Let* $f_n(z)$ *be a sequence of functions of class* E_p *interior to C, for which*

$$\int_C |(z-\alpha)^{-1} - f_n(z)|^p \, |dz| \leqq \int_C |(z-\alpha)^{-1} - F_0(z)|^p \, |dz| + \epsilon_n,$$

where $\epsilon_n \to 0$ *as* $n \to \infty$. *If D is any closed region lying entirely interior to C, there is a constant M, depending only on D, such that*

$$|f_n(z) - F_0(z)| \leqq M\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

As in §3, we may take $\epsilon_n \leqq M/R^{2n}$ for the minimizing polynomials, which enables us to prove the following theorem.

THEOREM 4.3. *The sequence of polynomials* $P_n(z)$ *of best approximation in the sense of least pth powers to the function* $(z-\alpha)^{-1}$ *on an analytic Jordan curve C converges maximally on* Γ *to the minimizing function.*

The method of proof for these results is indicated by the following. We can write

$$\int_C |(z-\alpha)^{-1} - f(z)|^p \, |dz| = \int_C |g(z)\cdot(z-\alpha)^{-1} - g(z)f(z)|^p \, |dz|.$$

The function within the absolute value signs on the right is now of class E_p interior to C, and at the point $z=\alpha$ takes on the value $g'(\alpha)$. If we apply Theorem 3.1, we get $F_0(z)$ as above.

5. **Introduction of a weight function.** We shall merely state a result, derived by methods as above.

THEOREM. *The minimizing function for*

$$\int_C n(z)\,|f(z)|^p \, |dz|, \qquad f(\alpha) = A,$$

where C is an arbitrary analytic Jordan curve, $z=\alpha$ *is a point interior to C,* $f(z)$ *is of class* E_p *interior to C, and* $n(z)$ *is the modulus on C of a function* $N(z)$ *analytic and nonvanishing in the closed region* Γ, *is*

$$F_0(z) = A\left[\frac{N(\alpha)}{N(z)}\cdot\frac{g'(z)}{g'(\alpha)}\right]^{1/p}.$$

Let $P_n(z)$ *be the corresponding minimizing polynomial of degree n. Then the sequence* $P_n(z)$, $n=0, 1, 2, \cdots$, *converges maximally to* $F_0(z)$ *on* Γ.

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UNIVERSITY OF WISCONSIN

NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

J. L. WALSH

The object of this note is to publish the statement of the following theorem.

THEOREM I. *In the extended* (x, y)*-plane let* R_0 *be a simply-connected region bounded by a continuum* C_0 *not a single point, and let the disjoint continua* $C_1, C_2, \cdots, C_n$ *lie interior to* R_0 *and together with* C_0 *bound a subregion R of* R_0. *By means of a conformal map of* R_0 *onto the unit circle we define in* R_0 *non-euclidean lines, the images of arbitrary circles orthogonal to the unit circle. Denote by* Π *the smallest closed non-euclidean convex region in* R_0 *which contains* $C_1, C_2, \cdots, C_n$.

Let the function $u(x, y)$ *be harmonic interior to R, continuous in the closure of R, with the values zero on* C_0 *and unity on* $C_1, C_2, \cdots, C_n$. *Then the critical points of* $u(x, y)$ *in R are* $n-1$ *in number and lie in* Π.

Critical points are of course to be counted according to their multiplicities.

A limiting case of Theorem I has already been established:[1] if $f(z)$

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[1] J. L. Walsh, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 462–470; see p. 465. The result was proved later by W. Gontcharoff, C. R. (Doklady) Acad. Sci. URSS. vol. 36 (1942) pp. 39–41.

is an analytic function whose modulus is constant on the boundary of a simply-connected region R, where $f(z)$ is analytic interior to R and continuous in the closure of R, then the zeros of $f'(z)$ in R lie in the smallest non-euclidean convex polygon in R containing the zeros of $f(z)$ in R. Theorem I is readily established by the use of this limiting case, and of methods developed elsewhere by the present writer;[2] details are left to the reader.

Theorem I admits an extension to the case where R_0 is bounded by C_0, and the subregion R of R_0 is bounded by C_0 and by further disjoint continua $C_1, C_2, \cdots, C_m, C_{m+1}, \cdots, C_n$ in R_0; the function $u(x, y)$ is supposed harmonic interior to R, continuous in the closure of R, with the values zero on C_0, unity on $C_1, C_2, \cdots, C_m$, and minus unity on $C_{m+1}, C_{m+2}, \cdots, C_n$; a non-euclidean line Λ in R_0 (if existent) which separates $C_1, C_2, \cdots, C_m$ from $C_{m+1}, C_{m+2}, \cdots, C_n$ cannot pass through a critical point of $u(x, y)$. If a Λ exists, the points of R_0 which do not lie on any such Λ form two disjoint non-euclidean convex point sets in R_0 which are closed with respect to R_0, which contain respectively $C_1, C_2, \cdots, C_m$ and $C_{m+1}, C_{m+2}, \cdots, C_n$, and which together contain all critical points of $u(x, y)$ in R. This extension of Theorem I may likewise be proved from a limiting case already formulated (loc. cit.) for a region R_0 bounded by a circle.

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[2] Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) pp. 551–554.

PROPERTIES EQUIVALENT TO THE COMPLETENESS OF $\{e^{-t}t^{\lambda_n}\}$

R. P. BOAS, JR., AND HARRY POLLARD

We are concerned with the following three properties which may be possessed by an increasing sequence $\{\lambda_n\}$ of positive integers.

(A) If $\{a_n\}$ is a sequence of complex numbers such that, for some β, $a_n = O(n^\beta)$ and $\Delta^{\lambda_n} a_0 = 0$ $(n=1, 2, \cdots)$, a_n is a polynomial in n; here

$$\Delta^n a_0 = \sum_{k=0}^{n} (-1)^k C_{n,k} a_k.$$

(B) The set $\{t^{\lambda_n} e^{-ct}\}$ is complete $L^2(0, \infty)$; that is,

$$\int_0^\infty t^{\lambda_n} e^{-ct} \phi(t) dt = 0 \qquad (n = 1, 2, \cdots; \phi \in L^2)$$

implies $\phi(t) = 0$ almost everywhere.[1]

(C) If $f(z)$ is regular and $O(|z|^\alpha)$ for some α in the half-plane $x > -\epsilon$, $\epsilon > 0$, and $f^{(\lambda_n)}(0) = 0$ $(n=1, 2, \cdots)$, $f(z)$ is a polynomial.[2]

W. H. J. Fuchs [3][3] showed that (A) and (B) are equivalent. We shall give a somewhat simpler proof, and show in addition that (C) is equivalent to (A) and (B).

Fuchs showed that (A) is true if $n(r) \geqq r/2 - \gamma$ for some constant γ, where $n(r)$ is the number of $\lambda_n \leqq r$. R. P. Agnew discovered independently [1] that (A) is true if $\lambda_n = 2n$; a simplified proof given by Pollard [5] was the starting point of this note. Boas [2] has shown by other methods that it is enough to have $n(r) \geqq r/2 - r\delta(r)$, where $\int^\infty r^{-1}\delta(r) dr$ converges and $\delta(r)$ satisfies some mild auxiliary conditions. (Fuchs, in a paper [3a] which appeared while this note was in the press, has shown that a necessary and sufficient condition for (A) is that $\int^\infty r^{-2}\psi(r) dr$ diverges, where $\log \psi(r) = 2\sum_{\lambda_n \leqq r} \lambda_n^{-1}$.)

Let $P(\lambda_n)$ mean that $\{\lambda_n\}$ has property (P); $P(\lambda_n - N)$, that the sequence $\{\lambda_n - N\}$ has (P), where $\lambda_n - N$ is replaced by 0 if $\lambda_n < N$. Our line of reasoning is schematically as follows: $A(\lambda_n) \to B(\lambda_n) \to C(\lambda_n + N) \to A(\lambda_n + N) \to A(\lambda_n - N) \to B(\lambda_n - N) \to C(\lambda_n) \to A(\lambda_n)$. It would be more direct to use $B(\lambda_n) \to B(\lambda_n - N)$; this can be quoted from the

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[1] Replacing ct by t, we see that (B) is independent of c.

[2] (C) thus concerns the analytic continuation of a function defined by a lacunary power series $\sum c_n z^{\mu_n}$, where $\{\mu_n\}$ is the sequence of positive integers complementary to $\{\lambda_n\}$.

[3] Numbers in brackets refer to the references at the end of the paper.

work of Fuchs, but the proof is rather involved, and we know of no really simple direct proof. One implication of our reasoning is that "$B(\lambda_n) \rightarrow B(\lambda_n - N)$" is actually equivalent to our other results, and not merely a convenient lemma. Carrying out the indicated scheme actually involves only four nontrivial steps.

(1) $A(\lambda_n) \rightarrow A(\lambda_n - N)$. It is sufficient to prove this when $N=1$. Suppose that $\{\lambda_n - 1\}$ does not have (A); we then have a nonpolynomial sequence $\{a_n\}_{n=0}^{\infty}$, $a_n = O(n^\beta)$, $\Delta^{\lambda_n-1}a_0=0$ if $\lambda_1>0$, $a_0=0$ if $\lambda_1=0$. Consider the sequence $\{b_n\}_{n=0}^{\infty}$, where $b_0=0$, $b_n=-\sum_{k=0}^{n-1}a_k$ for $n=1, 2, \cdots$. Then $a_n=b_n-b_{n+1}$, and for $p>0$, by a simple direct computation, $\Delta^p a_0 = \Delta^{p+1}b_0$. Consequently $\Delta^{\lambda_n}b_0=\Delta^{\lambda_n-1}a_0=0$ if $\lambda_n>0$, $\Delta^0 b_0=b_0=0$; furthermore, if $\{b_n\}$ were a polynomial sequence, $\{a_n\}$ would be one also; and $b_n=O(n^{\beta+1})$. Hence $\{\lambda_n\}$ cannot have (A) if $\{\lambda_n-1\}$ does not.

(2) $A \rightarrow B$. Suppose that $\phi(t) \in L^2$ and

$$\int_0^\infty e^{-t/2}t^{\lambda_n}\phi(t)dt = 0, \qquad n = 1, 2, \cdots.$$

We have to show that $\phi(t)=0$ almost everywhere if (A) is true. We define b_n by

$$n!b_n = \int_0^\infty e^{-t/2}t^n\phi(t)dt;$$

then

$$\text{(I)} \qquad \begin{aligned} a_n = \Delta^n b_0 &= \int_0^\infty e^{-t/2}\phi(t)\left\{\sum_{k=0}^{n}(-1)^k C_{n,k}t^k/k!\right\}dt \\ &= \int_0^\infty e^{-t/2}L_n(t)\phi(t)dt, \end{aligned}$$

where $L_n(t)$ is the nth Laguerre polynomial. Thus

$$|a_n|^2 \leqq \left\{\int_0^\infty e^{-t}L_n^2(t)dt\right\}\left\{\int_0^\infty |\phi(t)|^2 dt\right\} = \int_0^\infty |\phi(t)|^2 dt,$$

and so $a_n=O(1)$. Since, as is readily verified, $b_n=\Delta^n a_0$, (A) implies that $\{a_n\}$ is a polynomial sequence, which must be constant since $\{a_n\}$ is bounded. Hence $a_n=a_0$ for $n=1, 2, \cdots$. Since $e^{-t/2}L_n(t)$ is orthonormal, $\sum a_n^2$ converges, by (I). But this is possible only if all the a_n vanish. Hence $b_n=0$, $n=0, 1, \cdots$. But then $\phi(t)=0$ almost everywhere, since $B(n-1)$ is true.[4]

(3) $B(\lambda_n) \rightarrow C(\lambda_n+N)$, $N \geqq \alpha+1$. Suppose that $f(z)$ satisfies the hy-

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[4] This is equivalent to the completeness of the set $\{e^{-t}L_n(t)\}$; see [6, p. 104].

potheses of (C), with λ_n+N; we have to show that $f(z)$ is a polynomial if (B) is true for $\{\lambda_n\}$. For convenience, suppose $\epsilon=2$. If $P(z)$ is the sum of the terms through z^N in the Maclaurin series of $f(z)$, $z^{-N-1}\{f(z)-P(z)\}$ belongs to the class $H^2(-1)$ of functions $g(z)$ such that $g(x+iy)\in L^2$, qua function of y, uniformly in $x\geqq -1$, and consequently [4, p. 8]

$$f(z) = P(z) + z^N\int_0^\infty e^{-zt}e^{-t}\phi(t)dt, \qquad \phi\in L^2,\ x>-1.$$

Since $f^{(\lambda_n+N)}(0)=0$,

$$\int_0^\infty t^{\lambda_n}e^{-t}\phi(t)dt = 0, \qquad \lambda_n \geqq N.$$

Since (B) is assumed for $\{\lambda_n\}$, $\phi(t)=0$ almost everywhere and so $f(z)\equiv P(z)$.

(4) $C\to A$. Let $a_n=O(n^\beta)$, $\Delta^{\lambda_n}a_0=0$; we may assume that β is an integer. Define $b_n=\Delta^n a_0$, so that $b_n=O(n^\beta 2^n)$, $b_{\lambda_n}=0$. Consider

$$\begin{aligned} f(z) = \sum_{n=0}^\infty b_n z^n &= \sum_{n=0}^\infty z^n \sum_{k=0}^\infty (-1)^k C_{n,k}a_k \\ &= \sum_{k=0}^\infty (-1)^k a_k \sum_{n=k}^\infty C_{n,k}z^n \\ &= \frac{1}{1-z}\sum_{k=0}^\infty a_k\left(\frac{z}{1-z}\right)^k. \end{aligned}$$

The first series for $f(z)$ converges for $|z|<1/2$; the last, for $|z/(1-z)|<1$, that is, for $x<1/2$. Consequently $f(z)$ is regular in this half-plane. There is a number K such that $|a_n|\leqq Kn(n-1)\cdots(n-\beta+1)$, $n\geqq\beta$. We then have

$$\begin{aligned} |f(z)| &\leqq \left|\frac{1}{1-z}\sum_{k=0}^{\beta-1}a_k\left(\frac{z}{1-z}\right)^k\right| + K\sum_{k=\beta}^\infty \frac{k!}{(k-\beta)!}\left|\frac{z}{1-z}\right|^k \\ &\leqq O(1) + \frac{K\beta!}{|1-z|}\left|\frac{z}{1-z}\right|^\beta\left(1-\left|\frac{z}{1-z}\right|\right)^{-\beta-1} = O(|z|^\beta) \end{aligned}$$

in $x<1/2-\epsilon$, $\epsilon>0$. Since (C) is assumed, $f(-z)$ is a polynomial. Hence all b_n vanish from some n_0 on, and

$$a_n = \Delta^n b_0 = \sum_{k=0}^{n_0}(-1)^k b_k n(n-1)\cdots(n-k+1)/k!,$$

a polynomial in n.

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BROWN UNIVERSITY AND
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POLYNOMIAL DOMINANTS

PAUL CIVIN

Let $f(x)$ be a continuous function of period 2π, and let $F_n(x)$ be a trigonometric polynomial of degree n which dominates $f(x)$ at the points $x_j^n = 2j\pi/(2n+1)$ where $j = 0, \cdots, 2n$, that is, $F_n(x_j^n) \geqq |f(x_j^n)|$. The present paper is concerned with relations of inequality between various integral means of the functions $f(x)$ and $F_n(x)$. A similar problem was considered by Hardy and Littlewood [2][1] under the hypothesis that the Fourier series of a function $F(x)$ was a majorant of the Fourier series of $f(x)$. A further variation concerning the means of the Fourier coefficients under the hypothesis of functional dominance, that is, $F(x) \geqq |f(x)|$, was treated by the author [1]. In the present work, heavy reliance is made on the results of Marcinkiewicz and Zygmund in their paper *Mean values of trigonometric polynomials* [4]. This paper will be referred to in the sequel as MV.

The results which are obtained are of the type

$$M_a(f) \leqq AM_b(F_n) + B\omega(2\pi/n), \tag{1}$$

where (i) $M_\alpha(g)$ represents some integral mean of $g(x)$; (ii) A and B are constants independent of n and the functions $f(x)$ and $F_n(x)$, but dependent on the particular mean M_a; and (iii) $\omega(\delta)$ is the modulus of continuity of $f(x)$.

It can easily be seen that neither term on the right of inequality (1) can be omitted. Suppose first that, for some n, $f(x_j^n) = 0$ but $f(x)$ is not identically zero, and also that $F_n(x_j^n) = 0$ so that $F_n(x) = 0$. Then the left side of (1) will not be zero while the first term on the right vanishes, and consequently the second term on the right is essential. Finally, if the first term were not present in the right member of (1), the left side would be different from zero for non-null functions while the right side would tend to zero as $n \to \infty$.

Throughout the remainder of the paper we shall suppose that $x_j^n = 2j\pi/(2n+1)$; that $\phi_{2n+1}(t) = x_j$ when $x_j \leqq t < x_{j+1}$; and that a letter $A, B, \cdots$ with subscript will denote a constant depending only on those parameters appearing in the subscript. The same letter may have a different value at each appearance.

THEOREM 1. *For* $1 < r < \infty$,

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$(2)\quad \left(\int_0^{2\pi} |f(x)|^r dx\right)^{1/r} \leqq A_r\left(\int_0^{2\pi} |F_n(x)|^r dx\right)^{1/r} + B_r\omega(2\pi/n),$$

where A_r and B_r depend on r alone.

There is a trigonometric polynomial, $T_n(x)$, of degree n such that $|f(x)-T_n(x)| \leqq A\omega(2\pi/n)$, where A is an absolute constant [3, p. 7]. Therefore,

$$(3)\quad \begin{aligned}\left(\int_0^{2\pi} |f(x)|^r dx\right)^{1/r} &\leqq \left(\int_0^{2\pi} |f(x) - T_n(x)|^r dx\right)^{1/r} \\ &\quad + \left(\int_0^{2\pi} |T_n(x)|^r dx\right)^{1/r} \\ &\leqq A_r\omega(2\pi/n) + \left(\int_0^{2\pi} |T_n(x)|^r dx\right)^{1/r}.\end{aligned}$$

Theorem 1 of MV states that

$$(4)\quad \left(\int_0^{2\pi} |T_n(x)|^r dx\right)^{1/r} \leqq A_r\left(\int_0^{2\pi} |T_n(x)|^r d\phi_{2n+1}(x)\right)^{1/r}.$$

In turn,

$$(5)\quad \begin{aligned}&\left(\int_0^{2\pi} |T_n(x)|^r d\phi_{2n+1}(x)\right)^{1/r} \\ &\quad \leqq \left(\int_0^{2\pi} |T_n(x) - f(x)|^r d\phi_{2n+1}(x)\right)^{1/r} \\ &\quad\quad + \left(\int_0^{2\pi} |f(x)|^r d\phi_{2n+1}(x)\right)^{1/r} \\ &\quad \leqq A_r\omega(2\pi/n) + \left(\int_0^{2\pi} |f(x)|^r d\phi_{2n+1}(x)\right)^{1/r}.\end{aligned}$$

The hypothesis of dominance at the points x_j^n, together with Theorem 2 of MV, implies that

$$(6)\quad \begin{aligned}&\left(\int_0^{2\pi} |f(x)|^r d\phi_{2n+1}(x)\right)^{1/r} \\ &\quad \leqq \left(\int_0^{2\pi} |F_n(x)|^r d\phi_{2n+1}(x)\right)^{1/r} \leqq A_r\left(\int_0^{2\pi} |F_n(x)|^r dx\right)^{1/r}.\end{aligned}$$

The combination of (3) through (6) completes the demonstration of (2).

If $f(x)$ is a trigonometric polynomial of degree n, the approximation process is unnecessary and the terms in $\omega(2\pi/n)$ do not occur.

COROLLARY 1. *If $f(x)$ is a trigonometric polynomial of degree n, then for $1<r<\infty$*

$$\left(\int_0^{2\pi} |f(x)|^r dx\right)^{1/r} \leqq A_r\left(\int_0^{2\pi} |F_n(x)|^r dx\right)^{1/r},$$

where A_r depends on r alone.

THEOREM 2. *For $0<\mu<1$,*

$$\left(\int_0^{2\pi} |f(x)|^\mu dx\right)^{1/\mu} \leqq A_\mu \int_0^{2\pi} |F_n(x)|\, dx + B_\mu \omega(2\pi/n), \tag{7}$$

where A_μ and B_μ depend on μ alone.

Let $T_n(x)$ have the same connotation as above. Then by Theorem 6 of MV,

$$\begin{aligned} \Bigg(\int_0^{2\pi} |f(x)|^\mu dx &\leqq \int_0^{2\pi} |f(x) - T_n(x)|^\mu dx + \int_0^{2\pi} |T_n(x)|^\mu dx \\ &\leqq A_\mu[\omega(2\pi/n)]^\mu + B_\mu\left(\int_0^{2\pi} |T_n(x)|\, d\phi_{2n+1}(x)\right)^\mu. \end{aligned} \tag{8}$$

Theorem 1 of MV and the hypothesis of dominance show that

$$\begin{aligned} \left(\int_0^{2\pi} |T_n(x)|\, d\phi_{2n+1}(x)\right)^\mu &\leqq \left(\int_0^{2\pi} |T_n(x) - f(x)|\, d\phi_{2n+1}(x)\right)^\mu \\ &\quad + \left(\int_0^{2\pi} |f(x)|\, d\phi_{2n+1}(x)\right)^\mu \\ &\leqq A_\mu[\omega(2\pi/n)]^\mu + \left(\int_0^{2\pi} |F_n(x)|\, d\phi_{2n+1}(x)\right)^\mu \\ &\leqq A_\mu[\omega(2\pi/n)]^\mu + B_\mu\left(\int_0^{2\pi} |F_n(x)|\, dx\right)^\mu. \end{aligned} \tag{9}$$

Inequalities (8) and (9) therefore imply that

$$\int_0^{2\pi} |f(x)|^\mu dx \leqq A_\mu\left(\int_0^{2\pi} |F_n(x)|\, dx\right)^\mu + B_\mu[\omega(2\pi/n)]^\mu.$$

The extraction of the appropriate root and the corresponding relabelling of constants then gives (7).

THEOREM 3. *If* $|f(x)| \leqq M$, *then*

$$\int_0^{2\pi} |f(x)|\,dx \leqq A \log^+ M \int_0^{2\pi} |F_n(x)|\,dx + B + C\omega(2\pi/n)[1 + M + \log^+ M + C\omega(2\pi/n)],$$

where A, B, *and* C *are absolute constants.*

Again, let $T_n(x)$ have the connotation as above. Then by Theorem 5 of MV,

$$\begin{aligned}
&\int_0^{2\pi} |f(x)|\,dx \\
&\leqq \int_0^{2\pi} |f(x) - T_n(x)|\,dx + \int_0^{2\pi} |T_n(x)|\,dx \\
&\leqq A\omega(2\pi/n) + B\int_0^{2\pi} |T_n(x)| \log^+ |T_n(x)|\,d\phi_{2n+1}(x) + C \\
&\leqq B\int_0^{2\pi} \big|\, |T_n(x)| \log^+ |T_n(x)| - |f(x)| \log^+ |f(x)| \,\big|\, d\phi_{2n+1}(x) \qquad (10)\\
&\quad + A\omega(2\pi/n) + B\int_0^{2\pi} |f(x)| \log^+ |f(x)|\,d\phi_{2n+1}(x) + C \\
&= S_1 + A\omega(2\pi/n) + S_2 + C.
\end{aligned}$$

The application of Theorem 1 of MV shows that

$$\begin{aligned}
s_2 &\leqq B \log^+ M \int_0^{2\pi} |f(x)|\,d\phi_{2n+1} \\
&\leqq B \log^+ M \int_0^{2\pi} |F_n(x)|\,d\phi_{2n+1}(x) \qquad (11)\\
&\leqq B \log^+ M \int_0^{2\pi} |F_n(x)|\,dx.
\end{aligned}$$

To obtain an estimate for S_1, first suppose that for a particular value of j either $|T_n(x_j^n)| \leqq 1$ or $|f(x_j^n)| \leqq 1$. If the former holds, then $|f(x_j^n)| \leqq 1 + A\omega(2\pi/n)$, and $|f(x_j^n)| \log^+ |f(x_j^n)| \leqq MA\omega(2\pi/n)$. If the latter holds, then $|T_n(x_j^n)| \leqq \min[M + A\omega(2\pi/n), 1 + A\omega(2\pi/n)]$, so that $|T_n(x_j^n)| \log^+ |T_n(x_j^n)| \leqq [M + A\omega(2\pi/n)]A\omega(2\pi/n)$.

Suppose next that for a particular value of j, both $|T_n(x_j^n)| > 1$ and $|f(x_j^n)| > 1$. For brevity suppose also that $|T_n(x_j^n)| \geqq f(x_j^n)$. The other case will lead to the same conclusion. Then

$$\begin{aligned}
&| T_n(x_j^n) | \log^+ | T_n(x_j^n) | - | f(x_j^n) | \log^+ | f(x_j^n) | \\
&\qquad = | T_n(x_j^n) | [\log^+ | T_n(x_j^n) | - \log^+ | f(x_j^n) |] \\
&\qquad\quad + \log^+ | f(x_j^n) | [| T_n(x_j^n) | - | f(x_j^n) |] \\
&\qquad \leqq | T_n(x_j^n) | \log [1 + | T_n(x_j^n) | - | f(x_j^n) |] + A\omega(2\pi/n) \log^+ M \\
&\qquad \leqq [M + A\omega(2\pi/n)] A\omega(2\pi/n) + A\omega(2\pi/n) \log^+ M.
\end{aligned}$$

In any event, the individual terms comprising the mean in S_1 are less than $A\omega(2\pi/n)[M + \log^+ M + A\omega(2\pi/n)]$, and therefore S_1 has the same bound. The theorem then follows from (10) and (11).

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UNIVERSITY OF BUFFALO

GEORGE DAVID BIRKHOFF AND HIS MATHEMATICAL WORK

MARSTON MORSE

The writer first saw Birkhoff in the fall of 1914. The graduate students were meeting the professors of mathematics of Harvard in Sever 20. Maxime Bôcher, with his square beard and squarer shoes, was presiding. In the back of the room, with a different beard but equal dignity, William Fogg Osgood was counseling a student. Dunham Jackson, Gabriel Green, Julian Coolidge and Charles Bouton were in the business of being helpful. The thirty-year-old Birkhoff was in the front row. He seemed tall even when seated, and a friendly smile disarmed a determined face. I had no reason to speak to him, but the impression he made upon me could not be easily forgotten.

His change from Princeton University to Harvard in 1912 was decisive. Although he later had magnificent opportunities to serve as a research professor in institutions other than Harvard he elected to remain in Cambridge for life. He had been an instructor at Wisconsin from 1907 to 1909 and had profited from his contacts with Van Vleck. As a graduate student in Chicago he had known Veblen and he continued this friendsip in the halls of Princeton. Starting college in 1902 at the University of Chicago, he changed to Harvard, remained long enough to get an A.B. degree, and then hurried back to Chicago, where he finished his graduate work in 1907.

It was in 1908 that he married Margaret Elizabeth Grafius. It was clear that Birkhoff depended from the beginning to the end on her deep understanding and encouragement. Her varied talents and charm were reflected in the overflowing hospitality of their home. Their children, Garrett and Barbara (Mrs. Robert Treat Paine, Jr.) are well known to friends of Birkhoff.

Birkhoff admired Moore of Chicago, but not to the point of imitating him. He respected Bôcher no less, and did him the honor next to Poincaré of following his mathematical interests. F. R. Moulton's study of the work of Poincaré had something to do with Birkhoff's own intense reading of Poincaré. Poincaré was Birkhoff's true teacher. There is probably no mathematician alive who has explored the works of Poincaré in full unless it be Hadamard, but in the domains of analysis Birkhoff wholeheartedly took over the techniques and problems of Poincaré and carried on.

Birkhoff's loyalty to Harvard was complete and only occasionally critical. The response of the non-mathematical members of the faculty to his early successes was slow at first and he felt this. But he did not wait long before he was recognized by the University as a whole. It is a mark of the intellectual distinction of President Lowell that he hailed Birkhoff as one of Harvard's greatest men and honored him accordingly. He was made Perkins Professor of Mathematics and served at one time as Dean of the Faculty of Arts and Sciences. Birkhoff's colleagues in mathematics at Harvard were as one in their admiration of him and affection for him.

Unlike his colleague Osgood, Birkhoff thought of his contemporaries in Europe, particularly in Germany, as colleagues rather than as teachers. He held Klein lightly, was unenthusiastic over Weierstrass, but gave his full respect to Riemann. Through his papers on non-self-adjoint boundary value problems and asymptotic representations he probably influenced the Hilbert integral equation school as much or as little as it influenced him. His relations with the members of the French and Italian schools of analysis were close, both personally and scientifically. Levi-Civita and Hadamard were among his best friends. Birkhoff was at the same time internationally minded and pro-American. The sturdy individualism of Dickson, E. H. Moore and Birkhoff was representative of American mathematics "coming of age." The work of these great Americans sometimes lacked external sophistication, but it more than made up for this in penetration, power and originality, and justified Birkhoff's appreciation of his countrymen.

Much has been said about Birkhoff's teaching, and each of his pupils will have his tale. There is one thing which may be noted in general: that there is an excellent correlation between the more distinguished Ph.D.'s from Harvard and those who were inspired by Birkhoff's teaching. Among his "tutees" there was an unanalyzed appreciation of Birkhoff's ability to transform a subject into something alive, and a deep gratitude for his painstaking personal effort with them. He never consciously descended to the level of the untutored, nor gave them evidence that he was aware of his own relative elevation. He often improvised in an exposition, thereby illuminating many things besides the immediate theorem. He once said that if his livelihood depended upon it he would vie with those most perfect in exposition. Those who knew him best will agree that he could have done so and successfully. This demonstration however was never demanded.

In his social and political views, he was detached from the world about him and this was sometimes a source of misunderstanding. Proceeding from the dynamic individualism of the Middle West to an environment of tenacious self-sufficiency in New England, he understood and interpreted both as conditions for living. His philosophy of life was intellectual rather than emotional. His own conduct of life was extraordinarily well integrated, and free from manifestations of eccentricity. He showed his individualism most clearly in the broad range of suppositions which he could entertain and on which he could vividly speculate. The fact that these speculations were disassociated from his own life was one of the reasons why his extreme detachment was possible. He chose to be brief and simple, but not explanatory. He could examine a point of view as if it were his very own and later reflect with equal appreciation on an opposite point of view. A hearer who did not understand this range of his speculations could easily receive incorrect impressions of Birkhoff's views. To understand Birkhoff one had to weigh and average his conclusions over a long range of circumstances. Birkhoff's separation of thought and action was a part of his conscious desire to fight only one battle, the battle of mathematics.

Birkhoff was uncompromising in his appraisal of mathematics—by the test of originality and relevance. For him the systematic organization or exposition of a mathematical theory was always secondary in importance to its discovery. I recall his remarks on a mathematical treatise that had come to his attention and eventually had a wide circulation but which he did not regard as original. Birkhoff said "I read this book through in a half hour." His choice of topics of investigation could be called objective rather than subjective. His problems were not necessarily chosen from among those which he could solve; indeed many of his papers from 1920 on were on problems which he left unsolved. Some of the current mathematical theories were regarded by Birkhoff as no more than relatively obvious elaborations of concrete examples.

The major interest in Birkhoff in the future will undoubtedly center about his mathematical theories and this report is devoted to their exposition in brief.[1]

Popular opinion focuses attention on two striking achievements, his proof of Poincaré's Last Theorem and the Ergodic Theorem. By contrast much of his other work is obscured. Important as they are, it is to be doubted whether either of the above theorems is more substantial than other phases of his work. His magnificent theory of the generalized Riemann problem, his contributions to difference equations, his formal theory of stability in dynamics, and the consequent geometrical theory, all involve structures upon which much can and probably will be built.

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[1] In reviewing the research of Birkhoff the writer has had the assistance of Dr. Pesi Masani, particularly in the study of Birkhoff's early work.

In the domain of transitivity in which the Ergodic Theorem lies, the basic hypothesis of metric transitivity remains a conjecture as far as analytic dynamical systems of classical type are concerned. The relative importance in mathematics of statistical or symbolic dynamics can hardly be assessed until it is known whether or not topological transitivity in an analytic system implies metric transitivity. The importance which Birkhoff attached to symbolism in dynamics is made evident by the fact that his last papers on dynamics were largely concerned with the search for a general symbolism characterizing a dynamical system.

Birkhoff's first paper [1][2] was written with H. S. Vandiver under the title *On the integral divisors of* $a^n - b^n$. Birkhoff was eighteen and Vandiver twenty. An immediate consequence of the principal theorem was that there are an infinity of primes congruent to 1, mod n, a result known to Kronecker, Hilbert and Sylvester. The principal theorem is new. One consequence of this collaboration was the high regard for Vandiver's work which Birkhoff held throughout his life.

To continue with a systematic review of Birkhoff's work it will be convenient to divide his papers into six principal groups. Various subjects such as the four-color problem will be referred to separately.

I. ASYMPTOTIC EXPANSIONS, BOUNDARY VALUE PROBLEMS, STURM-LIOUVILLE THEOREMS

(a) *Asymptotic expansions and boundary value problems.* The years of Birkhoff's graduate study were major years in the development of integral equations. Parallel developments in terms of Green's functions were of great interest. Self-adjoint problems corresponding to symmetric kernels had been extensively studied, but when $n > 2$ the case of a non-self-adjoint operator

$$L(z) = \frac{d^n z}{dx^n} + * + p_2(x) \frac{d^{n-2} z}{dx^{n-2}} + \cdots + p_n(x) z \qquad (a \leqq x \leqq b)$$

had not been adequately treated.

The n boundary conditions in [3]

$$W_i(u) = 0 \qquad (i = 1, \cdots, n) \tag{1.1}$$

were linear and homogeneous in u and its first $n-1$ derivatives taken at $x = a$ and $x = b$, and were linearly independent. The functions $p_i(x)$

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[2] Numbers in brackets refer to the Bibliography at the end of this paper.

with all their derivatives were assumed continuous on the closed interval (a, b). The operator $M(z)$ adjoint to $L(z)$ was defined in the classical way. With $L(u)$ and the conditions (1.1) Birkhoff was the first[3] to associate the adjoint boundary conditions

$$V_i(v) = 0 \qquad (i = 1, \cdots, n)$$

in the now accepted manner. The parameter λ was introduced, giving the "problem"

$$L(u) + \lambda u = 0, \qquad W_i(u) = 0 \qquad (i = 1, \cdots, n) \tag{1.2}$$

and the adjoint problem,

$$M(v) + \lambda v = 0, \qquad V_i(v) = 0 \qquad (i = 1, \cdots, n). \tag{1.3}$$

In a paper [2] which constituted the first part of his thesis, Birkhoff had given an elegant asymptotic representation of solutions of differential equations of the form

$$\frac{d^n z}{dx^n} + \rho a_{n-1}(x, \rho) \frac{d^{n-1}z}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho) z = 0 \tag{1.4}$$

for large values of $|\rho|$. The differential equations in (1.2) and (1.3) took the form (1.4) on setting $\lambda = \rho^n$. With the aid of these asymptotic representations Birkhoff proved the existence of an infinite sequence $\lambda_1, \lambda_2, \cdots$ of characteristic values with corresponding solutions

$$u_1(x), \ u_2(x), \cdots,$$
$$v_1(x), \ v_2(x), \cdots$$

of (1.2) and (1.3) respectively. He was able to do this by imposing certain determinant conditions on the coefficients in the boundary conditions.

The function $f(x)$ to be expanded was assumed real and piecewise continuous with a continuous derivative. In the case in which the λ_i are zeros of the first order of the determinant $\Delta(\lambda)$ defining the characteristic roots, the expansion

$$\sum_{i=1}^{\infty} c_i u_i(x) \tag{1.5}$$

in which

$$c_i = \frac{\int_a^b f(x) v_i(x) dx}{\int_a^b u_i(x) v_i(x) dx}$$

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[3] As stated by Bôcher, *Applications and generalization of the conception of adjoint systems*, Trans. Amer. Math. Soc. vol. 14 (1913) pp. 403–420.

converges (as does a Fourier expansion) to

$$\frac{f(x+0)+f(x-0)}{2}$$

at each inner point of (a, b), to

$$a_1 f(a+0) + a_2 f(b-0)$$

at $x=a$ and to

$$b_1 f(a+0) + b_2 f(b-0)$$

at $x=b$ where a_1, a_2, b_1, b_2 are constants, independent of $f(x)$. In case λ_i is not a zero of the first order of $\Delta(\lambda)$ the corresponding term in (1.5) is to be replaced by a term readily defined with the aid of the Green's function. The proof of the convergence depended, as with Poincaré, on the method of contour integration in the λ-plane.

The interest in this work of Birkhoff is signaled by a historic interchange between Birkhoff and J. D. Tamarkin. Tamarkin had independently worked out asymptotic representations similar to those of Birkhoff without publishing them, and was prepared to treat the problem of Birkhoff when he received reprints from Birkhoff. Birkhoff's papers were published in 1907 and 1908 respectively and Tamarkin's paper[4] was published in 1912. In this paper Tamarkin stated that he found Birkhoff's reasoning brief, and on certain points insufficient. It should be stated that Birkhoff had treated the case of an even n in detail and had indicated the proofs when n was odd. In twelve pages in [6] Birkhoff showed that his earlier proofs could be carried through in general. In a note[5] in the Rendiconti, Tamarkin reviewed the matter at issue and accepted Birkhoff's supplementary proof.

Problems similar to those treated in [2] and [3] were taken up at length in Birkhoff's and Langer's paper [8] where the case of a system of ordinary differential equations was treated. In this paper may be found references to extensions of the theory by Hopkins and Wilder, pupils of Birkhoff.

(b) *Separation and comparison theorems.* The existence of characteristic roots in boundary value problems in the plane and the oscillation properties of characteristic functions are treated in [4] following the methods of Sturm, Liouville, and Bôcher ($n=1$). The existence of infinitely many characteristic roots is established in the self-adjoint case and a general oscillation theorem is obtained.

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[4] J. D. Tamarkin, *Sur quelques pointes de la théorie des équations différentielles linéaires ordinaries et sur la généralization de la serie de Fourier*, Rend. Circ. Mat. Palermo vol. 24 (1912) pp. 345–382.

[5] J. D. Tamarkin, *Sur un problème de la théorie des équations différentielles linéaires ordinaires*, Rend. Circ. Mat. Palermo vol. 37 (1914) pp. 376–378.

Around 1930 the new techniques auxiliary to the "variational theory in the large" opened up the separation-comparison-oscillation theory for self-adjoint systems of n second-order ordinary linear differential equations. The Sturm separation theorem was generalized by the theorem that the conjugate points on an arbitrary interval J of a point $x=a$ of the x-axis differ in number by at most n from the conjugate points on J of a point $x=b$. The boundary conditions were expressed in terms of a minimum set (u) of parameters and an accessory quadratic form $q(u)$ defined in the second variation. A necessary and sufficient condition for the existence of infinitely many characteristic roots was obtained.

In these terms it was possible for the first time to give a systematic classification of the possible separation, comparison and oscillation theorems, and with the aid of Morse's "index form" to establish the principal results. In [9] Birkhoff and Hestenes turned from the older methods of Sturm to the new methods, adding an isoperimetric interpretation to the proofs. A detailed review of [9] is here impossible. Fortunately, a recent "hour address" by W. T. Reid[6] covers this field in an excellent manner both historically and mathematically, and the interested reader is referred to this address.

In a short note [7] in 1917 Birkhoff gave a simple condition that a normalized set of orthogonal functions $u_i(x)$ be closed, and applied this to Sturm-Liouville series. The principle involved in the application could be stated as follows: any set of orthogonal vectors in a functional space lying near enough (in a sense defined by Birkhoff) to a complete set in that space is itself complete. Birkhoff was here sampling a great theory, a theory in which the name of his pupil and colleague Marshall Stone was later to become illustrious.

Perhaps the most remarkable of the contributions of Birkhoff in the domain of separation and comparison theorems is in [5] and concerns an ordinary third order linear differential equation. This equation does not lend itself to treatment by the Sturmian methods nor has it yet been effectively brought into the domain of the variational theory. A solution

$$y_i(x) \qquad (i = 1, 2, 3)$$

is thought of as a curve in a 2-dimensional projective space. Birkhoff utilizes the projective differential geometry of Darboux and Wilcynski and by elegant methods far removed from all other methods arrives at theorems very similar to the classical ones. This part of the field deserves further study.

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[6] W. T. Reid, *Boundary value problems of the calculus of variations*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 633–666.

II. LINEAR DIFFERENTIAL EQUATIONS AND THE GENERALIZED RIEMANN PROBLEM

With Gauss, Riemann, and Poincaré leading the way, ordinary linear differential equations of the second order of Fuchsian type have been studied by almost every analyst of note. In mathematical physics, equations of this type bear the name of Bessel and Legendre. In complex function theory such equations lead by way of conformal mapping to the elliptic modular and automorphic functions. In real function theory integral relations such as[7]

$$\int_0^\infty \{4(y')^2 - y^2/x^2\}dx > 0 \qquad (y(0) = 0)$$

are typical of a class of relations which arise[8] from Fuchsian differential equations with two singular points. Much of Bôcher's research was concerned with these equations. It was natural therefore that Birkhoff should try his hand in this central field.

Departing somewhat from the order in which Birkhoff's papers on this subject appeared we refer first to [11] on *A simplified treatment of the regular singular point.* At a regular singular point the canonical solutions have singularities (if any) of a well known type. This fact had been previously proved by the method of Frobenius, whereby the coefficients of a formal solution were first obtained and the solution then proved convergent. Birkhoff gives a direct proof, without use of a series expansion, and this proof is barely two pages in length.

Birkhoff was not content with this achievement. Linear differential systems with *irregular* singular points presented a field of great difficulty and next challenged his attention.

No mathematician of Birkhoff's day possessed greater facilitiy in combining formal algebraic methods with penetrating numerical analysis. Birkhoff needed all of this power in attacking the irregular case. Thomé was his predecessor in using formal solutions of a general type. Poincaré and Horn[9] had effectively introduced asymptotic representations. Birkhoff had used formal series and asymptotic representations in the study of boundary value problems. Hilbert and Plemelj, prior to Birkhoff and unknown to Birkhoff, had already solved one of the matrix problems relevant to the theory. Cunningham had generalized Poincaré's use of Laplace transformations in reducing the problem.

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[7] Hardy, Littlewood, Pólya, *Inequalities*, p. 175.

[8] Marston Morse and Walter Leighton, *Singular quadratic functionals*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 252–286.

[9] See Birkhoff's papers for references. References which may be found in the papers of Birkhoff are generally omitted.

There was thus a great technique available to Birkhoff when he began his research. But with all this power no one had carried the theory through to a triumphant end. Birkhoff in [10, 13] took up the problem with typical boldness. *He added the concepts of canonical systems of differential equations and of equivalence of such systems. He carried through the asymptotic representation of the matrix solution in the general irregular case; enumerated the essential "characteristics" of a singular point of arbitrary rank, defined and solved the "generalized Riemann problem."*

Such were Birkhoff's achievements in this field. Although any explicit formulation is complicated, it is needed if the reader is to sense the depth of Birkhoff's mastery and skill.

The differential equations can be represented in the form

$$\frac{dy_i}{dz} = a_{ij}(z)y_j \qquad (i, j = 1, \cdots, n) \tag{2.1}$$

using the summation convention for repeated indices. Here z is a complex variable, and $y_i(z)$ the unknown ith component of a solution (y). In any local study the singular point can be taken as the point $z=\infty$. The functions $a_{ij}(z)$ are assumed analytic at $z=\infty$ except at most for poles. The maximum order of these poles is denoted by q and $q+1$ is termed the *rank* of the singularity $z=\infty$. When $q<-1$ the singular point is *ordinary*. When $q=-1$ the point is *regular*, and when $q\geqq 0$ *irregular*. The irregular case $q=0$ had been reduced to the regular case $q=-1$ by Poincaré on using a Laplace transformation of the variables (y), and Poincaré had indicated a possible reduction of the irregular case $q>0$.

To simplify the system (2.1) Birkhoff uses linear transformations of the form

$$u_i = c_{ij}(z)y_j \qquad (i, j = 1, \cdots, n) \tag{2.2}$$

in which the functions $c_{ij}(z)$ are analytic at infinity and have there a nonvanishing determinant. Any system into which the system (2.1) goes under (2.2) is said to be *equivalent* to the system (2.1) at infinity. *Birkhoff shows that a system* (2.1) *of rank* $q+1\geqq 0$ *is equivalent to a canonical system*

$$(2.3)\qquad zy_i' = q_{ij}(z)y_j \qquad (i, j = 1, \cdots, n)$$

again of rank $q+1$ *in which* $q_{ij}(z)$ *is a polynomial of degree at most* $q+1$.

In proving this theorem Birkhoff used a lemma on analytic matrices [12]. Unknown to Birkhoff at the time, this lemma followed from general theorems of Hilbert and Plemelj. The canonical systems (2.3) are of great value in formally reducing the complexity of the system.

With Thomé, n formal solutions each of the type

$$(2.4)\qquad y_i = e^{p(z)}z^{\mu}B_i(z) \qquad (i = 1, \cdots, n)$$

are next determined. Here $B_i(z)$ is a formal series of nonpositive powers of z such that at least one of the n constant terms $(i=1, 2, \cdots, n)$ is not zero while $p(z)$ is of the form

$$(2.5)\qquad p(z) = \frac{\alpha z^{q+1}}{q+1} + \frac{\beta z^q}{q} + \cdots + \lambda z.$$

The coefficient α is a characteristic root of the matrix of coefficients of z^{q+1} in the polynomials $q_{ij}(z)$. These roots α_i are assumed distinct. For each of the n formal solutions there is thus a polynomial $p(z)$, an exponent μ, and a formal series $B_i(z)$. These solutions with $j=1, \cdots, n$ may be chosen so that their determinant $|B_{ij}(\infty)| \neq 0$.

By making generalized Laplace transformations of the variables y_i in (2.3) together with other linear transformations of elementary character, Birkhoff shows that the rank of the differential system at its different singular points can be reduced step by step until the system is regular. This process leads to solutions in the form of generalized Laplace integrals. Each of these integrals is asymptotically represented by a formal solution of the type (2.4) in an appropriate "sector" of a neighborhood of $z=\infty$. These sectors are bounded by the $m=n(n-1)(q+1)$ rays on which (R=real part)

$$R[z^{q+1}(\alpha_i - \alpha_j)] = 0 \qquad (i \neq j;\ i, j = 1, \cdots, n).$$

A base W_r of these solutions can be chosen in the rth sector $(r=1, \cdots, m)$ so that the base W_r^* belonging to the succeeding sector is obtained from the base W_r by multiplying W_r on the right by a matrix of the form $I+C_r$, where I is the unit matrix and C_r is a matrix with just one non-null element c_r.

These "transformation constants" $c_1, \cdots, c_m$, together with the n exponents μ_i and the $n(q+1)$ constants

$$(2.6)\qquad (\alpha_i, \beta_i, \cdots, \lambda_i) \qquad (i = 1, \cdots, n),$$

of which the ith set appears in the ith solution of a formal matrix solution, are termed by Birkhoff the *characteristic* constants of the singular point at infinity. With these constants given he shows that there exists a locally defined canonical system (2.3) to which these constants will be equivalent in the above sense.

The preceding theorem completes the local analysis and Birkhoff turns next to the problem in the large. Let S be a differential system of the type (2.1) with at most a finite number of regular or irregular singular points

$$a_1, \cdots, a_k \tag{2.7}$$

of ranks

$$q_1, \cdots, q_k \tag{2.8}$$

respectively. Corresponding to a base Y of solutions of S there exist matrices $T_1, \cdots, T_k$ of constants such that Y is replaced by YT_i when a positive circuit of a_i is made. The matrices T_i generate the *monodromic group* G with a relation

$$T_1 \cdots T_k = I \tag{2.9}$$

corresponding to a circuit which encloses the points (2.7).

Birkhoff says that two matrices $Y_1(z)$ and $Y_2(z)$ whose elements are analytic in the vicinity of $z=a$ are *properly equivalent* at $z=a$ if

$$Y_1(z) = A(z)Y_2(z) \qquad (z \neq a) \tag{2.10}$$

and $A(z)$ is a matrix of elements analytic at $z=a$ with $|A(a)| \neq 0$, and *improperly equivalent* if (2.10) holds and the elements of $A(z)$ are analytic at $z=a$ except at most for poles. He then proves the following theorem:

Let $a_1, \cdots, a_k$ *be k given points. Let* $T_1, \cdots, T_k$ *be matrices of constants such that* (2.9) *holds. Let* $Z_i(z)$ $(i=1, \cdots, k)$ *be matrices of functions, analytic of determinant not zero in the vicinity of* $z=a_i$ *and undergoing a transformation to* $Z_i(a)T_i$ *as z makes a positive circuit of* a_i. *There exists a matrix* $Y(z)$ *of functions not zero for* $z \neq a_1, \cdots, a_k$ *and analytic save at these points, which undergoes a transformation to* YT_i *as z makes a positive circuit of* a_i; *furthermore* $Y(z)$ *is properly equivalent to* Z_i *at* a_i $(i=1, \cdots, k-1)$ *and properly or improperly equivalent to* Z_k *at* a_k.

This theorem contains as a special case a theorem of Hilbert and Plemelj in which the T_i's are given but not the Z_i's, while $Y(z)$ is affirmed to exist and possess elements which are analytic to finite order. The above theorem of Birkhoff gives a first type of generalization of the Riemann problem. A second type of generalization which includes two problems proposed by Riemann is formulated by Birkhoff as follows.

To construct a system of linear differential equations of the first order with prescribed singular points (2.7) *and with a given monodromic group, the characteristic constants being assigned for each singular point.*

In order that a solution exist it is necessary that the relation (2.9) hold and that the elementary divisors of T_i be those of the transformation matrix determined at a_i by the "transformation constants" belonging to a_i. When these conditions are satisfied Birkhoff shows that the problem admits a solution provided one replaces the exponents μ_i $(i=1, \cdots, n)$ belonging to some one of the singular points by new exponents μ_i+s where s is a suitable integer independent of i.

This concludes Birkhoff's memorable work on differential systems. A byproduct was a number of theorems on the representation of analytic matrices. Birkhoff returned to this subject in a paper [14] on *Infinite products of analytic matrices* where he generalized the theorems of Weierstrass and Mittag-Leffler on the representation of an analytic function by an infinite product.

III. LINEAR DIFFERENCE EQUATIONS

In his study of the Riemann problem Birkhoff had mastered and extended the use of formal solutions and asymptotic representations. He had used the formal expansions not only to obtain asymptotic solutions but to find the invariant characteristics of the singular points and the a priori limitations on the monodromy group. This use of formal solutions was to remain characteristic of his mode of attack in at least three other main fields, *linear difference equations, stability in dynamical systems*, and *surface transformations with an invariant integral.* The underlying analogies were great but the innovations which remained to be made in each case were much greater and would discourage most mathematicians.

In his *Fifty years of American mathematics* [54] Birkhoff writes that it was the lectures of Van Vleck at Wisconsin on difference equations that led him to an appreciation of the open problems in this field. The thesis of Carmichael in 1911 under Birkhoff at Princeton was perhaps the first significant American contribution on difference equations. Birkhoff's own work began with [15], published in 1911 in a volume of the Transactions that also contained Carmichael's thesis.

Birkhoff [15] dealt[10] with a system of n linear homogeneous difference equations of the first order rather than with a single equation of the nth order studied by Carmichael. The system was assumed regular in the sense that the characteristic roots were simple and neither 0 nor ∞. In matrix form the system took the form

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[10] The writer is indebted to C. R. Adams for a summary of Birkhoff's work on difference equations.

$$Y(x+1) = A(x)Y(x) \qquad (|A(x)| \not\equiv 0)$$

with $A(x)$ a square matrix of rational functions. Two formal matrix solutions arose

$$A^{-1}(x)A^{-1}(x+1)\cdots,$$
$$A(x-1)A(x-2)\cdots$$

leading to two "principal solutions" $Y_+(x)$ and $Y_-(x)$. These principal solutions were analytic except for poles in the finite x-plane, and were asymptotically represented in the left and right hand planes respectively by formal solutions. Results similar in character to these had been established by Nörlund and Galbrun a few months earlier using methods based on Laplace transformations.

The matrices of constants which are used in the theory of linear differential systems are here replaced by matrices $P(x)$ of functions with a period 1. In particular a matrix $P(x)$ of this type was determined such that

$$Y_-(x) = Y_+(x)P(x).$$

The "characteristic constants" of the system include the constants in $P(x)$ and certain other constants defined by formal solutions in series form. The "generalized Riemann problem" was to determine a system of difference equations whose principal solutions have the given characteristic constants. Properly modified, this problem and its analogues for q-difference equations were solved in [13].

The irregular case remained unsolved at this time although Adams had carefully examined the extent to which Birkhoff's methods so far published were sufficient. In [16] (1930) Birkhoff presented a complete solution of the formal problem with strong analogies with the corresponding solution in differential equation theory. His last major paper [17] in this field was a significant collaboration with Trjitzinsky which extended and modified Birkhoff's earlier methods introducing new devices adequate for the analytic treatment of the general irregular case.

Birkhoff's writings in this field occurred in two short periods seventeen years apart. His interest in the field remained keen throughout his life as is shown by the doctoral theses written under him. His students in this field include R. D. Carmichael and K. P. Williams at Princeton; P. M. Batchelder, C. R. Adams, O. E. Lancaster, P. C.

Guenther and Jeanne S. LeCaine at Harvard. The last two students received their degrees in 1941.

IV. DYNAMICAL SYSTEMS

As with Poincaré the history of Birkhoff's researches in dynamics is one of successes which are partly complete and partly incomplete. The grand aim was to give a formal normal reduction of a dynamical system which distinguishes equilibrium points as to stability, and enables one to pass from these forms to a complete qualitative characterization of the system. Hamiltonian and Pfaffian systems formed the central core with the restricted problem of three bodies [**20**, **32**, **39**] a typical example. The formal aspects of Birkhoff's research in dynamics may be regarded as relatively complete. Normal forms of great generality were obtained.

On the qualitative side Birkhoff's success was unconditional in many important phases. His central and recurrent motions belong to a theory which is valid for systems with any degree of freedom. His ergodic theorem is, however, conditioned in its usefulness by the uncertainty as to the hypothesis of metric transitivity. By far the greater part of his qualitative analysis was restricted to systems with two degrees of freedom, $m=2$; this is inherent in his dependence on "regular surfaces of section" whose existence seems likely for $m=2$ but is in general unproved, and whose existence for $m>2$ is doubtful. Assuming metric transitivity and the existence of surfaces of section ($m=2$) he is able to reveal an astonishing range of ordered but intricate dynamical phenomena; in this he is close to realizing a complete, qualitative characterization.

As did Poincaré, Birkhoff seeks to make the periodic orbits central in the theory. Poincaré's conjecture that these orbits are everywhere dense in phase space turns out to be false in special cases but is believed true in general for $m=2$. Poincaré was stopped by the inadequacy of his method of analytic continuation of these orbits. In a series of studies Wintner[11] and others have penetrated deep into the mystery, but in the general case much remains to be done. It is the conjecture of the writer that the method of topological continuation[12] which Birkhoff does not use may be an essential key to the situation. We shall continue with a summary of various aspects of Birkhoff's theory.

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[11] A. Wintner, *Grundlagen einer Genealogie der periodischen Bahnen im restringierten Dreikörper Problem*, Math. Zeit. vol. 34 (1931) pp. 321–349, 350–402.

[12] See Marston Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications, vol. 18, New York, 1934, chap. 9.

(a) *Formal dynamics.* In general form a dynamical system is locally defined by differential equations [26]

$$\frac{dx_i}{dt} = F_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n). \tag{4.1}$$

The functions F_i are assumed analytic in the neighborhood of an initial point (x^0). Hamiltonian systems have the special form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, \cdots, m) \tag{4.2}$$

where H is a function of $2m$ variables

$$(p_1, \cdots, p_m; q_1, \cdots, q_m).$$

Pfaffian systems are defined by the Euler equations of an integral

$$\int_{t_0}^{t_1} [X_j(x_1, \cdots, x_{2m})x_j' + Z(x_1, \cdots, x_{2m})]dt \tag{4.3}$$

in the variational problem in nonparametric form in the space $(x_1, \cdots, x_{2m}, t)$. In order to be able to solve the Euler equations for the x_j' it is assumed that the determinant of coefficients of these variables in the Euler equations is not zero. The Hamiltonian and Lagrangian equations also admit the Hamilton and Jacobi variational forms.

Birkhoff speculates on the meaning of these variational forms. His views seem to have evolved somewhat with the passage of time. In [26] he points out that these variational forms have no "special significance" near a point which is not a point of equilibrium. At a point (x^0) of equilibrium

$$F_i(x^0) = 0 \qquad (i = 1, \cdots, n) \tag{4.4}$$

in (4.1). He indicates the formal convenience of the variational form when a transformation of coordinates is to be made. He later shows [25] that the Hamiltonian form is one to which (4.1) can be formally reduced neighboring a point of completely formal stability of general type (to be defined later) provided the formal transformations used are sufficiently general in character. Finally in 1943 [44, p. 310] he writes as follows:

"It is only the fact that there is a single explicit variational form available in the entire domain of the independent variables that is really significant. Possibly this interesting situation indicates that the basic importance of variational principles will be found to be *topological.*" (Italics are Birkhoff's.)

The situation is relatively clear in ordinary dynamics. The writer wonders in what form topology will first appear in quantum mechanics.

Most of Birkhoff's work is concerned with systems (4.1) which involve t in the functions F_i. This occurs whenever a periodic[13] orbit $x_i = x_i(t)$ is given and one then makes a change of coordinates

$$x_i = x_i(t) + \bar{x}_i \qquad (i = 1, \cdots, n).$$

Starting with (4.1) in its original form this transformation leads to a new function F_i of $(\bar{x})$ and of t as well, which vanishes at the origin in the space $(\bar{x})$ for all values of t. The origin $(\bar{x}) = 0$ is then termed a point of *generalized equilibrium.*

Neighboring such a point the modified system (4.1) is subjected to formal transformations in which x_i is replaced by a formal series in new variables y_i with coefficients which are analytic and periodic in t. These coefficients are permitted to have complex values but in such cases the transformation is to be such that the new variables y_i can be arranged in pairs which are formally conjugate in terms of the original real variables. In the case of the Hamiltonian system the new variables (p_i, q_i) are taken as conjugate.

Corresponding to an equilibrium point the equations[14] of variations of (4.1) can be brought to a canonical form. In the so-called *general*[15] *case* this canonical form is

$$\frac{dy_j}{dt} = \lambda_j y_j \quad (j \text{ not summed} = 1, \cdots, n) \tag{4.5}$$

where the λ_i's satisfy no relation

$$m_1\lambda_1 + \cdots + m_n\lambda_n + m_{n+1}(-1)^{1/2} = 0$$

in which the m_j's are integers not all zero. In the case of the Hamiltonian or Pfaffian system the multipliers can be grouped in m pairs

$$(\lambda_1, \ -\lambda_1), \cdots, (\lambda_m, \ -\lambda_m)$$

where the λ_i's are either real or pure imaginary. With this understood a prime result of Birkhoff is as follows [26].

By suitable formal (generally divergent) transformations a Hamiltonian or Pfaffian system with a generalized equilibrium point of "general" type at the origin may be taken into a normal Hamiltonian form in which

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[13] It will be convenient to take this period as 2π throughout.

[14] Involving the terms of first order in (4.1).

[15] The "general" case is referred to throughout in this sense.

$H(p, q)$ is a formal power series in the products (p_iq_i) starting with the quadratic terms

$$\lambda_j p_j q_j \qquad (j = 1, \cdots, m). \tag{4.6}$$

The case in which H reduces to the quadratic terms (4.6) is termed *degenerate.* Birkhoff is largely concerned with a "generalized equilibrium point" of "general" type with a "non-degenerate" Hamiltonian function H, as these terms have been defined.

The representations (4.1) which have been described are local. The whole of "phase space" is supposed "covered" by a finite number of such local representations each defined in a region of a space of parameters (x). *The phase space M is here supposed to be compact.* A transformation T of M is defined in which each point P corresponds under T to the point P_t on the trajectory through P. The origin for t is so selected that $P_0 = P$. This one-parameter family of transformations T is called a *flow.* In the case of certain systems (that is, Hamiltonian systems) there exists an integral of the form $f(V)$ where V is a measurable subset of M such that

$$f(V) = f(T(V))$$

where $T(V)$ is the image of V under T. The integral $f(V)$ is then termed *invariant* and the system *conservative.* The integrand of f is supposed positive and may ordinarily be taken as analytic.

Various types of stability are met, and these must be carefully distinguished. Historically, one of the most important is the property of a conservative system whereby the images of any open region R under the flow intersect R for values of t which cluster at ∞.

First order formal stability is defined relative to a generalized equilibrium point of a Hamiltonian or Pfaffian system. It requires that the multpliers λ_i be purely imaginary.

Formal trigonometric or *complete stability* occurs as follows. Let T be a time interval, m any positive integer, P_s any polynomial in (x) with coefficients analytic and periodic in t and of lowest degree s. It is then possible, for $e > 0$, to approximate P_s, for time intervals less than T, with an error numerically less than Me^{m+s} with a suitably chosen sum,

$$\sum_{j=0}^{N} (A_j \cos k_j t + B_j \sin k_j t) \qquad (k_i - k_j > k > 0)$$

in which M, N, k are constants dependent only on m and P_s $(k_0 = 0)$.

(A) *It is a capital result of Birkhoff that first order formal stability of Hamiltonian or Pfaffian systems in the case of a point of "generalized* *equilibrium" of "general" type implies formal trigonometric stability* [26].

(b) *Recurrence and transitivity.* A subset S of the phase space M which is carried into itself by the "flow" on M is called *invariant.* A point P of S is called *wandering* relative to S if there exists a neighborhood W of P relative to S which, for some $t_0 > 0$, and for $t > t_0$, has no image which intersects W. A maximal invariant subset S^* of S whose points are non-wandering relative to S^* is called *central* in S. Birkhoff shows by transfinite induction that every non-empty invariant subset S of M possesses a central subset S^*. Such a set S^* is closed relative to S and non-empty. If the system is conservative as defined earlier, the set which is central relative to M is M; but if M is non-conservative, the central set M^* of M is in general a proper subset of M. Let N be an arbitrary neighborhood of M^* on M. The fraction of time which the finite arc P_0P_t spends in N tends to 1 as t becomes infinite, independently of the choice of P_0 on M [26].

A closed invariant subset of M no proper subset of which is a closed invariant set is called *minimal* and any sub-motion (complete trajectory) of the set is called *recurrent* [18]. A periodic motion is a very special minimal set. For a motion P_t to be recurrent it is necessary and sufficient that for every $e > 0$ there exists a $T > 0$ so large that any arc P_tP_{t+T} of the motion has points within a distance e of every point of the motion. Limit points of a motion P_t as t becomes negatively or positively infinite respectively are called α- and *ω-limit points* respectively. Birkhoff shows that the α- (or ω-) limit motions of a given motion include at least one minimal set. In general, minimal sets do not reduce to a periodic motion. Recurrent motions on surfaces of negative curvature can be studied and classified with the aid of the topological symbolism of Hadamard,[16] as Morse first showed in his thesis.

In his desire to determine the ultimate significance of formal stability of Hamiltonian or variational forms, Birkhoff goes beyond the case of Hamiltonian or Pfaffian systems. In such general dynamical systems (A) is not true. Birkhoff defines even more general formal transformations and states that if these be admitted, any dynamical system can be formally reduced near an equilibrium point of general type to Hamiltonian form provided the conditions of first order and formal trigonometric stability are satisfied [25]. The meaning of this bold step is not altogether clear.

Departing from the domain of formal operations, *permanent stabil-*

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[16] For a general theory see Marston Morse and Gustav A. Hedlund, *Symbolic dynamics*, I and II, Amer. J. Math. vol. 60 (1938) pp. 815–866; vol. 62 (1940) pp. 1–42.

ity is said to occur when sufficiently small initial displacements at $t=t_0$ from an initial solution cause displacements from the initial solutions which are arbitrarily small for all time. Formal conditions for permanent stability are not known. In particular it is not known whether or not a point of generalized equilibrium which is formally "completely" stable is in general permanently stable. The theory in the large is tied up with the theory in the small because the validity of the hypothesis of metric transitivity in a system would imply that permanently stable periodic orbits do not exist.

A motion is termed *transitive* if it is everywhere dense in the phase space M. If such a motion exists the system is called *topologically transitive.* Birkhoff exhibits examples of such transitivity. The most general case of topological transitivity established up to this time in dynamical systems of classical type is that of geodesic motion on a surface of genus greater than 1 on which no point has a conjugate point.[17]

Under *metric transitivity*, as first defined by Birkhoff and P. A. Smith [27], the only invariant subsets of M are sets whose measure is 0 or $m(M)$. Metric transitivity implies topological transitivity. The converse is probably true in analytic systems or systems with some degree of analytic regularity. A proof or disproof of this converse is much needed.

Smith and Birkhoff [27] have defined central motions of iterates of a surface homeomorphism and characterized them in various ways. Metric transitivity is shown by them to be equivalent to the condition that no two "invariant integrals" on S be linearly independent. In the general analytic case at least two central motions are shown to exist.

B. O. Koopman [30] discovered that a conservative flow T_t on M with an invariant measure on M could be represented by a one-parameter family of unitary transformations in Hilbert space, thus opening up a new mode of investigation of dynamical systems. Following this, von Neumann established his "mean ergodic theorem" appropriate to Hilbert space. Under the stimulus of these ideas, Birkhoff [29] saw that in treating transitivity open sets should be replaced by sets of positive measure. Birkhoff's first theorem is what is commonly known as his *ergodic theorem*. We shall give a formulation due to Khintchine.[18]

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[17] Marston Morse and Gustav A. Hedlund, *Manifolds without conjugate points.* Trans. Amer. Math. Soc. vol. 51 (1942) pp. 362–386. For other aspects of the problem, see J. C. Oxtoby and S. M. Ulam, *Measure preserving homeomorphisms and metrical transitivity*, Ann. of Math. vol. 42 (1941) pp. 874–920, and P. R. Halmos, *In general a measure preserving transformation is mixing,* ibid. vol. 45 (1944) pp. 786–792.

The space M is assumed to have a finite measure m invariant under the flow. Let f be integrable over M and let P be a point of M. Then

$$\lim_{T\to\infty} \frac{\int_0^T f(P_t)dt}{T} \tag{4.7}$$

exists for almost all P on M.

Extensions and applications of the ergodic theory have been made by Wiener, Wintner, E. Hopf, Garrett Birkhoff and others. If f is the characteristic function of an arbitrary measurable subset V of M, the integral in (4.7) is the measure of the time spent by P_t in V for $0 \leq t \leq T$, and, in case the system is metrically transitive, Birkhoff shows that the limit in (4.7) equals the ratio of $m(V)$ to $m(M)$ for almost all P on M.

(c) *Periodic orbits.* In the case of a Hamiltonian or Pfaffian system with two degrees of freedom and with a periodic orbit λ (period 2π) one can use an integral to reduce the dimension of the phase space containing λ to $n=3$. Let S be an element of a regular analytic "surface of section" which cuts across λ at some point Q. If P is a point of S sufficiently near Q the orbit P_t will cut S again later at a point $T(P)$. Poincaré introduced these transformations T of S into itself in order to establish the existence of periodic orbits in the neighborhood of λ. It is clear that a necessary and sufficient condition that P represent an orbit with a period near 2π is that

$$P = T(P).$$

One can similarly use the kth iterate $T^k(P)$ of the transformation $T(P)$ to find orbits near λ with periods near $2k\pi$.

The orbit λ is termed *simple* if its equations of variation have no nonzero periodic solution. If λ is simple it cannot be a member of an analytic family of periodic orbits, and, bearing in mind no doubt that this excludes systems classically called "integrable," Birkhoff terms the problem *non-integrable* if every periodic orbit and its multiples are simple [26].

A generalized equilibrium point has been termed *degenerate* if the Hamiltonian function H can be formally reduced to quadratic terms. A system in which each periodic orbit is nondegenerate is termed *nondegenerate.* Non-integrable nondegenerate systems exist as Birkhoff shows. Let (u, v) be regular coordinates on S neighboring a fixed point (0, 0) of T. If the system is non-integrable and nondegenerate the terms of first order in T reduce either to a rotation about (0, 0) through an angle incommensurable with 2π (fixed point of elliptic type) or to the form

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[18] A. Khintchine, *Zu Birkhoff's Lösung des Ergodenproblems*, Math. Ann. vol. 107 (1933) pp. 485–488.

$$u' = \rho u, \qquad v' = \frac{1}{\rho} v \qquad (\rho \neq 0 \text{ or } 1)$$

where ρ is real (fixed point of hyperbolic type) [23].

Following Poincaré, Birkhoff makes use of the existence of an invariant integral in the original phase space to obtain an invariant integral [23, 33]

$$\int\int Q(u, v)dudv \qquad (Q > 0)$$

on S neighboring the origin, where Q is real and analytic in (u, v). Birkhoff shows that there exists a real power series $F(u, v)$ which may be convergent or divergent, and is formally invariant under T. The series F starts with terms of the second degree or higher. The series Q and F are formally related by differential equations

$$Q\frac{du}{dk} = F_u, \qquad Q\frac{dv}{dk} = - F_v$$

whose formal solutions u and v starting with terms of the first order yield T^k when k is replaced by an integer.

Formal series transformations of u and v reduce T to a transformation

$$u_1 + iv_1 = (u + iv)e^{h(r)i} \qquad (h(r) = a + cr^{2m}) \tag{4.8}$$

when λ is simple and nondegenerate and the fixed point (0, 0) elliptic. Here a and c are real constants with $c \neq 0$ and a incommensurable with 2π. The exponent m is a positive integer in general 1. Whenever (4.8) holds, T is actually as well as formally very close to a rotation through an angle a, for sufficiently small positive r. For a simple nondegenerate orbit and a hyperbolic fixed point, T has the normal form

$$u_1 = \rho u e^{c(uv)^m}, \qquad v_1 = \frac{1}{\rho} v e^{-c(uv)^m} \qquad (m > 0; c \neq 0)$$

where ρ is real and neither 0 nor 1. The use of these forms to obtain approximate representations of T is reminiscent of Birkhoff's use of asymptotic representations in his thesis, in the Riemann problem and in the formal Hamiltonian theory.

In [33] Birkhoff largely removes the condition that the orbit be "simple" although this task is one of great formal complexity. When the periodic orbit is "multiple" the terms of first order in the representation of T have equal characteristic roots. In such special cases the corresponding fixed point Q of T is termed *hyperbolic* if there exists a real invariant curve passing through Q. Otherwise Q is termed *elliptic*.

To establish the existence of periodic orbits in the restricted problem of three bodies Poincaré introduced his celebrated "last geometric theorem" but was unable to prove it.[19] This theorem may be stated as follows:

Given a ring $0<a\leqq r\leqq b$ *in the* r, θ, *plane and a* 1-1 *continuous area-preserving transformation* T *of the ring under which points on* $r=a$ *advance, and those on* $r=b$ *regress, there will exist at least two points of the ring invariant under* T.

A number of mathematicians had attempted to prove this theorem and at least one erroneous proof was published. Urged on by his colleagues, Birkhoff [19] gave a beautiful proof of the theorem that went directly to the heart of the problem. Modified forms of the theorem were also presented by Birkhoff [24].

Using theorems of a nature similar to that of Poincaré, Birkhoff proves the following in [26]:

(α) *In non-integrable, nondegenerate Hamiltonian systems with two degrees of freedom any neighborhood of a periodic orbit of elliptic type includes infinitely many periodic orbits both of the elliptic and hyperbolic types. At most a finite set of these orbits have periods less than a given constant.*

A generalization of the Poincaré theorem to higher dimensions is given in Birkhoff [28]. This generalization employs the theorem of Morse that the number of critical points, suitably counted, of an analytic function defined on the product of n-circles is 2^n. With D. C. Lewis, Birkhoff's theorem is applied in [31] to establish the existence of periodic orbits neighboring a periodic orbit of general, formally stable type.

A periodic orbit λ of hyperbolic type cannot be permanently stable. When λ in α is hyperbolic there exist [23] two analytic curves on S passing through (0, 0) and *invariant* under $T(P)$. These invariant curves on S imply the existence of analytic families of motions asymptotic to λ in either sense.

No example is known in which the elliptic orbit λ of (α) is "permanently stable" in the geometric sense and none such could exist if the system is metrically transitive. Nevertheless Birkhoff gives a relatively complete description of motions neighboring λ in case λ is in fact permanently stable. Since this description is similar to the one given by Poincaré, it is omitted.

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[19] H. Poincaré, Rend. Circ. Mat. Palermo vol. 33 (1912) pp. 375–407.

A priori existence theorems for periodic orbits are very necessary. Birkhoff turns to the "direct" method in the calculus of variations, and uses the Jacobi "action integral" J. In Lagrangian problems in which the constraints are independent of the time this integral will in general be regular and positive definite in the sense of classical variational theory. A periodic minimizing extremal will then exist in an "extremal convex" region. No periodic minimizing extremal exists in very general problems, as for example in the restricted problem of three bodies with a parameter μ sufficiently small. To fill in this gap in the theory Birkhoff applied a "minimax principle" [21].

In Birkhoff's applications this principle reduces to an existence theorem for critical points of an analytic function $F(x)$ of n-variables. If one supposes for the sake of definiteness that F is defined over a regular, compact, analytic manifold, then, suitably counted, there exist at least R_1+M_0-1 generalized saddle points, where R_1 is the linear connectivity of the manifold and M_0 the number of points (supposed isolated) of relative minimum of F. In similar or related forms this principle was known and applied by Poincaré, Maxwell, and Kronecker, and has an origin even more remote in the past. Birkhoff's bold step was to conceive of its application to functions of curves such as the integral J. He applied it in the billard ball problem [26] (motion on a convex table) and to obtain closed geodesics on a convex surface. On slipping an elastic band over the surface from one egg-shaped end to the other the process that requires the least stretching leads to a closed geodesic of minimax type. However, Birkhoff's conjecture that the closed geodesic analogously obtained for a topological n-sphere is of "minimax type," as characterized by him, is not true if $n>2$.

The a priori existence of periodic orbits is necessary for the dynamical theory as conceived by Poincaré and Birkhoff.

(d) *General qualitative dynamical theory*. This theory is limited almost wholly to the case of two degrees of freedom $m=2$ largely because a fundamental tool with Birkhoff as with Poincaré is a "regular surface of section." One presupposes a three-dimensional, analytic, compact, nonsingular phase space. A *regular surface of section* S is then an analytic surface crossed in the same sense by all trajectories without being tangent to S except along the boundaries. These boundaries are to be finite in number and consist of closed trajectories. Further every trajectory except the boundary orbits shall cut S at least once in any sufficiently large fixed interval of time.

It seems likely that such surfaces of section exist in general when $m=2$. As Poincaré indicated, such a surface exists in the restricted problem of three bodies when the mass μ (that is, earth) is sufficiently small. Although open hyper-surfaces of section exist when $m>2$ it has not yet been found possible to make profitable use of them.

As in (c) the "flow" in the phase space defines and is characterized by a conservative transformation T of S into itself. Birkhoff makes three tentative hypotheses none of which has been proved to hold in general, but each of which probably does so hold. These hypotheses as given in [**33**] are:

(A.1) The dynamical system is transitive.

(A.2) A regular surface of section S exists.

(A.3) There is at least one point which is fixed under T, but any such point, if elliptic, corresponds to a nondegenerate, periodic orbit.

This qualitative theory culminates in the paper [**33**] 130 pages long, crowned by the Pontifical Academy. This paper resumes and extends much of Birkhoff's earlier dynamical theories. Upon finishing this work Birkhoff commented to the author that it was an exhausting task. In this connection it is of historical interest to recall that Birkhoff stated about 1925 that he considered his Transactions paper [**21**] for which he received the Bôcher prize as good a piece of research as he would be likely to do. In the writer's opinion, the Rendiconti memoir [**20**], also a prize paper, should be placed near the top of Birkhoff's works, not because it is final but because one finds Birkhoff there first meeting the problems which he inherited from Poincaré, and taking the first clear cut concrete steps which he later generalized.

The Pontifical memoir can only be broadly summarized. After extensive formal studies already reported on, Birkhoff shows that under hypotheses (A) there exists an infinite number of hyperbolic points fixed relative to some of the iterates T^k of T. There may exist no elliptic points, but if one exists, there exist infinitely many such elliptic points. The general existence of the hyperbolic fixed points makes them a suitable instrument on which to base a qualitative characterization of T and this Birkhoff proceeds to do.

On S the invariant curves C through a hyperbolic fixed point P are divided into α-branches $E_\alpha(P)$ and ω-branches $E_\omega(P)$ according as the iteration of T^{-1} or T makes a point on C tend toward P on C as a limit. In the original phase space each point on an α-branch (ω-branch) defines a motion which becomes negatively (positively) asymptotic to the periodic orbit determined by P. Returning to S, Birkhoff shows that under hypotheses (A) these α- and ω-branches associated with a single hyperbolic fixed point P, if continued analytically, are *everywhere dense* on S. There is an exception which does not occur for most fixed points P when one of these branches belongs to a second hyperbolic fixed point. These α- and ω-branches intersect in infinitely many points termed "homoclinic" by Poincaré. Homoclinic points are proved to be cluster points of fixed points.

In the case of a metrically transitive system the set of all fixed points and of all α- or ω-branches attached to hyperbolic fixed points has a measure zero on the surface of section. The remaining points are grouped into sets Σ_Q termed *isomorphic* by Birkhoff. Let Q be a point which belongs to no set $E_\alpha(P)$ or $E_\omega(P)$. The maximal connected set of points which contains Q but no point of a set $E_\alpha(P)$ or $E_\omega(P)$ is independent of P among hyperbolic fixed points P and is denoted by Σ_Q. These sets are transformed into each other under T, hence the term isomorphic. It is probable that $\Sigma_Q = Q$ in general; this case is called *regular*.

Birkhoff shows in what sense the intersections of an $E_\alpha(P)$ and an $E_\omega(P)$ "characterize" the isomorphic sets Σ_Q. He invents a 2-dimensional symbol called the *signature* of the system which displays the topology of the intersections of $E_\alpha(P)$ and $E_\omega(P)$. *For Birkhoff such a signature is the ultimate in the qualitative description of a dynamical system.*

These α- and ω-branches belonging to a hyperbolic point P are properly called asymptotic to P. Birkhoff also defines connected sets asymptotic to elliptic points. These are sets and not curves, at least a priori. They were studied at length by Birkhoff before he recognized the importance of the α- and ω-branches belonging to hyperbolic points. He indicates the formal analogies between the sets asymptotic to elliptic points and the α- and ω-branches attached to hyperbolic points.

(e) *The restricted problem of three bodies.* This famous problem was studied by Birkhoff in three principal papers [20, 32, 34]. Much of the work in these papers is an application or exposition of general theories elsewhere reported. There remain several special achievements to be noted.

The two finite bodies (that is, sun and earth) have masses $1-\mu$ and μ respectively, and the third body (x, y) has a mass which is infinitesimal. The two principal bodies are originally supposed to rotate about their common center of gravity in the clockwise sense. In the representation of the problem in the (x, y)-plane the coordinates have been so transformed that the earth and sun rest fixed at points on the x-axis with their center of gravity at the origin. The problem then has the Jacobi integral

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega(xy) - C$$

where Ω is a positive function of special nature.

When C is sufficiently large the point (x, y) is confined in "Case I" to an oval about one of the finite masses. The phase space is then shown by Birkhoff to be the homeomorph of the three-dimensional projective plane. Topological models are given for the phase space for other values of C. Birkhoff keeps to Case I in general.

Direct periodic orbits had been established by Poincaré only for sufficiently small values of μ. Birkhoff shows that in Case I at least one retrograde periodic orbit symmetric with respect to the x-axis *always* exists. He replaces the ring transformation of Poincaré which led to the "last geometric theorem" by a transformation of a diskoid into itself. As long as this construction is possible there exists a direct periodic orbit. But such a construction is definitely established only for sufficiently small values of μ. However the existence of the diskoid seems to demand less by way of proof than the existence of the Poincaré ring.

Returning to the ring transformation of Poincaré, Birkhoff concerns himself with the existence of orbits which are symmetric with respect to the x-axis. He shows that the transformation of the ring surface of section S into itself is the product of two involutory transformations. He attaches *two characteristic integers* to a symmetric periodic orbit as follows. Suppose that this orbit cuts S at a point P. If T represents the ring transformation of Poincaré there is a least integer k such that the $T^k(P) = P$. The k-iterates of P on S will rotate about the Poincaré ring l times. The pair (k, l) characterize the orbit. Birkhoff shows that there are infinitely many pairs (k, l) which represent symmetric periodic orbits and he exploits this symbolism to the full.

After reading the preceding report on Birkhoff's advances in dynamics the reader may find it of interest to compare a summary [22] in 1920 and a prospectus [35] in 1941 by Birkhoff himself. The summary occurs under the title *Recent advances in dynamics* and the prospectus is entitled *Some unsolved problems of theoretical dynamics* and was presented at a fiftieth anniversary symposium at the University of Chicago in 1941. After a lapse of twenty-one years the problems remain essentially the same, but the approach and particularly the language of approach has become more topological and abstract. "Conservative flows" are to be studied both in the topological and the statistical sense, and abstract variational theory is to enter. There is no doubt about the challenge of the field, and the need for a powerful and varied attack.

V. PHYSICAL THEORIES[20]

The earliest evidence of Birkhoff's interest in relativity can be found in a review of *Books on relativity* [36], in 1922. In a few words, while commenting upon the development of Einstein's general theory, he formulates what has turned out to be his own ideal in the creation of physical theories: "In the first place, it illustrates afresh the importance of taking the simplest possible case as an abstract basis of departure. Secondly, Einstein uses mathematical analogy in passing, step by step, from the simple universe of the special theory to the most general universe, and at each step the mere sense of mathematical form is sufficient to point the way to a natural generalization. The mathematician may feel satisfied that the formal analogies supplied by classical dynamics and four-dimensional geometry furnish the very basis by which Einstein's generalization proceeds" (page 217).

In 1923, in his widely acclaimed monograph, *Relativity and modern physics* (written with the cooperation of R. E. Langer), Birkhoff carries out this ideal as closely as he can by establishing the electromagnetic equations, and the fundamentals of the special and general theories of relativity, showing how this can be done without indiscriminate appeal to physical intuition, and with a maximum of regard for geometrical symmetry, mathematical rigor and simplicity.

Feeling that "without a true model as a starting point, it does not seem likely that a final conception of the physical universe can be arrived at" [37, p. 70], Birkhoff selects a "perfect fluid," which satisfies the classical equations of hydrodynamics, slightly modified and unified so as to be invariant under the Lorentz transformations. The pressure-density relation was such that all disturbances would be propagated with the speed of light. This fundamental property of the substance of which all matter is composed was intended to eliminate those paradoxes in the theory of collision which Birkhoff pointed out in his presidential address of 1926 [40].

In this address, Birkhoff urged theoretical physicists to attempt to account for atomic phenomena using mainly the ideas of special relativity and abstract dynamics, for he felt that the recurrence properties of dynamical systems could eventually provide an intuitive explanation of quantum effects (cf. also [42, p. 317]). He made a modest attempt in this direction himself [39, 40]. He welcomed the return to the use of differential equations, due to Schrödinger, and justified these equations, from a general point of view, in [41].

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[20] This account of Birkhoff's physical theories has been written by Morse's assistant Dr. Richard Arens, who had the advantage of numerous conversations on this subject with Birkhoff in the last years.

Although it soon appeared that the "perfect fluid" could not be limited to a desirable behavior by its hydrodynamical equations alone [42, p. 324], Birkhoff retains it as raw material for "mathematical model-making," perhaps because he feels that any physical theory should contain some statement about the constitutive equations of "matter" [44, p. 301]. The best cross section of Birkhoff's ideas after ten years of contemplation can be obtained from his Franklin Institute Lecture of 1938, *Electricity as a fluid* [43], in which he states his ideas, opinions, and preferences with unusual directness, clarity and humor.

The final phase, which was undoubtedly intended to be only the beginning, of Birkhoff's work in relativity was his gravitational theory [43] of 1943. With such a degree of analogy to electrodynamics as the essence of things permits, this theory involves a gravitational tensor potential governed by a linear differential equation which is, as seems proper in view of the underlying flat but Minkowskian space-time, of hyperbolic type. Whereas in electrodynamics the acceleration of a charged particle is a linear function of the velocity with coefficients characterizing the field, here the acceleration depends quadratically on the velocity. The theory is completely linear, and homogeneous except for the presence of the "perfect fluid."

As regards experimental verification, Birkhoff's theory predicts the same rate of rotation of the apse line of planetary orbits as does the general theory of relativity. Recently, moreover, a practical method of successive approximation for treating the two body problem has been worked out.[21]

In Birkhoff's theory, phenomena such as the bending of light around the sun cannot be regarded as of electromagnetic nature, since there is no interaction of electromagnetic and gravitational field; but Birkhoff finds that by considering the world lines of "photons" as limits of world lines of particles on which the speed approaches that of light, the proper magnitudes for the bending of light and red-shift effect can be obtained [44a]. He has also discovered another approach to the red-shift effect which makes use of the wave-length-energy relation of Planck. These ideas merit careful investigation, even apart from the remaining theory.

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[21] C. Graef Fernandez, *El movimiento de los dos cuerpos en la teoria de la gravitación de Birkhoff*, Boletin de la Sociedad Matemática Mexicana vol. 1 (1944) p. 25.

It has been stated[22] that this theory does not provide for the identity of gravitational and inertial mass in as effortless a manner as the theory of Einstein, and this seems to be a defect from which linear theories must suffer.[23]

Birkhoff inherited from Poincaré the sentiment that no single mathematical theory of any phenomenon deserves the exclusive attention of physicists, or at least of mathematicians.

VI. MISCELLANEOUS WORKS

(a) *Fixed points in function space.* The paper [45] written jointly with O. D. Kellogg is undoubtedly one of Birkhoff's most important contributions. The first theorems are generalizations of the Brouwer theorem that a continuous transformation of an n-disc into itself has at least one fixed point. This is applied to a transformation of a space of uniformly bounded and equicontinuous functions $f(s)$ and of the space of functions $f(s)$ which are summable square and representable by a subset of points in Hilbert space for which $\sum x_i^2$ converges uniformly. Fixed elements $f(s)$ are inferred.

A second point of departure is the theorem that a continuous transformation of a real projective space of even dimensions into itself has at least one fixed point or "invariant direction." This theorem extends to integral equations.

Deformations of transformations are extensively used. A continuous transformation T_λ on an n-sphere H_n into the embedding $(n+1)$-space is given as depending linearly on a parameter λ in such a manner that as λ varies to 0, T_λ deforms into the identity without intersecting the origin. It is concluded that to each point (b) on H_n corresponds a point (a) ("inverse direction") on H_n whose image under T_λ is on the same ray as (b). This theorem has applications in the Fredholm theory.

It is of interest that the paper in the Transactions preceding [45] was an elegant treatment by Alexander of "transformations with invariant points," including many of the theorems in [45] concerning transformations of finite-dimensional spaces but not including applications to function spaces. The work of Birkhoff and Kellogg was the acknowledged stimulus of the noteworthy extensions of Schauder, and later of Schauder and Leray, including important applications to partial differential equations. The invariance of the Brouwer degree of a transformation under continuous deformations is a unifying element in the Schauder-Leray theory.

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[22] H. Weyl, *Comparison of a degenerate form of Einstein's with Birkhoff's theory of gravitation,* Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) pp. 205–210, especially p 205.

[23] Loc. cit. p. 206.

It is of interest to recall that Birkhoff's introduction to fixed point theory was by way of the lemma of Poincaré that the sum of the signed indices of an analytic vector field on an orientable surface of genus p is $2-2p$. Birkhoff used this theorem in his theory of elliptic and hyperbolic fixed points of the surface transformations arising in dynamics [21].

(b) *The 4-color problem.*[24] Birkhoff's interest in this problem was aroused in Veblen's seminar in "analysis situs" during his years at Princeton. Birkhoff brought out two papers [46, 47] on this subject during his stay at Princeton.

In [47] Birkhoff revived the qualitative approach based on Kempe chains. He found certain "reducible configurations," that is, figures whose presence in a regular map reduces the coloration problem to that of a simpler map. Of these the most important and the hardest to analyze was a ring of two regions not surrounding a regular pentagon. Of such reducible configurations later found by Franklin, Errera, and Winn, many are direct extensions of those of Birkhoff and most require for their proof Birkhoff's theorem on rings of five regions. The minimum number of regions in a map not capable of coloration in four colors was shown successively to be at least 25 (1922, Franklin), 27 (1927, Reynolds), 31 (1938, Franklin), and finally 35 (1940, Winn).

In [46] Birkhoff introduced the quantitative approach. "Chromatic polynomials" $P(x)$ equal to the number of ways a given map can be colored in x colors were found. Although the main objective of showing that $P(4)>0$ was not achieved, many properties of $P(x)$ were obtained by Birkhoff [46, 48] and later by Whitney whose thesis was on this problem. Birkhoff hoped that the theory of chromatic polynomials could be so developed that methods of analytic function theory could be applied. Birkhoff took the first real step in this direction in [49].

The quantitative approach was not as successful in the study of the classical problem as the qualitative. Nevertheless Birkhoff felt that it might be easier to establish a stronger result concerning $P(x)$ than a direct and weaker theorem of qualitative type. In a long joint paper with D. C. Lewis, submitted to the Transactions, a fusing of the qualitative and quantitative approaches is sought by way of a study of $P(x)$, borrowing and modifying the method of Kempe chains. In this study the reducibility of certain configurations previously established by Kempe chains is established by quantitative methods. Deeper insight is thus gained.

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[24] The writer is indebted to Philip Franklin and to D. C. Lewis for summaries of work on the 4-color problem.

This paper includes a conjecture to which Birkhoff attached the maximum importance. The map of the sphere under consideration consists of $n+3$ simply-connected regions containing only triple vertices, that is, points on the boundaries of just three regions. Let $P_{n+3}(x)$ be the number of ways of coloring this map in x colors. It is known that this polynomial is of degree $n+3$ and vanishes when $x=0$, 1 and 2. Hence

$$Q(x) = \frac{P_{n+3}(x)}{x(y-1)(y-2)}$$

is a polynomial of degree n in x. The conjecture is that

$$(x-3)^n \leqq Q(x) \leqq (x-2)^n$$

for $x \geqq 4$ and $n > 0$. The conjecture is readily established for $x \geqq 5$. It is also established for $n < 9$ even when $x=4$. It has been verified for special maps of 12, 14, 15, 16, and 17 regions with $x=4$.

This paper also contains an asymptotic formula for $P_n(x)$ conjectured by Birkhoff as a result of heuristic probability considerations.

(c) *General mean value and remainder theorems.* The paper [50] of Birkhoff written in 1904 is somewhat complicated in form as is perhaps necessary. There is given the value of $f(x)$ and some of its derivatives up to the $(n-1)$st at n points on an interval (a, b). Birkhoff introduces a polynomial F which together with its derivatives equals f and its derivatives at the respective points in question. The function $f^{(n-1)}(x)$ is assumed continuous, and $f^{(n)}$ is assumed to exist on (a, b). The general remainder theorem involves f and F and includes most known remainder theorems. It has applications in the fields of mechanical differentiation and quadratures.

(d) *Aesthetic measure.* Birkhoff's interest in the analysis of art and music forms was of long standing. It culminated in his famous treatise on aesthetic measure [51]. To oversimplify this work of Birkhoff would result in a misrepresentation. For this reason and because of the readability of [51] no extended account of Birkhoff's work in this field will be presented.

Within each class of aesthetic objects Birkhoff defines the order O and complexity C so that their ratio

$$M = \frac{O}{C}$$

yields the "aesthetic measure" of any object of the class. Equal at least in importance to the particular value of M attached to an aesthetic object is the fact that, such an attempt at aesthetic analysis leads to distinctions and comparison of a more refined nature than those ordinarily made. His objects of study include polygonal forms, ornaments and tilings, vases, melody and harmony, and musical quality in poetry. Birkhoff's work is in the spirit of many a great artist and artisan of bygone days. The writer recalls the impression made upon him by finding Sheraton, the great creator of 18th century furniture, starting a treatise on design with descriptive geometry and ending with sideboards.

In addition to this principal work Birkhoff wrote a number of papers on the subject, and spent a half year on leave of absence in the Far East and Europe to gather material. His study has been of interest to artists, musicians, psychologists, historians, and even mathematicians. The writer has found Birkhoff's theories on melody of practical value in the problem of effective improvisation on the organ. The high value which Birkhoff put on the aesthetic is illustrated by the answer he gave a professional musician as to why one should study mathematics. "One should study mathematics," said Birkhoff, "because it is only through mathematics that Nature can be conceived in harmonious form."

(f) *Basic geometry.* Birkhoff and Ralph Beatley joined forces in the interest of the teaching of elementary geometry around 1929. In 1932 Birkhoff [52] wrote a set of postulates for plane geometry based on the scale and protractor. These studies were capped by a textbook [53] on *Basic geometry,* written with Beatley and published in 1940. The necessity of undefined terms and assumptions is emphasized from the outset, doing away with "self-evident truths." There are five fundamental postulates, seven basic theorems and nineteen other theorems together with seven in loci. The system of real numbers is incorporated into three of the five axioms leading at once to the heart of geometry. Reports on the use of the book in practice are favorable. In any case the introduction of new ideas from the pen of a man as eminent as Birkhoff should be a great stimulus to the subject of elementary geometry.

In summarizing the mathematical work of Birkhoff a sentence of Poincaré comes to mind. Of the periodic orbits in dynamical systems he says that they are "la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputé inabordable." Poincaré would be amazed to see the extent to which Birkhoff has widened this breech and opened many others. His zeal never flagged.

During the major part of his life Birkhoff was the acknowledged leader of American mathematics. His nomination as President of the International Congress of Mathematics that was to have been held at Cambridge, Massachusetts in 1940 was symbolic of this fact. Learned societies and universities the world over honored him. His work was crowned with four prizes. In the case of a man such as Birkhoff, a ranking at this time in the hierarchy of the great, in anticipation of the verdict of history, seems of doubtful value. The author can find no words adequate to define him except that he was profoundly unique. Those who knew him best can add very simply that he was a kind and courageous friend for whom they will have a lasting affection.

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50. *General mean value and remainder theorems with applications to mechanical differentiation and quadrature*, Trans. Amer. Math. Soc. vol. 7 (1906) pp. 107–136.

51. *Aesthetic measure*, Cambridge, 1933.

52. *A set of postulates for plane geometry based on scale and protractor*, Ann. of Math. vol. 33 (1932) pp. 329–345.

53. With Ralph Beatley, *Basic geometry*, New York, 1940.

54. *Fifty years of American mathematics*, Amer. Math. Soc. Semicentennial Publications, vol. 2, 1938, pp. 270–315.

PHYSICS AND THE WAVE EQUATION

J. C. SLATER

Often in the history of physics a guiding line of mathematical thought has permeated the whole of the science for years, tying together apparently unrelated branches of the subject, contributing to the unity of physics, but at the same time stimulating philosophical thought, and focussing attention on a branch of mathematics, and leading to its development. The Newtonian mechanics was such a guiding principle. From Kepler through Newton and up into the nineteenth century, more and more of physics was explicable in mechanical form. Even philosophy and economics and history felt the impact of rationalism. Mathematics felt the tide; calculus and the theory of ordinary differential equations grew up under the impetus of the physicist, who needed the mathematical methods to explain his physical facts. A second guiding principle was the variation principle. D'Alembert, Lagrange, Hamilton expressed the laws of mechanics in variational form. As time went on, more and more branches of physics could be formulated in similar language. We had not merely the principle of least action in mechanics, but Fermat's principle in optics, and variational formulations of electromagnetic theory. Here again there were impacts on both philosophy and mathematics. The philosophers grasped at the principle of least action as a proof of the existence of the deity, who used the simplest and most effective means to accomplish his purposes. The mathematicians were led to the development of the calculus of variations, and to such related fields as the theory of continuous groups and of contact transformations. Several similar developments have come since that time; two conspicuous ones are statistics, as seen in statistical mechanics, in the philosophical ideas associated with the second law of thermodynamics, and in the mathematical development of the theory of statistics; and relativity, with its obvious philosophical accompaniments, and its relation to the theories of the absolute differential calculus and tensor analysis. In all of these cases, I believe one could make out a case for the thesis that each succeeding line of thought in physics enriched and supplemented, but never supplanted, those which had gone before; that the philosophical applications were in general superficial and ephemeral, the embodiment not of fundamental truth but of the

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prejudices of the men of the time; and that the impact on mathematics stimulated some of the most vital and valuable developments of that science.

I should like this evening to call to your attention the guiding principle which, more than any other, is running through the theoretical physics of the present, and to suggest that, though the mathematicians are well aware of it, they could nevertheless profit by following it even more closely. This is the wave equation. I shall not try to draw any philosophical implications from the fact that more and more parts of physics are now being explained in terms of the wave equation. I do not believe it has any philosophical implications at all. We are merely in that stage of the great development of physical thought in which we are exploring and exploiting the wave equation and its uses, as we have in the past explored and exploited Newtonian mechanics, the variational principle, statistics, tensor analysis, and many other lines. It will fit into a continually developing mosaic, enriching it and making it more powerful. We should be as wrong to base our philosophy on the principle of uncertainty and the wave mechanics as our ancestors were in basing theirs on Newtonian determinism; each is only a partial phase of the broader physical theory which is gradually unfolding itself. But mathematics may well be stimulated by studying the efforts of physicists to handle problems in the wave equation which are beyond the range of our present mathematical techniques, but not beyond the problems whose solution is not only useful but pressing.

The present importance of the wave equation is the culmination of a century and a half of continually increasing emphasis on partial differential equations. Laplace, Poisson, Fourier, Bernoulli, and many others, discovered the extraordinary similarity of the mathematical theory underlying the propagation of waves in strings and membranes and fluids and solids, the flow of heat and electricity and fluids, and the behavior of the gravitational and electrostatic potentials. These problems all have a mathematical framework in common. In each one there is a single dependent variable, the displacement or velocity, the temperature, the potential, and so on, and four independent variables, the three coordinates of space, and the one of time. The formulation of their fundamental laws in terms of differential equations leads us then at once to partial differential equations, relations between the derivatives of the dependent variable, which for the sake of illustration we may call u, with respect to the four independent variables x, y, z, and t. The remarkable feature of the problem is not this mathematical framework, but the fact that in every one of them the dependence on x, y, and z comes through the Laplacian expression

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

which we may write as $\nabla^2 u$, using the notation of Gibbs, in whose honor we are meeting this evening, and who more than any other clarified the relations between these branches of physics and vector analysis. The Laplacian appears in different ways in different problems. The form which we are particularly interested in tonight is the wave equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

in which v is a constant, which proves to be the velocity of propagation of the wave. But we have as well Laplace's equation, $\nabla^2 u=0$, Poisson's equation, $\nabla^2 u=f(x, y, z)$, where f is a function of x, y, z, and the heat flow or diffusion equation, $\nabla^2 u=a\partial u/\partial t$, where a is a constant. These can be handled, as far as their time variation is concerned, in a common manner: we assume that u varies with time according to the exponential function e^{ipt}, so that $u=w(x, y, z)e^{ipt}$. Thus the wave equation becomes $\nabla^2 w+(p^2/v^2)w=0$, Laplace's equation is $\nabla^2 w=0$, Poisson's equation is $\nabla^2 w=f$, and the heat flow equation is $\nabla^2 w=aipw$. All of these can be considered as special cases of the general equation

$$\nabla^2 w + k^2 w = f(x, y, z)$$

which we may call the inhomogeneous wave equation. By setting $k=0$, or $f=0$, we can obtain our various special cases.

We have here a partial differential equation for a function $w(x, y, z)$, whose solution is one of the classical problems of mathematical physics. Two broad approaches to its solution have been made, the method of expansion in orthogonal functions, and of Green's functions. Both are based first on a consideration of the related homogeneous equation $\nabla^2 w+k^2 w=0$. It is perfectly simple to obtain solutions, in fact an indefinitely large assortment of solutions, of this equation. The trick is to find solutions satisfying certain desired boundary conditions. If we desire solutions of the equation, for instance within a closed volume, satisfying the condition that $w=0$ over the surface of the volume, then we can show, if k^2 is positive, that solutions are possible only for certain discrete values of k, the characteristic values or proper values or eigenvalues, which we may denote by a subscript, as k_n; the corresponding functions w_n, the characteristic functions or proper functions or eigenfunctions, may then easily be shown to possess a property of orthogonality,

$$\int w_n w_m dv = 0 \qquad \text{if} \quad n \neq m$$

where the integral is extended over the volume. The general solution of our homogeneous wave equation is then shown to be expressible as a sum of particular solutions,

$$u = \sum_n A_n w_n(x, y, z) e^{ik_n vt}$$

where the A_n's are arbitrary, and may be used to satisfy initial conditions at $t=0$. To satisfy these initial conditions, we must expand the initial value of u, say $W(x, y, z)$, in series of the form

$$W(x, y, z) = \sum_n A_n w_n$$

and so determine the constants A_n. Such an expansion is easily carried out, in virtue of the orthogonality of the w_n's: we merely multiply both sides of the equation by a particular w_m, integrate over the volume, and have

$$A_m = \int W w_m dv \Big/ \int w_m^2 dv.$$

This is an expansion analogous to a Fourier expansion, and many of the properties characteristic of the Fourier expansion hold here too.

We may set up a space of an infinite number of dimensions in which the A_m's are plotted as the coordinates of a point. Such a point represents a function; the space is called a function space, or a Hilbert space, and the property of orthogonality can be interpreted geometrically in function space. Through such paths we are led to a sort of vector analysis in a space of an infinite number of dimensions; that is, we are led to a form of linear algebraic equations formally equivalent to the solution of the partial differential equation. This is a transformation of great power in all problems of this type, a transformation which for example allows us to solve problems in perturbations: if we know the solution of a problem near to the desired one, we can expand the solution of the desired problem in terms of the characteristic functions of the similar soluble problem, and, using the algebraic form of the problem, we can find the expansion coefficients as power series in the small perturbation.

By using the method of orthogonal expansion, we can solve the inhomogeneous wave equation, for we can expand both w and the inhomogeneous function $f(x, y, z)$ in series in the w_n's, and equate terms on both sides of the equation. An interesting case is found where the inhomogeneous part of the equation varies sinusoidally with time. If, for instance, we have the equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = f(x, y, z)e^{i\omega t}$$

and if we let $u = \sum_n U_n w_n e^{i\omega t}$, $f = \sum_n F_n w_n$, then, remembering that $\nabla^2 w_n + (\omega_n^2/v^2)w_n = 0$ (where ω_n^2/v^2 is written for the earlier k_n^2), we have

$$\sum_n U_n\left(\frac{\omega^2 - \omega_n^2}{v^2}\right) w_n = \sum_n F_n w_n$$

or, equating terms,

$$U_n = F_n v^2/(\omega^2 - \omega_n^2),$$

showing that the coefficients U_n show a dependence on ω similar to that of the amplitude of a linear oscillator of natural frequency ω_n, forced by an external frequency ω. We are led by following this argument to a theory of the normal modes of oscillation, closely related to the normal coordinates of a vibrating system of a finite number of degrees of freedom. We find, in fact, that we can build up the whole theory of the orthogonal functions and normal modes of the wave equation by a generalization of the theory of mechanical oscillations of a system of a finite number of degrees of freedom, to the case where the number of degrees of freedom becomes infinite.

The extensive development of the theory of orthogonal functions, which we have merely sketched, is very beautiful and very general, but its usefulness is severely limited by the very narrow range of problems for which we can actually obtain the solutions and find the orthogonal functions. The problems which are soluble by classical means are only those in which a separation of variables is possible; that is, in which w can be written as a product of functions $X(\xi)H(\eta)Z(\zeta)$, where ξ, η, ζ are three variables, functions of x, y, z, such that the boundaries are determined by setting either ξ, η, or ζ equal to constants. It can be shown that there are only eleven sets of coordinates in which separation can be effected, the familiar ones being rectangular, cylindrical, spherical, and ellipsoidal coordinates. These soluble problems are important, but the insoluble ones, with more complicated boundary conditions, are more so, and at present we know no elegant ways of getting solutions in such cases. Physicists attempt to devise approximate ways of computing the characteristic functions and characteristic numbers of such problems, but these methods lack generality. No problem which the mathematician could attack would be more useful to the physicist at present than a real advance in the solution of the non-separable wave equation.

The other method of solving the wave equation, which shows as much generality as the method of orthogonal functions, is Green's method. The most familiar example of this is found in the solution of Poisson's equation, $\nabla^2\phi = -4\pi\rho(x, y, z)$, for the potential $\phi(x, y, z)$ determined by a density of charge $\rho(x, y, z)$. The solution is $\phi = \int(\rho/r)dv$, where r is the distance from the point where ρ is computed to the place where we are finding ϕ. This holds if the boundary condition is that the potential is to vanish at infinite distance. If instead the potential is to have specified values over a finite boundary, surface integrals must be added to the volume integral, arising from the so-called Green's distribution over the surface. If we have the wave equation instead of Poisson's equation, we then meet the Kirchhoff solution, in which we must use the retarded potential, computing the ρ which appears in the integral not at the time t at which we want to find ϕ, but at a time earlier by the amount r/v, such that a disturbance leaving the point of integration at the earlier time, traveling with the velocity v, would reach the point xyz at time t. These solutions are very general and very powerful, but often not very convenient for computation. We can naturally show the equivalence of the method of orthogonal functions and of Green's function, and in this way can exhibit interesting interrelationships between the theory of integral equations (to which Green's method naturally leads us) and the theory of orthogonal functions and of function space. The mathematical interrelationships of all these ways of handling the wave equation should be the familiar stock in trade of every physicist. Unfortunately they are not; for the conventional mathematical texts and college and university courses concentrate almost entirely on the soluble cases of potential theory rather than on the powerful general methods and the relationships between them. The physicist, rather than the mathematician, is left with the main burden of expanding this branch of mathematics.

Now let us inquire why and how these methods of handling the wave equation have come to have so much more importance in the last few years. This has come about largely through the wave mechanics. Everyone realizes by now that it has been found that the correct formulation of mechanical problems is through Schrödinger's equation, an equation which is essentially like the wave equation, only with a velocity which is a function of x, y, z. Such an equation can perfectly well occur in other branches of physics, for instance in optics, if the index of refraction varies from point to point, but it had been largely neglected by the mathematicians. Schrödinger's equation, however, has forced the physicist to hunt for ways of solving this case, approximately or exactly. The problem is further complicated by the fact that there are not merely the three coordinates x, y, z in addition to time, but $3n$ coordinates, where n is the number of particles concerned in the problem. In spite of this great added difficulty, great progress has been made in discussing the practically important cases of Schrödinger's equation. The general method of orthogonal functions, with all its ramifications, still applies. Green's method, on the other hand, has had almost no study as related to Schrödinger's equation; maybe there are unsuspected advances to be made in that direction. At any rate, wave mechanics has widened the horizons and strengthened the techniques of the physicist when dealing with the wave equation. It has also clearly shown that the importance of the method of variations, and the method of classical mechanics, which I mentioned earlier, was exaggerated in earlier periods. For it shows that the principle of least action is an approximate principle giving the path of a ray, as Fermat's principle is in optics, and is merely a derived result of the wave equation; and that classical mechanics is likewise a limiting case, describing the limiting motion of a so-called wave packet, in the case of large-scale objects.

Wave mechanics has greatly broadened and strengthened the grasp of the physicist on the wave equation, and has made him more anxious than ever to learn more and more powerful techniques for handling it. This broadened grasp has reacted in other applications of physics. My colleagues Morse and Bolt have been applying the methods learned in wave mechanics to the study of acoustics, to sound waves in rooms and other enclosures, with the result that they are changing the whole line of development of that science. Many of us working on radar during the war have been applying our knowledge of wave mechanics to help us in the solution of electromagnetic problems, with very satisfactory results. For instance, I have been using the method of expansion in orthogonal functions for a general study of electromagnetic oscillations in a cavity, leading to formulas like the ones I mentioned earlier for the solution of an inhomogeneous wave equation, which resembles the formula for a forced oscillator. In electromagnetic language, this becomes the formulation of the behavior of an oscillating cavity in terms of an equivalent circuit, and it gives a complete theory of the circuit-like behavior of a cavity, a result of great practical importance in microwave theory. This is an example of a case where the general theory has important uses, even in the absence of exact solutions for special cases; for the general theory tells us the type of functional behavior we can expect, in terms of certain undetermined constants, and in our practical work we have determined those constants experimentally, in cases where we could only estimate them crudely from numerical calculation.

The applications of wave theory will extend far beyond such examples, however. For it becomes clear that wave mechanics is furnishing at least a beginning for the study of nuclear forces, and the behavior of particles of very high energies. We are then at the threshold of a great new field of theoretical physics, for which we are very conscious that we do not at present possess the answer. Such theory as we have is a development of the types of theory that I have sketched this evening. On the other hand, it is very tentative and unsatisfactory. For one thing, there are points of the utmost importance in which the theory, which takes the form of an expansion in power series, gives series that fail to converge. In some cases, these divergences seem to result from an unhappy choice of the form of expansion to use; in other cases they are inherent. In those cases, more than in any other which physicists have yet had to face, convergence or divergence is a major problem. Here it is the physicists who are likely to elucidate the inner nature of the problem, as Gibbs, in a very similar case, pointed out the true nature of the so-called Gibbs phenomenon in Fourier expansion. It seems likely that the real solution of our physical problems will not be accomplished without at the same time bringing about real extensions of our mathematical knowledge.

At this point, I return to my introduction. More and more of physics is being found to be subject to the wave equation. The physicist needs to know more than he does about it, both in the matter of the general theory, and of the solution of special problems. Yet the physicist finds very little help from the mathematician. For every mathematician like von Neumann who realizes these problems, and contributes practically to them, there are twenty who seem to have no interest in them, who either work in fields of only remote interest to physics, or who stress the older and more familiar parts of mathematical physics. Is it any wonder that in such a situation the physicist, looking at the mathematicians, feels that they have strayed from the path which has led to the past greatness of mathematics, and feels that they will not regain this path until they again resolutely enter the main current of progress of mathematical physics, the current which in the past has led to the most fruitful development of mathematics? Gibbs was great because he kept close to earth, always knew what his theory was for. That, the physicist firmly feels, is the only path through which the mathematician of the present can achieve greatness.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THE FEBRUARY MEETING IN NEW YORK

The four hundred fourteenth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 23rd in conjunction with a meeting of the Association for Symbolic Logic. The attendance was approximately three hundred including the following two hundred forty-two members of the Society:

Leonidas Alaoglu, C. B. Allendoerfer, R. L. Anderson, R. F. Arens, H. A. Arnold, H. E. Arnold, L. A. Aroian, W. A. Asprey, Silvio Aurora, Melvin Avramy, E. G. Baker, S. F. Barger, I. L. Battin, E. G. Begle, Stefan Bergman, P. G. Bergmann, Lipman Bers, Archie Blake, Gertrude Blanch, M. L. Boas, R. P. Boas, H. W. Bode, G. L. Bolton, C. B. Boyer, C. C. Bramble, A. B. Brown, C. T. Bumer, R. S. Burington, A. W. Burks, Hobart Bushey, J. H. Bushey, H. H. Campaigne, G. C. Campbell, W. B. Campbell, Herman Chernoff, D. E. Christie, Alonzo Church, E. P. Churchill, R. V. Churchill, M. D. Clement, G. R. Clements, A. B. Coble, Carl Cohen, I. S. Cohen, T. F. Cope, Franco Croci, D. R. Crosby, H. B. Curry, M. D. Darkow, Norman Davids, F. H. Davidson, D. J. Dickinson, C. H. Dowker, Y. N. Dowker, T. L. Downs, T. C. Doyle, R. J. Duffin, Nelson Dunford, J. C. Durand, W. D. Duthie, Jacques Dutka, Samuel Eilenberg, W. H. Fagerstrom, Ky Fan, A. B. Farnell, Herbert Federer, J. M. Feld, N. J. Fine, W. B. Fite, F. H. Fowler, R. H. Fox, Gerald Freilich, Bernard Friedman, R. E. Fullerton, Aaron Galuten, Hilda Geiringer, Abe Gelbart, David Gilbarg, P. W. Gilbert, A. M. Gleason, A. W. Goodman, R. O. Goodman, A. M. C. Grant, M. C. Gray, George Grossman, C. C. Grove, D. W. Hall, F. C. Hall, Marshall Hall, H. K. Hammer, K. E. Hazard, E. A. Hedberg, G. A. Hedlund, M. H. Heins, Erik Hemmingsen, Edwin Hewitt, L. S. Hill, Einar Hille, Abraham Hillman, Banesh Hoffmann, T. R. Hollcroft, Harold Hotelling, E. M. Hull, H. D. Huskey, (L.) C. Hutchinson, Nathan Jacobson, Herbert Jehle, Fritz John, R. A. Johnson, R. E. Johnson, A. W. Jones, Edward Kasner, M. E. Kellar, J. L. Kelley, D. E. Kibbey, W. J. Klimczak, J. R. Kline, E. G. Kogbetliantz, R. E. Langer, J. A. Larrivee, J. R. Lee, D. H. Lehmer, R. A. Leibler, Howard Levi, Norman Levinson, Marie Litzinger, W. R. Longley, L. H. Loomis, E. R. Lorch, A. N. Lowan, N. H. McCoy, S. W. McCuskey, L. A. MacColl, G. W. Mackey, Saunders MacLane, H. F. MacNeish, A. J. Maria, M. H. Martin, W. T. Martin, A. E. Meder, Karl Menger, D. D. Miller, D. S. Miller, Don Mittleman, Deane Montgomery, T. W. Moore, D. J. Morrow, Marston Morse, Ernest Nagel, C. A. Nelson, O. E. Neugebauer, Morris Newman, G. E. Noether, P. B. Norman, L. R. Norwood, C. O. Oakley, Rufus Oldenburger, L. F. Ollmann, J. C. Oxtoby, J. S. Oxtoby, L. G. Peck, B. J. Pettis, C. R. Phelps, R. S. Phillips, Everett Pitcher, Harry Polachek, Harry Pollard, E. L. Post, M. H. Protter, R. G. Putnam, Hans Rademacher, Moses Richardson, R. G. D. Richardson, D. E. Richmond, C. E. Rickart, John Riordan, J. F. Ritt, E. K. Ritter, H. E. Robbins, R. E. Roberson, V. N. Robinson, P. C. Rosenbloom, Arthur Sard, L. J. Savage, S. A. Schaaf, Robert Schatten, I. J. Schoenberg, Lowell Schoenfeld, E. J. Scott, J. B. Secrist, I. E. Segal, C. E. Shannon, R. H. Shaw, Max Shiffman, James Singer, M. M. Slotnick, D. M. Smiley, M. F. Smiley, Harold E. Smith, P. A. Smith, E. R. Stabler, E. P. Starke, F. M. Stewart, J. J. Stoker, M. H. Stone, W. J. Strange, E. G. Straus, L. M. Straus, R. C. Strodt, W. C. Strodt, A. C. Sugar, M. M. Sullivan, E. G. Swafford, A. H. Taub, Feodor Theilheimer,

J. M. Thomas, Leonard Tornheim, J. I. Tracey, C. A. Truesdell, Y. W. Tschen, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, Annita Tuller, H. E. Vansant, G. L. Walker, A. D. Wallace, W. R. Wasow, J. V. Wehausen, Harry Weingarten, Louis Weisner, A. P. Wheeler, G. W. Whitehead, D. V. Widder, John Williamson, H. P. Wirth, W. D. Wray, Bertram Yood, J. W. Young, Leo Zippin, M. F. Zucker, Antoni Zygmund.

The meeting Saturday morning was divided into two sections, one for papers in Analysis, Geometry, and Topology, at which Dr. R. P. Boas presided, and one for papers in Algebra, Logic, Applied Mathematics, and Statistics, at which Professor Marshall Hall presided.

At the general session Saturday afternoon, Professor A. D. Wallace gave an address entitled *Spheroidal properties of normal spaces.* Professor Saunders MacLane presided.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Papers numbered 1–10 were presented in the section for Analysis, Geometry, and Topology, papers 11–19 in the section for Algebra, Logic, Applied Mathematics, and Statistics, papers 20–50 were presented by title. Paper 6 was read by Dr. Schatten, paper 10 by Professor Pitcher, paper 14 by Professor Tukey. Professor Hua was introduced by Dr. H. F. Tuan, Mr. Kohn by Professor Alexander Weinstein, and Professor Schiffer by Dr. Stefan Bergman.

1. Erik Hemmingsen: *Dimension theory for normal Hausdorff spaces.* Preliminary report. (Abstract 52-3-100.)

2. I. E. Segal: *Topological groups in which multiplication on one side is differentiable.* (Abstract 52-3-75.)

3. C. H. Dowker: *Lebesgue dimension of a normal space.* (Abstract 52-3-99.)

4. G. W. Whitehead: *On families of continuous vector fields over spheres.* (Abstract 52-3-102.)

5. T. C. Doyle: *Tensor theory of invariants for the projective differential geometry of a ruled surface.* (Abstract 52-3-90.)

6. Robert Schatten and John von Neumann: *The cross-space of linear transformations.* II. (Abstract 52-1-22.)

7. R. E. Fullerton: *Compactness in certain Lebesgue spaces.* (Abstract 52-3-68.)

8. Herman Chernoff: *Complex solutions of partial differential equations.* I. (Abstract 52-1-16.)

9. Harry Pollard: *Integral transforms.* (Abstract 52-3-72.)

10. J. L. Kelley and Everett Pitcher: *Applications of natural homomorphism sequences.* II. (Abstract 52-5-209.)

11. Leonard Tornheim: *A method for determining the number of compositions of certain types.* (Abstract 52-3-61.)

12. D. H. Lehmer: *Some non-existence theorems on partitions.* (Abstract 52-3-59.)

13. Archie Blake: *A Boolean derivation of the Moore-Osgood Theorem.* (Abstract 52-3-94.)

14. G. W. Brown and J. W. Tukey: *Some distributions of sample means.* (Abstract 52-3-96.)

15. Herbert Jehle: *Transformation of hydrodynamical equations of stellar dynamics.* (Abstract 52-3-80.)

16. R. J. Duffin: *Nonlinear networks.* II. (Abstract 52-3-79.)

17. R. S. Phillips: *rms error criterion in servo system design.* (Abstract 52-3-81.)

18. Harry Polachek: *Solution of the differential equations of motion of a projectile in a medium of quasi-Newtonian resistance.* (Abstract 52-3-82.)

19. Einar Hille: *Analysis in a noncommutative Banach algebra without unit element.* (Abstract 52-5-153.)

20. A. A. Albert: *On Jordan algebras of linear transformations.* (Abstract 52-3-53-*t*.)

21. Reinhold Baer: *Polarities in finite projective planes.* (Abstract 52-3-89-*t*.)

22. R. H. Bing: *Generalization of a theorem of Janiszewski.* (Abstract 52-3-97-*t*.)

23. R. H. Bing: *Sets cutting the plane.* (Abstract 52-3-98-*t*.)

24. G. D. Birkhoff and Garrett Birkhoff: *Distributive postulates for systems like Boolean algebras.* (Abstract 52-3-93-*t*.)

25. Gertrude Blanch: *On the computation of Mathieu functions.* (Abstract 52-3-62-*t*.)

26. R. C. Buck: *On a class of entire functions.* I. (Abstract 52-3-63-*t*.)

27. R. C. Buck: *On a class of entire functions.* II. (Abstract 52-3-64-*t*.)

28. Herman Chernoff: *Complex solutions of partial differential equations.* II. (Abstract 52-3-65-*t*.)

29. Paul Civin: *Polynomial dominants.* (Abstract 52-3-66-*t*.)

30. Nelson Dunford: *Resolutions of the identity.* (Abstract 52-5-143-*t*.)

31. Leonard Greenstone: *Mapping by analytic functions of multiply connected domains.* (Abstract 52-3-69-*t*.)

32. Edwin Hewitt: *A characterization of rings of continuous functions.* (Abstract 52-5-204-*t*.)

33. Edwin Hewitt: *A generalization of the concept of complete regularity.* Preliminary report. (Abstract 52-5-205-*t*.)

34. Edwin Hewitt: *An effective construction of the Stone-Čech beta for a countable discrete space.* (Abstract 52-5-206-*t*.)

35. J. D. Hill: *Summability of the sequence of Rademacher functions.* (Abstract 52-3-70-*t*.)

36. L. K. Hua: *Orthogonal classification of Hermitian matrices.* (Abstract 52-3-56-*t*.)

37. Edward Kasner and John DeCicco: *Problems on perspective maps.* (Abstract 52-1-35-*t*.)

38. G. K. Kalisch: *Completion of topological rings and fields.* (Abstract 52-3-58-*t*.)

39. Walter Kohn: *Distortion by simple functions.* (Abstract 52-3-71-*t*.)

40. D. P. Ling: *Geodesics on surfaces of revolution.* (Abstract 52-3-92-*t*.)

41. Harry Pollard: *Necessary and sufficient conditions for the representation of an analytic function as a Laguerre series.* Preliminary report. (Abstract 52-3-73-*t*.)

42. Eric Reisner: *Stresses and small displacements in shallow spherical shells.* (Abstract 52-3-83-*t*.)

43. L. B. Robinson: *Solution of an integral equation by alternating successive approximations.* I. (Abstract 52-1-20-*t*.)

44. L. B. Robinson: *Solution of an integral equation by alternating successive approximations.* II. (Abstract 52-1-21-*t*.)

45. H. E. Salzer: *Coefficients for repeated integration with central differences.* (Abstract 52-1-28-*t*.)

46. H. E. Salzer: *Table of coefficients for obtaining the first derivative without differences.* (Abstract 52-1-29-*t*.)

47. Menahem Schiffer: *On the kernel function of an orthonormal system.* (Abstract 52-3-74-*t*.)

48. J. M. Thomas: *Division sequences.* (Abstract 52-3-60-*t*.)

49. Alexander Weinstein: *On Stokes' stream function and Weber's discontinuous integral.* (Abstract 52-3-88-*t*.)

50. H. J. Zimmerberg: *On a self-adjoint differential system of even order.* (Abstract 52-3-77-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEW

Vectors and matrices. By C. C. MacDuffee. (Carus Mathematical Monographs, no. 7.) Ithaca, N. Y., Mathematical Association of America, 1943. 11+192 pp. $2.00.

Professor MacDuffee's book is a clear and careful introduction to the theory of vector spaces and matrices. It should prove extremely useful not only to the student of mathematics, but also to the ever increasing circle of other scientists who show an interest in these fields. The more advanced undergraduate student will have no difficulty reading the book. The material is given in easy, almost leisurely steps. The book sets out from familiar facts, formulating the well known aspects in a new manner, gradually approaching new ideas, and almost inadvertently the reader will have become familiarized with the more abstract ways of mathematical thinking. There is, of course, another way of reaching the same aim: to push the reader into the cold waters of mathematical abstraction right away and let him swim around as well as he can. The student who follows Professor MacDuffee's book will not experience a sudden shock of this kind. Unless he ventures out into the last chapter prematurely (and in a footnote he is given permission to do so if he desires), he will learn about the more abstract concepts such as groups with operators, endomorphisms, and rings of endomorphisms only after he has digested the more "concrete" theories of vectors and matrices.

The book starts with a treatment of linear equations. Determinants are avoided at this stage. The author feels that determinants have been vastly overrated and that most parts of the theory of linear equations can be developed much better without determinants. This is a view with which at least most algebraists will concur. Fifty years ago it was a sign of a progressive attitude in a book to use determinants; to-day it is a sign of progressive thinking to avoid determinants as far as possible.

In the second chapter, vector spaces are defined and matrices are introduced, first in a rather formal manner. The reader is made familiar with the concept of rank which is at once made useful for the theory of linear equations. Determinants can now be approached in the next chapter. The author emphasizes the fact that the idea of matrix precedes that of determinant and suggests that to discuss determinants without matrices is like having the feline grin without the Cheshire cat. Determinants are studied as polynomials $d(X)$ in the coefficients of a square matrix X, which satisfy the equation $d(X)d(Y) = d(XY)$ and which are of minimal positive degree. No completeness is attempted, but the most important properties of determinants are treated briefly.

Chapter 4 deals with matrix polynomials. The characteristic and the minimum equation of a matrix are studied. In the following chapter the union and the intersection of vector spaces are introduced. Application to the spaces spanned by the rows of matrices leads to a theory of greatest common divisors and least common multiples of matrices.

Chapter 6 forms, in a sense, the center piece of the whole book. It brings a complete and detailed account of the rational canonical form for a matrix. The normal form used is somewhat simpler than that usually given; for the proofs, ideas of M. H. Ingraham have been used. Readers who study matrices out of necessity because of their usefulness in physics, statistics, or economics will perhaps not learn much of "practical" value in this chapter. Still, let us hope that at least some of them have become sufficiently interested in the subject so that they read these pages for their own sake. The treatment of the normal form is completed in the following chapter by a discussion of elementary divisors and their significance for the question of equivalence of matrices. A short chapter deals with orthogonal and symmetric matrices. The book concludes with a brief introduction to the concept of endomorphisms of abelian groups.

This discussion of the contents will show that the author did not attempt to give a complete treatment of the subject. This would, indeed, have been an impossible task for a book of this size. The selection of the material in a case like this will, of course, depend on the personal taste of the author.

Our remarks will indicate that the book can be used very well as a textbook for certain classes, specially for students at the difficult stage of transition from undergraduate to graduate work. Some instructors may object to the lack of exercises but this would be an unjust criticism in the eyes of the reviewer. At this level of mathematical education there is no need to suggest exercises which consist essentially in substituting numerical values for the indeterminates appearing in the considerations. As regards exercises and problems of a more valuable type, the instructor will have no difficulty in suggesting them himself according to his personal tastes. Alternate proofs for statements in the text can often be found. Material omitted in the book may be given. Geometrical and other applications can be discussed.

There are a few minor misprints, but it does not seem necessary to mention them here. The proof on page 52 for the equation $dW(l) \equiv l^i$ holds only in the case that the underlying field F is algebraically closed. This assumption is not made in the book and it is superfluous. The expression $dW(l)$ represents a polynomial $f(l)$ of a certain degree i, and it is known that $f(l)f(m) = f(lm)$ for all l, m in F. Comparing the terms of degree i in m in this equation, the equation $f(l) = l^i$ is obtained at once, provided that F contains infinitely many elements. The case of a finite field F (if it is to be included) can be treated directly.

Many students of mathematics will be extremely grateful to Professor MacDuffee for his well planned and lucid exposition.

RICHARD BRAUER

NOTES

Princeton University plans to hold its Bicentennial Celebration starting September, 1946, and continuing through the academic year 1946–1947. In connection with this Celebration there will be a series of Bicentennial Conferences. The subject of one of these is *The role of mathematics.*

A conference on algebra will be held at the University of Chicago during the week of July 15–19, 1946. Some of the sessions will emphasize the recent contacts of algebra with other branches of mathematics such as topology, function theory, and geometry. It is hoped that the meetings will be held in Eckhart Hall. Reservations should be made directly with Chicago hotels. Hotels in the vicinity of the University of Chicago are the Broadview, 5540 South Hyde Park Boulevard; Del Prado, 5307 South Hyde Park Boulevard; Shoreland, 5454 South Shore Drive; Windermere, 1642 East 56th Street; Mira-Mar, 6218 South Woodlawn. Because of the acute shortage of hotel space, reservations should be made as soon as possible.

The Royal Society is planning a celebration, to be held July 15, 1946, in commemoration of the three-hundredth anniversary of the birth of Isaac Newton.

The Mathematical Society of São Paulo was founded in 1945 in São Paulo, Brazil. Professor Omar Catunda of the University of São Paulo was elected president of the society.

The French Academy of Sciences has announced the award of the following prizes in mathematics and statistics for 1945: Carrière Prize, to André Lichnerowicz of the University of Strasbourg, for his mathematical works; Montyon Prize, to Robert Fortet of the University of Caen, for his work in the calculus of probabilities; D'Ormoy Prize, to Szolem Mandelbrojt of the College of France, for his mathematical works; Saintour Prize, to Marcel Brelot of the University of Grenoble, for his mathematical works and particularly for his papers on subharmonic functions; Charles Dupin Prize, to Paul Vincensini of the University of Besançon, for his work in higher geometry.

Dr. M. J. Herzberger of the Eastman Kodak Company has been awarded the Cressy-Morrison Prize by the New York Academy of Sciences for a paper on *Limitations of optical image formation.*

Professor Élie Cartan of the Sorbonne has been elected president of the French Academy of Sciences.

Captain H. T. Engstrom, U.S.N.R., has been awarded the rank of Officer in the Military Division of the Order of the British Empire.

Dr. H. W. Alexander has been appointed to a professorship at Adrian College, Adrian, Michigan.

Assistant Professor W. J. Dixon of the University of Oklahoma has been promoted to an associate professorship.

Associate Professor Samuel Eilenberg of the University of Michigan has been appointed to a professorship at Indiana University.

Dr. R. W. Erickson has been appointed to an associate professorship at Carleton College, Northfield, Minnesota.

Dr. G. E. Forsythe has accepted a position as research engineer with the Boeing Aircraft Company, Seattle, Washington.

Mother Ethelyn Fox of Barat College of the Sacred Heart has been appointed Director of Humanities at Duchesne College, Omaha, Nebraska.

Mr. S. G. Hacker has been appointed to an associate professorship at Washington State College.

Dr. Theodore Hailperin has been appointed to an assistant professorship at Lehigh University.

Professor H. K. Justice of the University of Cincinnati has been appointed assistant dean and director of admissions.

Dr. E. S. Kennedy of the University of Alabama has been appointed to an adjunct professorship at the American University of Beirut, Beirut, Lebanon.

Dr. P. A. Lagerstrom of Bell Aircraft Corporation has accepted a position as research aerodynamicist with Douglas Aircraft Company of Santa Monica, California.

Professor Rafael Laguardia has returned to the University of Uruguay after having spent two years in the United States as a Rockefeller and Guggenheim Fellow.

Mr. W. I. Layton of Amarillo College has been appointed to a professorship at Austin Peay State College, Clarksville, Tennessee.

Dr. Walter Leighton of the Rice Institute has been appointed to a professorship at Washington University.

Assistant Professor A. P. Morse of the University of California has been promoted to an associate professorship.

Dr. M. J. Norris has been appointed to an assistant professorship at the College of St. Thomas, Saint Paul, Minnesota.

Mr. G. P. Oakland of the University of Saskatchewan has been appointed to an assistant professorship of actuarial science at the University of Manitoba.

Dr. P. V. Reichelderfer has been appointed to an assistant professorship at the University of Wisconsin.

Assistant Professor Henry Scheffé of Syracuse University has been appointed to an associate professorship of engineering at the University of California at Los Angeles.

Assistant Professor I. J. Schoenberg of the University of Pennsylvania has been promoted to an associate professorship.

Dr. R. W. Shephard of the University of California has been appointed to an assistant professorship at Purdue University.

Assistant Professor M. F. Smiley of Lehigh University has been promoted to an associate professorship.

Mr. W. F. Smith has been appointed to an assistant professorship at the University of Detroit.

Dr. C. F. Stephens has been appointed to a professorship at Prairie View University, Prairie View, Texas.

Assistant Professor F. J. Taylor of the College of St. Thomas, Saint Paul, Minnesota, has been promoted to an associate professorship.

Mr. M. A. Treuhaft has accepted a position as engineer with the National Union Radio Corporation of Newark, New Jersey.

Associate Professor R. M. Trimble of the University of North Carolina has been promoted to a professorship.

Assistant Professor George Whaples of the University of Pennsylvania has been appointed to an assistant professorship at the University of Wisconsin.

Dr. P. M. Whitman has been appointed to an assistant professorship at Tufts College.

In the March issue of the Bulletin it was announced that Professor Oscar Zariski had been appointed to a professorship at the University of Illinois. This announcement should have read "research professorship."

The following appointments to instructorships are announced: Georgia School of Technology: Mr. W. A. Martin; University of Houston: Dr. Albert Newhouse; Lehigh University: Mr. S. D. Bernardi, Dr. H. K. Brown; University of Michigan: Dr. L. V. Toralballa; Michigan State College: Mr. Stanley Johnson, Mrs. Madelyn Kintner, Miss J. E. Master, Mr. Nicholas Musselman, Mr. Frank Saidel; University of Missouri: Mr. L. M. Kelly; Missouri School of Mines: Mr. Neil Lockwood; Oregon State College: Dr. J. C. R. Li; University of Pennsylvania: Dr. Y. C. Wong; Purdue University: Dr. Bernard Dimsdale; Randolph Macon Woman's College: Mr. M. E. Terry; University of Rochester: Mr. G. M. Wing; Rutgers University: Dr. Lee Byrne; College of St. Thomas, St. Paul, Minnesota: Mr. Eric Clamons, Mr. D. J. Lewis, Dr. Takashi Terami; Swarthmore College: Dr. A. W. Burks; Temple University: Miss Marie A. Wurster; Agricultural and Mechanical College of Texas: Dr. R. E. Basye; University of Wisconsin: Mr. S. C. Snowdon.

Mr. Joseph Allen, former associate professor of the College of the City of New York, died March 3, 1946, at the age of ninety-five years. He had been a member of the Society for thirty-five years.

Mr. A. A. Blumberg of the Agricultural and Mechanical College of Texas died October 21, 1945.

Professor Emeritus F. H. Clutz of Gettysburg College died December 30, 1945, at the age of seventy-two years. He had been a member of the Society for fourteen years.

Dr. Fletcher Durell died March 25, 1946, at the age of eighty-six years. He had been a member of the Society for forty years.

Professor C. W. Emmons of Simpson College, Indianola, Iowa, died December 29, 1945, at the age of sixty-four years. He had been a member of the Society for twenty-three years.

Professor C. F. Gummer of Queen's University, Kingston, Ontario, Canada, died January 21, 1946, at the age of sixty-three years. He had been a member of the Society for twenty-three years.

Professor Laurence Hadley of Purdue University died March 21, 1946, at the age of seventy years. He had been a member of the Society since 1941.

Professor Althéod Tremblay of Laval University died in February, 1946, at the age of sixty-nine years. He had been a member of the Society since 1932.

Professor Emeritus E. E. Whitford of the College of the City of New York died in February, 1946, at the age of eighty-six years. He had been a member of the Society for twenty-six years.

Professor Emeritus H. R. Willard of the University of Maine died March 26, 1946, at the age of seventy-one years.

The following twenty-six doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1945 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

N. R. Amundson, Minnesota, December, minor in chemical engineering, *Solution of a non-linear partial differential equation of the parabolic type.*

T. W. Anderson, Jr., Princeton, June, *The non-central Wishart distribution and its application to problems in multivariate statistics.*

R. F. Arens, Harvard, June, *Topologies for spaces of transformations.*

Winifred A. Asprey, Iowa, August, *Some families of total oscillators and their derived functions.*

Miriam C. Ayer, Ohio State, August, *On convergence in length.*

Joshua Barlaz, Cincinnati, June, *On some triangular summability methods.*

R. H. Bing, Texas, June, minor in education, *Concerning simple plane webs.*

Frances Campbell, Michigan, June, *A study of truncated bivariate normal distributions.*

Jeremiah Certaine, Harvard, March, *Lattice-ordered groupoids and some related problems.*

W. M. Chen, California, June, *Power function of the analysis of variance and covariance of a normal bivariate population.*

J. B. Díaz, Brown, June, *A class of partial differential equations of even order.*

Evelyn Frank, Northwestern, June, minors in education and physics, *C-fractions.*

R. E. Fullerton, Yale, June, *Representations of linear operators.*

R. A. Good, Wisconsin, May, *On the theory of clusters.*

Corinne R. Hattan, Illinois, June, minor in zoology, *Lebesgue-Stieltjes integral equations of the first kind.*

J. W. Lawson, Chicago, September, *Minima of integrals over hypersurfaces in the calculus of variations.*

J. J. Livers, Michigan, February, *Use of partitions in multivariate moment sampling theory.*

R. J. Marcou, Massachusetts Institute of Technology, June, minor in physics, *Rectilinear congruences in Euclidean four space.*

M. E. Munroe, Brown, June, *Certain generalizations of the fundamental theorem of integral calculus.*

L. F. S. Ritcey, Chicago, September, *Index theorems for discontinuous problems in the calculus of variations.*

J. C. Smith, Cornell, June, *Asymptotic distributions of sums of Rademacher functions and of cosines with big gaps.*

H. K. Sohl, Johns Hopkins, June, *Theory of visco-plastic flow.*

W. B. Stiles, Iowa State, December, *Solution of clamped plate problems by means of functions derived from membrane characteristic functions.*

J. S. Stubbe, Cincinnati, June, *Cesàro and logarithmic summability of double Fourier series and their derived series.*

C. T. Taam, Harvard, June, *On the zeros of the solutions of homogeneous ordinary linear differential equations of the second order in the complex domain.*

H. J. Zimmerberg, Chicago, June, *A class of definite boundary value problems.*

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

103. A. A. Albert: *The Wedderburn principal theorem for Jordan algebras.*

The author has recently given a general theory of Jordan algebras, that is, linear spaces A of linear transformations a, b, and so on, such that A is closed with respect to the operation $a \cdot b = (ab+ba)/2$. In the present article the author proves that the Wedderburn theorem holds for such algebras. This is the theorem stating that if N is the radical of a Jordan algebra A then $A = S+N$, where S is an algebra, and so S is isomorphic to the semisimple Jordan algebra $A-N$. (Received February 11, 1946.)

104. Reinhold Baer: *Absolute retracts in group theory.*

The subgroup R of the group G has been termed a retract of the group G whenever there exists an idempotent endomorphism of G which maps G upon R. As this definition is in strict analogy to the topological concept of retract, one may be tempted to define absolute retracts in like similarity to topological usage. But it shall be shown in the present note that the identity is the only group which is a retract of every containing group. Consequently only modifications of the topological concept will be useful, and it will be shown in this note that each of the following classes of groups may in some sense be termed absolute retract: the complete groups, the abelian groups the orders of whose elements are finite and square free, and the free groups. (Received February 25, 1946.)

105. Grace E. Bates: *Free loops and nets and their generalizations.*

In generalizing to loops the group concept of "freeness," the following definition is used: A loop L is free over its sub-half-loop K (a set of elements in L having the same composition as L, but which is not necessarily closed under this composition), if every homomorphism of K into a loop may be extended to a homomorphism of L into the same loop. The main theorem is the following: If a loop L is free over and generated by its sub-half-loop K, and if S is a sub-loop of L, then S is the free sum of a free loop F and the loop generated by $S \cap K$. Applications of this theorem yield loop analogues to such theorems as Schreier's theorem on subgroups of a free group and the refinement theorem for free products. In the proofs the author uses nets and net constructions, taking advantage of the well known equivalence of net and loop theory. A direct application of one net construction proves the imbeddability of any half-loop into a loop which is free over and generated by the half-loop. (Received March 11, 1946.)

106. H. W. Becker: *Combinatory interpretations of Bell's numbers.*

These are evolved by iterated matrix multiplication of the table of Stirling's numbers, second kind (E. T. Bell, Amer. J. Math. vol. 61 (1939) p. 89; Ann. of Math. vol. 39 (1938) p. 539). They enumerate: (1) the number of distributions of n individuals into crews (or circuits), with m levels of hierarchy (or insulation, or jumpers) in the organization chart; (2) the number of rhyme schemes, or logical relatives (C. S. Pierce, Amer. J. Math. vol. 3 (1880) p. 48) with m plexes of nuance possible to each letter. (Specialized, these yield compositions and partitions of numbers under m modes of addition, as normal, weak, strong, and so on.) The readily formulated breakdowns are formulated according to (1) number of crews, or, equivalently, (2) number of different letters, and to (1) size of first crew, or, equivalently, (2) number of a's. The above may further be interpreted as a generalization of the substitution cycles and inscribed polygons of Touchard (Acta Math. vol. 70 (1939) p. 249): in which the cycles are separated by up to m types of parenthesis; or the polygons are bounded by lines of up to m different colors simultaneously. (Received March 21, 1946.)

107. H. W. Becker: *The general theory of rhyme.*

Prosodic rhyme schemes are called sequations, abstractly, of umbra @. They classify according to range (number of different letters), terminal (last letter), singletons (unrhymed letters, more generally any partition of letters), and quantum (number of a's, more generally any composition of letters). Generalizations are the multipolar, multinomial, multinary, multiplex, and multilinear sequations, having respectively m types of accent, alphabet, partial identity, nuance of nuance, and parallelism. Specialized patrices (expansions of a matrix, with given path restrictions) are those in which: each letter rhymes s times or more, or less; some letter, or none, occurs s-ply; no rhyme repeats within s adjacencies; the differences of successive letters are restricted; given letters are forbidden, or fixed, in certain positions; all numbers in the partition, or range, are odd, or even; there are no nonconsecutive rhymes (these are isomorphic with compositions); the previous is further specialized, to be isomorphic with partitions. In permuted sequations, the debuts of the different letters are in nonalphabetic order: enumerated by differences of zero, integer powers, and the polynomials $(@M+X)^n$. These polynomials kernel the lexicon theorems, by which the patrices are well-ordered. All are generated by variations of $\exp t@ = \exp (e^t - 1)$. (Received March 21, 1946.)

108. A. T. Brauer: *On a theorem of M. Bauer.*

In generalization of a theorem of M. Bauer (J. Reine Angew. Math. vol. 131 (1906) pp. 265–267) the following theorem is proved. Let $f(x)$ be a polynomial with integral rational coefficients which has at least one real root. Let $G(k)$ be the group of the residue classes relatively prime to k, and H a subgroup which does not contain the class of numbers congruent to -1 (mod k). Then $f(x)$ contains infinitely many prime divisors which do not belong to the classes of H. It follows for instance that each such polynomial contains an infinite number of prime divisors which are quadratic non-residues for a given prime of form $4n+3$. (Received March 20, 1946.)

109. A. T. Brauer and Gertrude Ehrlich: *On the irreducibility of certain polynomials.*

Pólya has proved the following theorem (Jber. Deutschen Math. Verein. vol. 28 (1919) pp. 31–40): If for n integral values of x, the integral polynomial $P(x)$ of degree

n has values which are different from zero and, without regard to sign, less than a certain constant $G_1(n)$, then $P(x)$ is irreducible in the field of rational numbers. Tatuzawa (Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 253–254) proved that this theorem holds for a larger bound $G_2(n)$. Moreover, he proved that $P(x)$ is also irreducible if $|P(x_\mu)|$ is different from zero and less than a certain constant $H_1(m)$ for m integral values x_μ with $n>m>n/2$. In this paper these results are improved further. Instead of $G_2(n)$ and $H_1(m)$ larger bounds $G(n)$ and $H(m)$ are obtained. These results contain theorems of Schur (Archiv für Mathematik und Physik (3) vol. 13 (1908) p. 367) and of Dorwart and Ore (Ann. of Math. vol. 34 (1933) pp. 81–94). Moreover, an irreducibility criterion of a new type is obtained. If $0<|P(x_\nu)|<S$ with $S>G(n)$ for n integers x_ν, but less than another smaller constant T for m of these x_ν, then $P(x)$ is irreducible. (Received March 20, 1946.)

110. I. S. Cohen: *Groups and rings of finite rank.* Preliminary report.

An integral domain R is said to be of finite rank if there exists a number k such that every ideal of R has a basis of k elements. It is shown that a necessary condition that R be of finite rank is that every proper residue class ring satisfy the descending chain condition. This condition is also sufficient if R is a local ring, and more generally, if the quotient rings R_P are integrally closed for all but a finite number of prime ideals P; this latter condition is satisfied if the integral closure of R is a finite R-module. These results permit an extension of a theorem of I. Kaplansky concerning R-modules of finite rank. (Received March 23, 1946.)

111. M. A. Coler: *Observations on primes.*

A type of function designated as a *cushion* function is defined. It is shown that certain conjectures, such as Goldbach's Theorem and those concerned with the distribution of twin primes, can be restated conveniently as conjectures regarding the distribution of roots of the same simple cushion function, $\mathcal{C}_1$. (Received February 15, 1946.)

112. H. S. M. Coxeter: *A new extreme form in seven variables.*

Let M be the smallest value taken by a given positive definite quadratic form $F(x_1, \cdots, x_n)$ for integers x_i, not all zero, and let D be the determinant of the coefficients. The form is said to be extreme if the ratio M^n/D decreases for every small change in the coefficients. Extreme forms for $n \leqq 6$ have been completely enumerated. (See Hofreiter, Monatshefte für Mathematik und Physik vol. 40 (1933) pp. 129–152.) Three septenary extreme forms were discovered by Korkine and Zolotareff in 1873. They may be expressed as $\sum_1^7 x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_5x_6 - x_jx_7$, where $j=6$, 5, or 4, and $M^7/D = 2^4$, 2^5, or 2^6. Geometrical considerations now suggest a fourth extreme form $\sum_1^7 x_i^2 + (2/3)\sum_{i<j} x_ix_j - (4/3)(x_5+x_6)x_7$, for which $M^7/D = 3^7/2^6$. In the corresponding packing of equal spheres, each sphere touches 56 others, as in the case $j=6$; and yet this packing is denser than in the $j=5$, where each sphere touches 84 others. (The densest possible lattice packing occurs in the case $j=4$, where each sphere touches 126 others.) (Received March 6, 1946.)

113. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in abstract groups.* IIa. *Kernels and three-dimensional cohomology.*

In a factor group $E/K=Q$ the elements of Q operate as automorphisms on K, but these automorphisms are defined and multiply only modulo inner automorphisms. A group K together with such operators is called a kernel over Q. For a given abelian group Z with Q as (ordinary) operators, kernels K are considered with center Z. Such a kernel is called extendible if there exists a group E with $E/K=Q$. With the multiplication of kernels suitably defined it is proved that the kernels taken modulo extendible kernels form a group isomorphic with the three-dimensional cohomology group $H^3(Q, Z)$ (Ann. of Math. vol. 46 (1945) pp. 480–509). In particular, the extendible kernels are characterized by the fact that a certain 3-cocycle is a coboundary. (Received March 8, 1946.)

114. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in abstract groups.* IIb. *Group extensions with a non-abelian kernel.*

Let the non-abelian group K with center Z be an extendible kernel over Q, in the sense of the preceding abstract. The totality of all extensions E such that $E/K=Q$ will be enumerated. According to Baer (Math. Zeit. vol. 38 (1934) pp. 375–416) the kernel K determines uniquely a group R with $R/S=Q$ where $S=K/Z$, and every extension E may be regarded as an extension $E'/Z=R$. It is shown that the extensions E' of R by Z so obtained form a coset in the group $H^2(R, Z)$ modulo a well-determined subgroup. Furthermore, different extensions E may lead to identical E'. Each equivalence class of E's obtained in this manner has the structure of a specific group constructed from the one-dimensional cohomology group $H'(Q, Z)$. (Received March 8, 1946.)

115. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in abstract groups.* III. *Non-associative systems and cohomology.*

It is known that the two-dimensional cohomology group $H^2(Q, G)$ (Ann. of Math. vol. 46 (1945) pp. 480–509) can be interpreted as the group of group extensions of the group Q by the abelian group G, realizing the prescribed operators of Q on G. A similar interpretation of the higher dimensional cohomology groups is obtained by the use of loops (for definition, see A. A. Albert, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 507–519). The associator $A=A(a, b, c)$ of three elements in a loop L is defined by $a(bc)=A\left[(ab)c\right]$, and the higher associators $A(a_1, \cdots, a_{2n}, b)=A(a_1, \cdots, a_{2n-2}, A(a_{2n-1}, a_{2n}, b))$ are obtained by iteration. A prolongation of Q by G is a loop L with a homomorphism ϕ of L onto Q such that (i) the kernel K of ϕ contains the given group G; (ii) $A(k, b, c)=A(a, k, c)=1$; (iii) each $A(a, b, c)$ lies in the center of the group K; (iv) $ag=\left[(\phi a)\cdot g\right]a$. Under a suitable multiplication, the product of two prolongations is again a prolongation. If $A(a_1, \cdots, a_{2n}, b)$ is in G for all b, and is zero for $b\in K$, this associator is essentially a $(2n+1)$-dimensional cocycle f of Q in G. The correspondence $f\longleftrightarrow L$ establishes an isomorphism of $H^{2n+1}(Q, G)$ to a suitably defined group of classes of prolongations. A similar correspondence is established for even dimensions. (Received March 8, 1946.)

116. R. A. Good: *On the construction of clusters.*

In this paper the author continues the study of clusters and discusses their construction by the method of introducing a multiplication operation into an additive group. A suitable multiplicative rule is described if the group is homomorphic to a cyclic subgroup of itself. Clusters whose order is a prime or the product of two primes are characterized. (Received March 20, 1946.)

117. R. A. Good: *On the theory of clusters.*

A cluster is defined as a certain generalization of a (non-associative) ring. A cluster contains its derived ring; the homomorphism concept is applied; ideals are studied, especially the commutator and annihilator ideals. One method of construction is extension of an arbitrary ring by an arbitrary group to form a cluster $\mathfrak{R}$ whose elements are couples. Rules of operation in $\mathfrak{R}$ are described in terms of mappings which are defined in the given systems and subjected to certain interrelating postulates. After the properties and structure of the resulting clusters have been examined, two special cases are treated in detail. Numerous examples illustrate the diversity of properties in various clusters. (Received March 21, 1946.)

118. Lois W. Griffiths: *Linear homogeneous diophantine equations.*

In the system $a_{i1}x_1 + \cdots + a_{in}x_n = 0$ $(i=1, \cdots, r)$ the coefficients are constant rational integers and the rank of the matrix of the coefficients is r. It is proved that a complete solution of the system in integers is $x_j = (-1)^j tE_j/e$ $(j=1, \cdots, n)$, in which t is an arbitrary integer and $E_1, \cdots, E_n, e$ are written down directly from the equations in the system in the following way. The equations $\xi_{i1}x_1 + \cdots + \xi_{in}x_n = 0$ $(i=1, \cdots, n-r-1)$, in which the ξ_{ij} are arbitrary integers, are adjoined to the original system. Then E_j is the determinant obtained by deleting the jth column of the matrix of the resulting $n-1$ equations, and e is the greatest common divisor of $E_1, \cdots, E_n$. This result is more simple and usable than those in the literature (Th. Skolem, *Diophantische Gleichungen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 5, no. 4, 1938; D. N. Lehmer, Proc. Nat. Acad. Sci., U.S.A. vol. 4 (1919) pp. 111–114). (Received March 12, 1946.)

119. L. K. Hua: *Geometries of matrices.* III. *Fundamental theorems in the geometries of symmetric matrices.*

The paper contains a proof of the theorem that a continuous mapping carrying symmetric matrices into symmetric matrices and leaving arithmetic distance invariant is either a symplectic transformation or an anti-symplectic transformation. Other results on continuous mappings in the geometry of symmetric matrices are also obtained. The latter half of the paper is concerned with analytic mappings. (Received March 1, 1946.)

120. A. W. Jones: *Semimodular finite groups and the Burnside basis theorem.*

Group theoretic characterizations are given for the following finite groups: those for which the lattice of all subgroups (1) of the entire group is lower semimodular, (2) of the ϕ quotient group is complemented and modular and simple, (3) of the ϕ quotient group is complemented and modular, (4) of the entire group is upper semimodular. As a corollary to one characterization of the groups of (1), it is shown that the direct product of lower semimodular groups, and hence also of modular groups, is always lower semimodular. The groups of classes (2) and (3) are shown to be a proper subclass of those in (1). The groups in (3) are shown to be direct products of groups in (2) having relatively prime orders. By a result of Dilworth it is shown that the groups in (3) constitute the largest class of groups G having G/ϕ lower semimodular to which the Burnside basis theorem applies. A dual Burnside basis theorem is shown to apply to all the groups in (4). (Received March 21, 1946.)

121. G. K. Kalisch: *Uniform spaces and topological groups as generalized metric spaces.*

A *general metric space* has a distance function satisfying the usual conditions, but where the distance is an element of a partially ordered group. It is proved that uniform spaces (cf. A. Weil, *Sur les espaces à structure uniforme*) are general metric spaces whose value group is a certain direct sum of the real numbers indexed according to a directed set. Next it is proved using a theorem of Kakutani (*Ueber de Metrisation der topoligischen Gruppen*, Proc. Imp. Acad. Tokyo vol. 12 (1936) p. 82) that topological groups have a left invariant general metric. This, in turn, implies that topological rings and fields and vector spaces have norms. It is shown by an example that in general the norm of a product need not be less than or equal to the product of the norms (in case the value group is the group of real numbers). It is noted, as a consequence of the first mentioned theorem, that uniform spaces are also sequence spaces where every sequence is indexed according to a fixed directed set (cf. Bourbaki, *Topologie générale: filters*; J. Tukey, *Convergence and uniformity*). (Received March 25, 1946.)

122. Irving Kaplansky: *On a problem of Kurosh and Jacobson.*

In analogy with Burnside's problem, Kurosh (Bull. Acad. Sci. URSS. Sér. Math. vol. 5 (1941) pp. 233–240) has raised the question whether a finitely generated algebraic algebra necessarily has a finite basis. In conjunction with results of Jacobson (Ann. of Math. vol. 46 (1945) pp. 695–707), it is shown that the answer is affirmative if the degrees of the elements are bounded say by n, and if the coefficient field has at least n elements. (Received March 21, 1946.)

123. Fred Kiokemeister: *A note on the Schmidt-Remak theorem.*

A group G with operator domain Ω is said to satisfy the modified maximal condition if every proper Ω-subgroup of G satisfies the maximal condition. A proof of the Schmidt-Remak theorem such as that given by Jacobson (*The theory of rings*, Mathematical Surveys, vol. 2, New York, 1943) yields with few changes the following theorem: Let G be a group with operator domain Ω, and let Ω contain the inner automorphisms of G. Let $G=A_1\times A_2\times\cdots\times A_n$ where the Ω-subgroups A_i are directly indecomposable and satisfy the minimal condition and the modified maximal condition. If $G=B_1\times B_2\times\cdots\times B_n$ is a second decomposition into the direct product of indecomposable factors, then $m=n$. Further the B_i may be so rearranged that $A_i\cong B_i$, and, for $k\leqq n$, $G=B_1\times\cdots\times B_k\times A_{k+1}\times\cdots\times A_n$. (Received March 21, 1946.)

124. Fred Kiokemeister: *A theory of normality for quasigroups.*

A normal relation on a quasigroup G is defined to be an equivalence relation ρ such that (i) $ab\,\rho\,ac$ implies $b\,\rho\,c$, (ii) $ba\,\rho\,ca$ implies $b\,\rho\,c$, (iii) $a\,\rho\,b$ and $c\,\rho\,d$ imply $ab\,\rho\,cd$. The set of elements of G equivalent to an element a under ρ is $R_a(\rho)$. If $R_a(\rho)$ is a quasigroup, it is a normal divisor of G, and ρ is called a-normal. For fixed a, the set of a-normal relations is a modular lattice and is isomorphic with the lattice of normal divisors of G containing a. The Ore theorem on partially ordered sets (Bull. Amer. Math. Soc. vol. 49 (1943) p. 563) is shown to imply the Jordan-Hölder theorem for composition series in G. If e is an idempotent element of G, every normal relation is e-normal. If $A_1, A_2, \cdots, A_n$ are normal divisors of G such that $A_i\cap A_j=e(i\neq j)$ and if $G=A_1\times A_2\times\cdots\times A_n$, then G is said to be the direct product over e of the subquasigroups $A_1, A_2, \cdots, A_n$. The modularity of the lattice of normal relations of G implies the Schmidt-Remak theorem for direct decompositions of G over e through reference to the corresponding structure theoretic theorem by Ore (Ann. of Math. vol. 37 (1936) p. 272). (Received March 21, 1946.)

125. Fred Kiokemeister: *Reducibility in a class of homogeneous polynomials.*

The following theorem is established: Let $H(x_1, x_2, \cdots, x_n)$ be a polynomial, homogeneous of degree n in the variables $x_1, x_2, \cdots, x_n$, with coefficients in the perfect field P. If H factors into a product of linearly independent linear factors over some extension field Δ of P, then H is reducible over an algebraic extension field F of P if and only if there exists in F a nontrivial zero of H. The theorem is proved by exhibiting a commutative linear algebra of which μH is a parastrophic form for some μ in P. The stated property of H follows from a relation between the zeros of H and the ideals of the linear algebra. (Received March 21, 1946.)

126. H. E. Salzer: *Further empirical results on tetrahedral numbers.*

I. The list of integers not greater than n which are the sum of no less than 5 positive tetrahedral numbers was extended from $n=1000$ to $n=2000$. (30 more such integers were found.) The "density" of such numbers decreases at first, but then increases again, showing no tendency to diminish to zero. To illustrate, the number of such integers among the first four groups of 500 are 30, 15, 12 and 18. Only one ends in 4, two in 1, two in 9 and the rest in 2, 3, 7 or 8. II. With the aid of that list, Pollock's conjecture that every integer is the sum of 5 positive tetrahedrals was verified for $n \leqq 20000$ and ending in 2, 3, 7 or 8. With Richmond's statement that every multiple of $5 \leqq 20000$ requires only 4 positive tetrahedrals, this verifies Pollock's conjecture for every $n \leqq 20000$. III. The verification of the author's conjecture that every n^2 is the sum of 4 positive tetrahedrals was extended from $n=200$ to $n=300$. (Received March 13, 1946.)

127. R. D. Schafer: *Equivalence in a class of division algebras of order* 16.

Let $\mathfrak{C}$ be a Cayley-Dickson division algebra over $\mathfrak{F}$. Let $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$ with multiplication $(a+vb)(x+vy)=(ax+g\cdot ybS)+v(aS\cdot y+xb)$ where S is the involution $x\leftrightarrow xS=t(x)-x$ of $\mathfrak{C}$ and g is some fixed element of $\mathfrak{C}$. The author has shown (Bull. Amer. Amer. Math. Soc. vol. 51 (1945) pp. 532–534) that $\mathfrak{A}$ is a division algebra of order 16 over $\mathfrak{F}$ for any choice of g in $\mathfrak{C}$ such that $n(g)$ is not the square of an element in $\mathfrak{F}$. Let $\mathfrak{A}_0=\mathfrak{C}_0+v_0\mathfrak{C}_0$ with multiplication defined similarly in terms of a fixed element g_0 of $\mathfrak{C}_0$. The algebra $\mathfrak{A}_0$ is equivalent to $\mathfrak{A}$ if and only if (1) $\mathfrak{C}_0$ is equivalent to $\mathfrak{C}$, the equivalence being given by $x\leftrightarrow xA$, x in $\mathfrak{C}_0$, xA in $\mathfrak{C}$, and (2) $g_0A=\delta^2 g$ for some $\delta\neq 0$ in $\mathfrak{F}$. The equivalence between $\mathfrak{A}_0$ and $\mathfrak{A}$ is $x+vy\leftrightarrow xA+\delta v(yA)$. From this it follows that H is an automorphism of $\mathfrak{A}$ if and only if H induces an automorphism A on $\mathfrak{C}$, $(x+vy)H=xA+\delta v(yA)$, and $gA=\delta^2 g$. If $t(g)\neq 0$, then $gA=g$, $\delta=\pm 1$; if $t(g)=0$, then $\delta^4=1$. (Received February 16, 1946.)

128. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* I.

This paper is an algebraic development of a theory of ordered dyads in chapter 2 of the author's *Theory of geometrical relations* (Amer. J. Math. vol. 31). The concept, configuration set of ordered dyads (Bull. Amer. Math. Soc. Abstracts 44-11-474 and 45-9-375) is formally expressed: (1) as an associative and commutative sum of dyads; (2) as an associative and commutative product of dyads. These expressions are respectively generalized to (1) a linear function $\Lambda(\alpha)$ of a complete set of ordered dyads $(\alpha_i\alpha_j)$ on $(n+1)$ elements $(n=1, 2, 3, \cdots)$ with coefficients in a suitable domain D (field F): $\sum\lambda_{ij}\alpha_i\alpha_j$; (2) a "quasi-determinant" $\Delta(\lambda;\ \alpha)=|\lambda_{ij}\alpha_i\alpha_j|$ whose relation to $\Lambda(\alpha)$ is given by $\Delta(\lambda;\ \alpha)=\{\Lambda(\alpha)\}^{n+1}/(n+1)!$ where in the product $\{\Lambda(\alpha)\}^{n+1}$ it is assumed that $\alpha_i\alpha_j\times\alpha_k\alpha_l=0$ if and only if $i=k$ or $j=l$. For $n=1$, $\Delta(\lambda;\ \alpha)=\lambda_{11}\lambda_{22}\alpha_1\alpha_1\cdot\alpha_2\alpha_2+\lambda_{21}\lambda_{12}\alpha_2\alpha_1\cdot\alpha_1\alpha_2$. If the λ's are all equal to unity, then $\Delta(\lambda;\ \alpha)$ reduces to the symmetric group of configurational sets of dyads (substitutions) expressed as a formal sum. The linear function $\Lambda(\alpha)=\sum\lambda_{ij}\alpha_i\alpha_j$ is used to represent the matrix $\|\lambda_{ij}\|$; representation of the product of two matrices is obtained by multiplying the corresponding linear functions in the usual formal manner and assuming that $\alpha_i\alpha_j\times\alpha_k\alpha_l=0$ or α_{il} according as $j\neq k$ or $j=k$. (Received March 20, 1946.)

129. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* II.

The author discusses properties of quasi-determinants defined in the preceding paper and the relation of the latter to determinants. Quasi-determinants are expanded in two ways: (1) in analogy with the usual expansion of determinants in terms of minors, (2) by multiplying the rows (columns) expressed as formal sums and assuming that $\alpha_i\alpha_j\times\alpha_k\alpha_l=0$ if and only if $i=k$ or $j=l$. The product of two configurational sets of ordered dyads expressed as quasi-determinants is defined to be the quasi-determinant of the product of the two sets expressed as sums corresponding to matrices. This product is generalized to the product of two quasi-determinants: $\Delta(\lambda;\ \alpha)\cdot\Delta(\mu;\ \alpha)$ by multiplying the corresponding expansions in the usual formal manner and replacing formal products of configurational sets by their values. Necessary and sufficient conditions are investigated that the product of two quasi-determinants of given order be again a quasi-determinant of the same order. Elementary properties of quasi-determinants are pointed out, such as formal invariance under interchange of rows and columns, commutativity of any two rows (columns) and so on. (Received March 20, 1946.)

130. I. E. Segal: *Representation of certain commutative Banach algebras.*

A theorem of Gelfand and Neumark (Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp. 197–213, Lemma 1) is given a full proof and extended to the case of an algebra which does not necessarily contain an identity, this case being of interest in the theory of group algebras. The extended theorem is as follows: Let A be a complex commutative Banach algebra on which there is defined a semi-linear operation $a\rightarrow a^*$ which is involutory and bounded, and such that $\|aa^*\|=\|a\|\cdot\|a^*\|$. Then there is an algebraic isomorphism between A and the algebra of all continuous complex-valued functions on a locally compact Hausdorff space Γ, which are continuous and vanish at infinity (in the sense that the subset of Γ on which the absolute value of the function is not less than any preassigned positive number is compact); and if one denotes the function corresponding to a by $a(\gamma)$, $a^*(\gamma)=\overline{a(\gamma)}$ and $\|a\|=\sup_{\gamma\in\Gamma}|a(\gamma)$. The proof includes a simple derivation of the extension of the Weierstrass approximation theorem to functions on compact spaces (due to M. H. Stone), this extension being based on the Riesz-Markoff representation theorem for linear functionals on a space of continuous functions. (Received March 22, 1946.)

131. I. E. Segal: *The spectrum of a commutative subalgebra of a group algebra.* Preliminary report.

The following theorem is proved and applied to the theory of group algebras: Let A be a commutative self-adjoint algebra of bounded operators on a Hilbert space, and suppose that A is also a Banach algebra with respect to a norm N. Assume further that $\|a\| \leqq N(a)$ for every $a \in A$, where $\|a\|$ designates the operator bound of a, and that $aa^*(i+aa^*)^{-1} \in A$, where i is the identity operator. Then every maximal ideal in A is the intersection of A with a maximal ideal in the uniform closure of A. Moreover, $\lim_{n\to\infty}[N\{(aa^*)^n\}]^{1/n} = \|a\|$. The proof is based partly on the characterization of the algebra of continuous functions stated in the above abstract, and partly on the fact that A is adjoint semi-simple (that is, the intersection of all ideals in A which are maximal with respect to being self-adjoint is the null ideal). As applications, several theorems about harmonic analysis on groups which are either locally compact abelian or compact are derived, notably a representation theorem of positive definite functions which in the locally compact abelian case is equivalent to the Herglotz-Bochner-Weil theorem. (Received March 22, 1946.)

132. I. M. Sheffer: *Note on multiply-infinite series.*

It is natural to inquire what properties of simple series extend to multiply-infinite series. In each case the answer depends on the definition of convergence adopted. The present note considers Pringsheim convergence and σ-convergence (cf. Amer. Math. Monthly vol. 52 (1945) pp. 365–376), relative to these two properties: (a) the Riemann rearrangement theorem; (b) the Cauchy product theorem. It is shown that for the former definition neither of these properties is true, whereas both are valid for σ-convergence. (Received March 21, 1946.)

133. J. D. Swift: *Certain power series with coefficients in a finite field.* Preliminary report.

Series of the type $\sum a^{k(k-1)/2}x^k$ where a is a nonzero mark of a finite field are generated from the usual series for e^x by ordering, to the integer k, the mark a^{k-1} in a manner analogous to that used by Morgan Ward in *A calculus of sequences* (Amer. J. Math. vol. 18 (1936) pp. 255–266). These series display properties analogous to those of the exponential series and can also be related to the power series forms of the theta-functions. Since these series may be formally expressed as the ratio of two polynomials, the analogous theta functions and their derivatives may also be expressed in this form. The polynomial appearing in the numerator, $\sum_0^{q-1} a^{k(k-1)/2}x^k$, where q is the number of nonzero marks of the field, has no zeros in the field if a is a primitive mark. Formulas are also obtained for the value of the polynomial when x is replaced by a mark of the field. Polynomials which factor completely in a field are obtained, exhibiting the zeros of the corresponding theta functions. (Received March 19, 1946.)

134. Bernard Vinograde: *Sfield (division ring) composites.*

A decomposition into a ring R and set N, invariant under automorphisms and non-intersecting, of a ring which is sfield modulo its radical (or an ideal) corresponds to the notion of the ring as a configuration in a space where a point or spectrum is a sheaf $\cdots A, B, C, \cdots$ of formally distinct isomorphic sfields subject to $a_ib_j - c_ic_j$

being nilpotent (or in the ideal), where the subscripts tag corresponding images a in A, and so on. Two spectra S and Σ may be subjected to a generalized calculus guided by a distinct master sfield M. A cleft R is generated by one spectrum. If R properly contains a one-sfield spectrum it must be uncleft. The same holds if two spectra (relatively unordered) have overlapping images in M. The singular case of one element generation is easily handled in the case of algebras. The spectra fall into invariant subspaces, and an M without automorphisms is equivalent to one-spectrum subspaces. A spectrum may sometimes be generated by a function $k(x, t)$, where the parameters x generate a sfield and the t generate the equivalence classes. (Received March 22, 1946.)

135. Morgan Ward: *Note on the order of the free distributive lattice.*

If r_n denotes the order of the free distributive lattice on n elements, and if we set $\log_2 r_n$ equal to $2^n\phi(n)$, then for large n, $1/n^{1/2}<\phi(n)<1/4$ so that $\log_2\log_2 r_n \sim n$. Computational evidence and combinatorial arguments suggest that $n^{1/2}\phi(n)\rightarrow\infty$, but the exact order of $\phi(n)$ is unknown. Incidentally the value of r_6 was computed. It is 7,828352. The method of computation devised easily verified Randolph Church's value 7579 for r_5 (Duke Math. J. vol. 6 (1940) pp. 732–734) but is not powerful enough to evaluate r_7 without prohibitive labor. (Received March 22, 1946.)

ANALYSIS

136. R. H. Bing: *Converse linearity conditions.*

An example is given of a bounded function $f(x)$ $(a<x<b)$ having a derivative on its range and being nonlinear on every subinterval of its range which is such that each point of the graph of $f(x)$ and each point between two points of the graph of $f(x)$ is halfway between some two points of the graph of $f(x)$. (Received March 16, 1946.)

137. Garrett Birkhoff and L. J. Burton: *A weakening of the Hölder conditions for Newtonian force fields.*

Let $\rho(x)$ be a continuous density function of position $x=(x_1, \cdots, x_n)$ near a point $a=(a_1, \cdots, a_n)$ in Euclidean n-space. It is shown that the improper integrals $\int\int\int\rho(x_i-a_i)dR/r^n$ defining the force components for Newtonian attraction exist as improper Riemann multiple integrals (that is, are absolutely integrable) if and only if $\int\int\int\rho d\omega dr/r^{n-1}<+\infty$, where $d\omega$ denotes infinitesimal spherical area. The sufficiency of the usual Holder conditions for convergence is a weak corollary of this. If $\rho dr d\omega=dm$, the corresponding result for Stieltjes integrals is obtained. (Received March 25, 1946.)

138. D. G. Bourgin: *Approximate isometries.*

The Hilbert space results of Hyers and Ulam (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 288–292) are extended to the spaces $L_p(0, 1)$, $1<p<\infty$. (Received March 22, 1946.)

139. R. C. Buck: *An extension of Carlson's theorem.*

Let $K^*(a, c)$ be the class of functions regular and of order 1 in $R\{z\}\geqq 0$, and of type a on the whole positive real axis and type c on the imaginary axis. If A is a subset of the set I of all integers, denote by $\gamma(A)$ the least number for which the following theorem is true: if $f(z)\in K^*(a, c)$, $c<\gamma(A)$, and if $f(z)$ vanishes in A then it vanishes identically. A theorem of Carlson asserts that $\gamma(I)=\pi$; more generally, if A possesses a density $D(A)$, then $\gamma(A)=\pi D(A)$. In this paper these results are extended to the case where A has a mean density or a Poisson density. For general A it is shown that $\pi \inf D_P(A) \leqq \gamma(A) \leqq \pi \sup D_P(A)$. Results are also obtained on the location of singularities of power series with missing coefficients; in these gap theorems, ordinary density and Pólya minimum density are replaced by more general densities. (Received March 18, 1946.)

140. R. H. Cameron and W. T. Martin: *The behavior of measure and measurability under change of scale in Wiener space.*

In this paper it is shown that the measured (outer measure, inner measure, measurability) of a set in Wiener space is completely independent of its measure (outer measure, inner measure, measurability) after change of scale. In fact, given two functions $f(\lambda)$, $g(\lambda)$ defined on $0<\lambda<\infty$ and completely arbitrary except that $0 \leqq f(\lambda) \leqq g(\lambda) \leqq 1$, there exists a set E in Wiener space such that inner meas.$(\lambda E)=f(\lambda)$ and outer meas.$(\lambda E)=g(\lambda)$ for all positive λ. If $f(\lambda)\equiv g(\lambda)$, the set E (which is now measurable in every magnification) can be constructed explicitly (without the use of Zermelo's axiom). (Received March 19, 1946.)

141. W. B. Caton: *An inequality for analytic functions which are represented by their Fourier-Laguerre series.*

Let $f(z)$ be analytic, and let the nth Fourier-Laguerre coefficient be defined as $a_n=\int_0^\infty l_n(z)f(z)dz$, where $l_n(z)=e^{-z/2}L_n(z)$, with $L_n(z)$ the normalized polynomial of Laguerre of degree n. Now suppose that $-\limsup n^{-1/2}\log|a_n|=\kappa>0$, and assume that $f(z)$ is holomorphic in $2R\left[(-z)^{1/2}\right]=\kappa$, and that $f(z)=\sum_0^\infty a_n l_n(z)$ for any $z\equiv x+iy$ in this parabola. It is shown that for any γ, $0<\gamma<2^{-1}\kappa$, and any λ, $0\leqq\lambda\leqq\gamma$, one has $|f(z)|\leqq K|z|^{-1/8}\exp\left\{(x-|z|)/4-(\lambda/2)(|z|+x)^{1/2}\right\}$ for all values of z for which $|z|-x\leqq\gamma^2-\lambda^2$. The inequality obtained here is the analogue of an inequality of Hille for analytic functions represented by their Hermitian series. See E. Hille, *Contributions to the theory of Hermitian series*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 80–94. (Received March 18, 1946.)

142. W. M. Chen: *Distortion theorems in the theory of pseudo conformal mappings.*

Let P denote a mapping by a pair of two complex variable, $w_k(z_1, z_2)$, $k=1, 2$, which are regular in a domain M. M is assumed to be bounded by two analytic hypersurfaces: $z_1=h(z_2, \exp i\theta_2)$, $z_2=\exp i\theta_1$; $0\leqq\theta_k\leqq 2\pi$, $k=1, 2$. Let B be a proper subdomain of M. Using the integral formula proved by Bergman (Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936) p. 851) and some of his results on complex orthogonal functions (Math. Zeit. vol. 36 (1933) p. 171), the author gives an upper (lower) bound for the variation of Euclidean volume of $P(B)$. These bounds depend upon B and the minimum (maximum) in $P(B)$ of the Kernelfunction of $P(M)$. Similar results are obtained for B-area (Bergman, Rendiconti della Reale Accademia Nazionale dei Lincei (6) vol. 19 (1934) p. 474) of a hypersurface $P(S)$, $S'\subset M$, from which bounds for the variation of ordinary area of $P(S)$ follow readily if an additional point function over $P(S)$ is introduced. (Received March 21, 1946.)

143. Nelson Dunford: *Resolutions of the identity.*

A formula of the type $\int\Psi(s)dxE_e^*x^*=x^*\psi(T)x$ is derived for an arbitrary linear operator on an arbitrary Banach space. The formula reduces to the customary one in the case of a self-adjoint operator in Hilbert space. It also throws some light on questions of strong and weak convergence of sequences of functions of certain operators on other Banach spaces. (Received February 23, 1946.)

144. Bernard Epstein: *Integral representation of solutions of the harmonic and wave equations.*

Using the method of integral operators (see Bergman, Math. Zeit. vol. 24 (1926) p. 641; Math. Ann. vol. 99 (1928) p. 629; Bull. Amer. Math. Soc. vol. 49 (1943) p. 163), the author investigates the singularities of some harmonic functions in three variables and "harmonic" vectors $\mathfrak{S}$ (that is, vectors for which $\nabla\cdot\mathfrak{S}=\nabla\times\mathfrak{S}=0$). Algebraic functions with singularities along circles and other algebraic curves are obtained and investigated. A similar operator is introduced for the wave equation and the functions corresponding to the same "associates" as those previously studied in the harmonic case are investigated. Finally, a pair of operators is introduced which generates vector fields, $\mathfrak{E}$ and $\mathfrak{H}$, which constitute solutions of Maxwell's equations for the electric and magnetic fields in free space. (Received March 20, 1946.)

145. Herbert Federer: *The (ϕ, k) rectifiable subsets of n-space.*

Suppose $H_n^k(A)$ is the k-dimensional Hausdorff measure of the subset A of Euclidean n-space, and $F_n^k(A)=\beta(n, k)^{-1}\int_{G_n}\int_{E_k}N(P_R^k, A, y)dL_k y d\phi_n R$ is the k-dimensional integralgeometric Favard measure of A. Here L_k is Lebesgue measure over k-space, E_k, and ϕ_n is the Haar measure over the group G_n of all orthogonal transformations of n-space such that $\phi_n(G_n)=1$. $N(P_R^k, A, y)$ is the number (possibly ∞) of points x in A for which $P_R^k(x)=y$. P_R^k is the projecting function on E_n to E_k such that $P_R^k(x)=(z_1, z_2, \cdots, z_k)$ whenever $R(x)=z=(z_1, z_2, \cdots, z_n)$, $R\in G_n$. Further $\beta(n, k)=\Gamma((k+1)/2)\Gamma((n-k+1)/2)/\Gamma(1/2)\Gamma((n+1)/2)$. The following theorem is proved: If A is an H_n^k measurable set for which $H_n^k(A) < \infty$, then $F_n^k(A)\leqq H_n^k(A)$. A necessary and sufficient condition for equality is that A be contained in a countable number of rectifiable (Lipschitzian) k-dimensional surfaces, except for a subset of H_n^k measure zero. Another result is as follows: The Lebesgue area of every continuous nonparametric two-dimensional surface in three space is equal to the Hausdorff measure H_3^2, and also to the Favard measure F_3^2, of the associated point set. (Received February 13, 1946.)

146. K. O. Friedrichs: *An inequality for potential functions.*

Let ϕ denote potential functions of the N variables $x_1, \cdots, x_N$ in a domain R of a rather general type. To every such domain R there is a constant Γ such that the inequality $\int_R(\partial\phi/\partial x_1)^2dx\leqq\Gamma\int_R\left[(\partial\phi/\partial x_2)^2+\cdots+(\partial\phi/\partial x_N)^2\right]dx$ holds provided that ϕ satisfies the side condition $\int_R(\partial\phi/\partial x_1)dx=0$. The existence of the right-hand member implies in particular the existence of the left-hand member. For $N=2$, the inequality reduces to one between real and imaginary parts of an analytic function. The main tool of the proof is the identity $\int_R\sum_{k=1}^N(\partial\Omega_k/\partial x_k)(\partial\phi/\partial x_1)^2dx$ $=\int_R\left[\sum_{\mu=2}^N A_\mu(\partial\phi/\partial x_\mu)(\partial\phi/\partial x_1)+\sum_{\mu,\nu=2}^N B_{\mu\nu}(\partial\phi/\partial x_\mu)(\partial\phi/\partial x_\mu)\right]dx$, where A_μ and $B_{\mu\nu}$ are linear combinations of the first derivatives of the functions Ω_k, $k=1, \cdots, N$. These functions Ω_k are to vanish at the boundary, to possess bounded derivatives, and to satisfy $(\partial\Omega_1/\partial x_1)+\cdots+(\partial\Omega_N/\partial x_N)\geqq 1$ in the neighborhood of the boundary. Such functions Ω_k are constructed for domains whose boundary possesses a continuous normal except for a finite number of corners or edges. (Received March 29, 1946.)

147. K. O. Friedrichs: *On a theorem of Lichtenstein.*

Let ϕ denote quadratically integrable functions of the N variables $x_1, \cdots, x_N$ which possess quadratically integrable Laplacian derivatives $\Delta\phi$ defined in an appropriate generalized sense. Then ϕ possesses all first and second derivatives in the same generalized sense. To every sphere S_0 and subsphere S there is a constant Γ such that the inequality $\int_S \Delta\phi^2 dx \leqq \int_{S_0} \Delta\phi^2 dx + \Gamma \int_{S_0} \sum_{k=1}^{N} (\partial\phi/\partial x_k)^2 dx$ holds. This theorem is used to generalize to N dimensions the theorem of Lichtenstein that the function $\phi(x, y) = (2\pi)^{-1} \int_R (\log r) \rho(x', y') dx' dy'$, with $r^2 = (x - x')^2 + (y - y')^2$, possesses almost everywhere quadratically integrable derivatives if ρ is quadratically integrable. (Received March 29, 1946.)

148. K. O. Friedrichs: *On the boundary-value problems of the theory of elasticity and Korn's inequality.*

Let $[u_1, \cdots, u_N]$ denote functions of the N variables $[x_1, \cdots, x_N]$ in a domain R of a rather general type. To any such domain R there is a constant κ such that the inequality $\int_R \sum_{\kappa,\lambda} (u_{\kappa,\lambda} - u_{\lambda,\kappa})^2 dx = \int_R \sum_{\kappa,\lambda} (u_{\kappa,\lambda} + u_{\lambda,\kappa})^2 dx$, $u_{k,l} = (\partial u_k/\partial u_l)$, holds provided that the functions u either vanish at the boundary of R or satisfy the side conditions $\int_R (u_{k,l} - u_{l,k}) dx = 0$, $k, l = 1, \cdots, N$. This inequality was formulated by A. Korn in 1907 for $N = 3$. A new more direct proof is given in the present paper. Korn's inequality is a decisive tool in the treatment of boundary value and characteristic value problems of elasticity by minimum problems. These problems can then be completely treated without using the characteristic singularity of the differential equation. (Received March 29, 1946.)

149. K. O. Friedrichs and W. R. Wasow: *On singular perturbations of nonlinear oscillations.*

Many questions in the theory of elastic and electric vibrations lead to systems of differential equations of the form $dX_i/dt = F_i(X_1, \cdots, X_n)$ $(i = 1, 2, \cdots, n-1)$, $\epsilon dX_n/dt = F_n(X_1, \cdots, X_n)$. Of particular interest in this connection is the determination of periodic solutions for small values of the parameter ϵ. The authors prove that if for $\epsilon = 0$ a periodic solution of such a system is given, then there exists for every small ϵ a unique periodic solution in its neighborhood, provided $\partial F_n/\partial X_n \neq 0$, and a certain nondegeneracy condition is satisfied. The latter states in essence that the given periodic solution for $\epsilon = 0$ is an isolated periodic solution. This result justifies the use of the perturbation method for the calculation of periodic solutions of such differential systems. These systems are of a singular type in as much as the order is reduced for $\epsilon = 0$. Although an approach related to that of Poincaré's classical investigations has been used, this singular character of the system necessitates considerable additions and modifications. (Received March 21, 1946.)

150. W. S. Gustin: *Convex boundary sets.*

Two topics are discussed in this paper. 1. A set E in a euclidean ν-space is a *convex boundary set* if E is a subset of the boundary of a convex set in space. Some characterizations of convex boundary sets are obtained. 2. A set E in a euclidean ν-space is of *order* τ (*exact order* τ) if every euclidean subspace of dimension $\nu - 1$ intersects E in at most τ (exactly τ) points. It is shown that: the closure of a connected set of finite order is of finite order; the closure of a connected set of order ν is of order ν; a connected set of exact order ν is totally disconnected. These two topics are linked by the theorem that a connected set of order ν in a euclidean ν-space is a convex boundary set. (Received March 21, 1946.)

151. P. R. Halmos. *Invariant measures.*

Necessary and sufficient conditions are found for the existence of an invariant σ-finite measure m^* stronger than m, for one-to-one measurable transformations T of a measure space X with σ-finite measure m. Hopf's theorem on the existence of a finite m^* is freed from its unnecessary nonsingularity condition; an example is constructed of an incompressible transformation which preserves no finite measure. The extent of uniqueness of the invariant measure (in case it exists) is determined. A structure theorem for nonpathological transformations is derived: it asserts that the general measurable transformation is the product of a measure preserving transformation and a purely dissipative transformation. (Received March 18, 1946.)

152. M. R. Hestenes: *Theorem of Lindeberg in the calculus of variations.*

The present paper is concerned with an analogue of the theorem of Lindeberg for the parametric problem of Bolza. It is shown that if C_0 is a nonsingular arc along which the strengthened condition of Weierstrass holds relative to an integral $I(C)$ and if near C_0 the E-function of $I(C)$ dominates the E-function of a second integral $J(C)$, then there is a constant $b>0$ such that given a constant $\epsilon>0$ one can select a neighborhood $\mathcal{J}$ of C_0, such that $I(C)-I(C_0)\geqq b|J(C)-J(C_0)|-\epsilon$ for every admissible arc C in $\mathcal{J}$. A number of consequences of this and similar results are given. It is shown in particular that the problems of Bolza and Mayer are equivalent, that parametric and nonparametric problems are equivalent, and that properimetric and non-isoperimetric problems are equivalent. (Received March 22, 1946.)

153. Einar Hille: *Analysis in a noncommutative Banach algebra without unit element.*

In the analytical theory of a complex (B)-algebra $\mathfrak{A}$ with unit element, the resolvent $R(\lambda; x)$ is basic. If no unit element exists, one may introduce instead the function $A(\lambda; x)$ with $x+\lambda A(\lambda; x)=xA(\lambda; x)=A(\lambda; x)x$. Thus $A(\lambda; x)$ is the adverse (quasi-inverse) of x/λ when it exists. This leads to a definition of resolvent sets and spectra which is consistent with usual conventions when $\mathfrak{A}$ has a unit element. $A(\lambda; x)$ is a holomorphic function of λ in each of the components of the resolvent set for fixed x and an F-differentiable function of x for fixed λ. If $f(\lambda)$ is holomorphic in a simply connected domain of the λ-plane containing the spectrum of x and if $f(0)=0$, then we may define $f(x)=-(1/2\pi i)\oint f(\lambda)A(\lambda; x)d\lambda/\lambda$ and this definition is consistent with the definition in terms of $R(\lambda; x)$ when $\mathfrak{A}$ has a unit element. Here $f(x)$ is F-differentiable and the mapping $f(\lambda)\rightarrow f(x)$ is a continuous isomorphism taking λ into x and is uniquely determined by these properties. (Received February 13, 1946.)

154. Alfred Horn: *The asymptotic behavior of solutions of integral equations.* Preliminary report.

In the case of the Volterra integral equation $u(x)=\lambda\int_\alpha^x K(x, y)u(y)dy+f(x)$, where $K(x, x)=-A(x)<0$ for $\alpha\leqq x\leqq\beta$, the author proves that if K and f are of class C^{n+1}, then $u(x)=\exp\left(-\lambda\int_\alpha^x A(t)dt\right)\sum_{i=0}^{n-1}F_i(x)/\lambda^i+\sum_{i=1}^{n}G_i(x)/\lambda^i+v(x, \lambda)/\lambda^n$, where $v(x, \lambda)=-G_n(\alpha)\exp\left(-\lambda\int_\alpha^x A(t)dt\right)+R(x, \lambda)/\lambda$, and $R(x, \lambda)$ is uniformly bounded for $\alpha\leqq x\leqq\beta$, $|\arg\lambda|\leqq\pi/2-\delta$. For F_i and G_i are independent of n and λ, and are obtained by solving recurrent differential equations and Volterra equations of the first kind respectively (each G dependent on the previous F_i and G_i). $G_1(x)$ is identically zero if and only if $f(x)$ is a multiple of $K(x, \alpha)$. Hence the resolvent kernel $R(x, y, \lambda)$ has a similar representation with $G_1(x, y) \equiv 0$. Also $G_2(x, y) \equiv 0$ if and only if $K(x, y)$ is of the form $-B(x)C(y)$, in which case $R = B(x)C(y) \exp\left(-\lambda\int_y^x B(t)C(t)\,dt\right)$. Using this representation, it is shown that if K and f are of class C^2, $K(x, x) < 0$, $f(\alpha) = 0$, then the solution of $u(x)/\lambda = \int_\alpha^x K(x, y)u(y)dy + f(x)$ approaches the solution of $\int_\alpha^x K(x, y)u(y)dy + f(x) = 0$ uniformly in $\alpha + \eta \leqq x \leqq \beta$, $|\arg \lambda| \leqq \pi/2 - \delta$. (Received March 13, 1946.)

155. R. D. James and W. H. Gage: *A generalized integral.* Preliminary report.

Let D^2_+g and D^2_-g denote the upper and lower limits, respectively, as $h \to 0$ of $\{g(x+h) - 2g(x) + g(x-h)\}/h^2$. In this paper a three-point integral, $\int_{a,b,c} f(x)dx$, is defined in terms of major and minor functions $M(x)$ and $m(x)$, which are continuous and satisfy the conditions: (1) $M(a) = M(c) = m(a) = m(c) = 0$; (2) $D^2_-M \geqq f(x)$, $D^3_-M > -\infty$, $D^2_+M \leqq f(x)$, $D^2_+m < +\infty$ for all x in (a, c). Certain properties of the integral are studied, and, in particular, it is shown that the function defined by $F(x) = -\int_{a,x,c} f(x)dx$ has, almost everywhere in (a, c), a generalized second derivative equal to $f(x)$. (Received March 12, 1946.)

156. S. Minakshisundaram and Otto Szász: *On absolute convergence of multiple Fourier series.*

The authors consider integrable functions of k variables, periodic of period 2π in each variable, and establish a close relationship between the mean modulus of continuity of a function and the absolute convergence of its Fourier series, thus generalizing certain results for simple Fourier series (cf. Szász, Trans. Amer. Math. Soc. vol. 42 (1936) pp. 366–396). In particular if the function satisfies a Lipschitz condition of degree α, and if $a_{n_1 \cdots n_k}$ are its Fourier coefficients, then $\sum |a_{n_1 \cdots n_k}|\beta < \infty$ for $\beta > 2k/(k+2\alpha)$, while the series may become divergent for $\beta = 2k/(k+2\alpha)$. (Received March 5, 1946.)

157. M. E. Munroe: *A second note on weak differentiability of Pettis integrals.*

In a recent paper (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 167–174) the author proved that if Ω is any compact metric space containing nondenumerably many points and $C(\Omega)$ is the Banach space of all continuous functionals over Ω, then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure. Using a different construction, the same result is obtained, assuming only that Ω contains infinitely many points. It then follows that if M is an abstract (M)-space with unity, a necessary and sufficient condition that every Pettis integral defined to M be almost everywhere weakly differentiable is that M be finite dimensional. (Received March 13, 1946.)

158. Leopoldo Nachbin: *On linear expansions.* I.

Trigonometric series $\sum a_n \cos nx + b_n \sin nx$ possess a number of properties embodied in the very well known theorems of Cantor-Lebesgue, Denjoy-Lusin, and so on. Mutual relations of these properties are investigated for general orthogonal systems. (Received February 25, 1946.)

159. George Piranian: *A summation matrix with a governor.*

For every formal series (1) $\sum a_n$ the relation $p_n\{a\} = \sum_{k=0}^{n} |a_k|$ defines a Norlund sequence p_n provided (2) $\lim_{n\to\infty} p_n\{a\} / \sum_{r=0}^{n} p_r\{a\} = 0$. The series (1) is defined to be summable (G) to S if condition (2) is satisfied and the series is summable $(N; p_n\{a\})$ to S. The method (G) is regular; it is included in the Abel method; and if $f(z)$ is a polynomial with real coefficients, the series $\sum(-1)^n f(n)$ is summable (G). (Received March 13, 1946.)

160. Maxwell Reade: *On averages of Newtonian potentials.*

The problem considered in this paper is that of determining relations that exist between the mass distribution associated with a Newtonian potential and the mass distribution associated with a particular average (mean-value) of the potential; this problem had been considered by Thompson (Bull. Amer. Math. Soc. vol. 41 (1935)) only for circular (spherical) mean-values. Although more general averages are considered in the paper, the following illustrates the results of the paper. Let $\mu(e)$ be a distribution of positive mass, defined for all Borel sets e contained in a fixed, bounded closed plane set F, such that $\mu(F) < \infty$. Define $\mu(E) \equiv \mu(E \cdot F)$, for all Borel sets E in the plane W. Let $U(x, y)$ be the Newtonian potential due to $\mu(E)$ at (x, y), and define $V(x, y) \equiv (1/4h^2) \int_{-h}^{h} \int_{-h}^{h} U(x+\xi, y+\eta) d\xi d\eta$. Then it is proved that $V(x, y)$ is a Newtonian potential with associated mass distribution $\sigma(E) \equiv (1/4h^2) \int_{-h}^{h} \int_{-h}^{h} \mu(E_{x,y}) dx dy$, where $E_{x,y}$ is the image of E when the origin is translated to (x, y). Moreover, if $\delta(x, y)$ is the density of $\sigma(E)$ at (x, y), it is proved that $\delta(x, y) = \mu(Q_{x,y})/(4h^2)$ holds almost everywhere; here $Q_{x,y}$ is the square with vertices $(x \pm h, y \pm h)$. It should be noted that the method used here is different from that used by Thompson; the method used here depends upon the approximation methods developed by Riesz (Acta Math. vol. 54 (1930)). (Received March 21, 1946.)

161. L. A. Ringenberg: *On the extension of interval functions.*

In this paper a necessary and sufficient condition is obtained for a function of intervals to have a completely additive extension. This result was proved in a previous paper (*The extension of rectangle functions* by Reichelderfer and Ringenberg, Duke Math. J. vol. 8 (1941) pp. 231–242) using the results of Radon. The present proof is based on the theory of outer measure in the sense of Carathéodory; it permits a simple characterization of the extension. The second part of the paper is concerned with B-extensions. The properties of the B-extension were suggested by a result of Burkill on the extension of the indefinite integral of a function of intervals. Burkill obtained his extension using absolute continuity and integrability as sufficient conditions, while the present paper gives conditions which are both necessary and sufficient. (Received March 16, 1946.)

162. E. H. Rothe: *Gradient mappings in Hilbert space.*

Mappings of the form $y = x + F(x)$ of the real Hilbert space into itself are considered under the assumption that F is the gradient of a scalar $I(x)$. Necessary conditions and sufficient conditions for I are given in order that F be completely continuous. If these conditions are satisfied an approximation theorem for such mappings is proved which does not hold if F is not a gradient. The second part of the paper contains applications to nonlinear integral equations, especially to the eigenvalue problem. Since in case of the classical linear integral equation the symmetry of the kernel is the necessary and sufficient condition for the corresponding integral operator to be a gradient, the equation $x+F(x)=0$, where F is a gradient, appears as a natural nonlinear generalization of the linear symmetric kernel case. (Received March 21, 1946.)

163. A. C. Schaeffer and D. C. Spencer: *The coefficients of schlicht functions.* III. Preliminary report.

For functions $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ which are regular and schlicht in the unit circle the coefficients $a_2, a_3, \cdots, a_n$ define a point in $2n-2$ dimensional real euclidean space. The properties of $f(z)$ restrict the point $(a_2, a_3, \cdots, a_n)$ to a well defined region of this space which is called the nth region of variability V_n. A method has been found which gives V_n implicitly for general n and explicitly in terms of elementary functions in the case $n=3$. Let $F(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ be a real function whose gradient is not zero in a domain containing V_n. There exist functions F which have a maximum in V_n at an everywhere dense set of boundary points. By using a variational method developed by the authors in previous papers it may be shown that any function $f(z)$ maximizing F satisfies a differential equation of a certain type. There is a one-to-one correspondence between solutions $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ of this differential equation which are regular in $|z|<1$ and boundary points $(a_2, a_3, \cdots, a_n)$ of V_n. (Received March 23, 1946.)

164. Robert Schatten (National Research Fellow): *On projections with bound* 1.

The present notation is that of Bull. Amer. Math. Soc. Abstract 51-9-162. A Banach space $\mathfrak{B}$ is proved to be unitary if, and only if, $\mathfrak{M}\otimes_\gamma\mathfrak{M}^*\subset\mathfrak{B}\otimes_\gamma\mathfrak{M}^*$, for every two-dimensional linear manifold $\mathfrak{M}\subset\mathfrak{B}$. (Received February 13, 1946.)

165. Y. C. Shen: *Interpolation to certain analytic functions by rational functions.*

Let $f(z)$ be a function of class S_2, that is, a function analytic in the interior K of the unit circle $|z|=1$, integrable together with its square on K, and (hence) capable of the integral representation $f(z)=\pi^{-1}\iint_K(f(t)/(1-z\bar{t})^2)dS$, $|z|<1$. Let a_{ni}, $i=1, 2, \cdots, n$; $n=1, 2, \cdots$, be a set of preassigned points on K such that (i) a_{ni} have no limit point on K; (ii) $a_{ni}\neq 0$; and (iii) $a_{ni}\neq a_{nj}$ if $i\neq j$. Finally, let $f_n(z)$ be the rational function of the form $\sum_{i=1}^{n}A_{ni}/(1-\bar{a}_{ni}z)^2$, found by interpolation to $f(z)$ at the points a_{ni}: $f_n(a_{ni})=f(a_{ni})$, $i=1, 2, \cdots, n$. The following results are obtained: (A) *If* $\lim_{n\to\infty} n\prod_{i=1}^{n}|a_{ni}|^2=0$, *then, for every function* $f(z)$ *of class* S_2, *the sequence* $f_n(z)$ *converges to* $f(z)$ *on* K, *uniformly on any point set on* K. (B) *The condition in* (A) *can be replaced by the condition* $\lim_{n\to\infty}\prod_{i=1}^{n}|a_{ni}|=0$, *if, for each* n, *the* n *points* a_{ni} *all lie on a circle* C_n *orthogonal to the unit circle and passing through a fixed point* P *on* K. The restrictions (i), (ii), and (iii) on a_{ni} can be easily removed. (Received March 27, 1946.)

166. P. A. Smith: *Foundations of Lie groups.*

Let G_r be an r-parameter local group with coordinate system $a^1, \cdots, a^r$. Assume that the composition function ab can be represented in the form (written vectorially): $ab=a+b+|a|F(a, b)$ where $|a|=(\sum(a^i)^2)^{1/2}$ and $|F|\to 0$ when $|a|\to 0$, $|b|\to 0$. Then G_r is a local Lie group—that is, G_r admits a coordinate system in which ab is analytic. (Received March 28, 1946.)

167. Fred Supnick: *A class of functions with associated circle manifolds.*

The author treats functions of the form (1) $\sum(\pm 1)/(rn^2+sn+t)$ summed over a subset of the integers, where r, s and t are functions of one or two variables. These functions include the first and second derivatives of the log of the gamma function, $\pi^2/\sin^2 \pi x$, $(\psi(x)-\psi(y))/(x-y)$ and a number of definite integrals and double series, for example, $\sum_{n=2}^{\infty}\sum_{s=0}^{\infty}1/[(ps+r)^n-1]$. Let $C(x)$, $C(y)$, $C(z)$ be three mutually tangent circles not all tangent at one point. Let $C(1)$ and $C(-1)$ be the circles tangent to $C(x)$, $C(y)$, $C(z)$ ($C(1)$ being the smaller one if they are not both equal). Let $C(n)$ and $C(-n)$ be the circles tangent to $C(n-1)$, $C(y)$, $C(z)$ and $C(1-n)$, $C(y)$, $C(z)$ respectively, for $n=1, 2, \cdots$; $C(x)=C(0)$. The circles $C(x)$, $C(y)$, $C(z)$ are called the generators. A non-null subset of the set $\{C(i)\}$ $(i=0, \pm 1, \pm 2, \cdots)$ is called a circle manifold. In this paper it is proved that each function $f(x)$ of the class (1) has an associated circle manifold such that the value of $f(x)$ is $\sum(\pm$ radii) of the circle manifold. The radii of the generators are given as explicit functions of x. Also certain areal inequalities relating to circle manifolds are discussed. (Received March 16, 1946.)

168. Fred Supnick: *On certain limiting sequences of polygons.*

Let a polygonal line P_0 joining the noncollinear vertices u, v, w in the stated order be given. Let a polygonal line P_n be obtained from P_{n-1} by "cutting off" the corners of P_{n-1} according to a given rule R for $n=1, 2, \cdots$. Let R be such that the sequence of polygonal lines P_i $(i=1, 2, \cdots)$ has a limit P. Suppose it is required that P should have a first derivative at each (non-end) point. The question arises—how must R be defined? In this paper it is shown how R can and in certain cases must be defined so that P should have a derivative at each point. Other properties are also considered. On the other hand, the elements of the sequence can be obtained by building triangles on each side of the successive polygonal lines. The sequence P_i does not always have a limit in this case. If it does, it might be "pathological." Certain properties of the latter are derived. (Received March 18, 1946.)

169. Fred Supnick: *On Vitali's covering theorem.*

A synthetic proof (non-Vitalian methods) of the following special case of the Vitali covering theorem is given: Let some one maximal circle C_1 be packed into a Jordan domain J, C_2 into $J-C_1$ and C_n into $J-\sum_{i=1}^{n-1}C_i$ for $n=1, 2, \cdots$. Then $m(\sum C_i)=m(J)$. Vitali's theorem and some generalizations thereof relating to sphere packing problems are also discussed. (Received March 16, 1946.)

170. H. S. Wall: *Bounded J-fractions.*

A J-fraction is bounded and has norm N if the associated J-form is bounded and has norm N. There is determined a convex set K contained in the circle $|z|=N$ such that the J-fraction converges uniformly over every domain whose distance from K is positive. The J-matrix has a unique bounded reciprocal for any z not in K. If the coefficients in the J-fraction are real, the set K reduces to a real interval contained in the interval $(-N, +N)$. (Received March 5, 1946.)

171. Alexander Weinstein: *On a generalization of an axially symmetric potential.*

This paper deals with the differential equation (*) $y\Delta\phi+p\phi_y=0$, previously considered by H. Bateman (*Partial differential equations*, Cambridge University Press, 1932, pp. 408, 409) and L. Bers and A. Gelbart (Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188; Bull. Amer. Math. Soc. vol. 50 (1944) p. 56). For all real $p \geqq 1$, $\phi(x, y)$ is interpreted as the potential of a $(p+2)$-dimensional axially symmetric flow with a stream function $\psi(x, y)$ given by $\psi_x = -y^p\phi_y$, $\psi_y = y^p\phi_x$. It is shown that the fundamental solution $\phi_0(x, y)$ of (*), with a singularity at (x_0, y_0), can be expressed in terms of Bessel functions, and that the corresponding $\psi_0(x, y)$ is many-valued, except for $y_0 = 0$. An application of the divergence theorem yields the period of ψ_0 and leads to the determination of the various limits of the integral $\int_0^\infty \exp(-|x|s)J_q(as)J_{q+1}(ys)ds$ for $x \to 0$ and $y \to b$, where q denotes a non-negative real number. Corresponding results can be formulated in terms of generalized Laplace integrals. (Received February 28, 1946.)

172. J. E. Wilkins: *A note on the general summability of functions.*

There is a result of Titchmarsh (*Introduction to the theory of Fourier integrals*, Oxford, 1937) which gives sufficient conditions for the relation $\int K(x, y, \delta)f(y)dy = f(x) + o(1)$ as $\delta \to 0$. In case $K(x, y, \delta)$ has the form $\{\Gamma(\lambda+1)/2\pi^{1/2}\Gamma(\lambda+1/2)\} \cos^{2\lambda} (1/2)(y-x)$, where $\lambda = 1/\delta$, then Natanson (*On some estimations connected with singular integral of C. de la Vallée Poussin*, C. R. (Doklady) Acad. Sci. URSS. vol. 45 (1944) pp. 274–277) has shown that the remainder term $o(1)$ may be written as $\delta f''(x) + o(\delta)$. The author proposes to provide an extensive generalization of Titchmarsh's theorem to give sufficient conditions for the existence of an asymptotic expansion for the integral of $K(x, y, \delta)f(y)$. The author will also apply the general theory thus developed to several interesting kernels $K(x, y, \delta)$, and in particular will obtain the asymptotic expansion of which Natanson gave the first two terms. (Received March 21, 1946.)

173. H. J. Zimmerberg: *Two general classes of definite integral systems.*

In this paper the notions of definite integral systems considered by Reid (Trans. Amer. Math. Soc. vol. 33 (1931) pp. 475–485), Wilkins (Duke Math. J. vol. 11 (1944) pp. 155–166) and the author (Bull. Amer. Math. Soc. Abstract 52-3-77) are extended to integral systems written in matrix form $y(x) = \lambda\{A(x)y(a) + B(x)y(b) + \int_a^b K(x, t)y(t)dt\}$, where the $n \times n$ matrix $K(x, t) \equiv H(x, t)S(t)$ and the $n \times 2n$ matrix $\|A(x)B(x)\| \equiv \|H(x, a)H(x, b)\|\mathcal{G}$, $\mathcal{G}$ denoting a $2n \times 2n$ constant matrix. These integral systems include the system of integral equations to which a system of first-order linear definite differential equations containing the characteristic parameter linearly in the two-point boundary conditions is equivalent. It is also shown that an integral system of the above form is equivalent to a system of $3n$ homogeneous equations of Fredholm type. (Received February 27, 1946.)

APPLIED MATHEMATICS

174. Nathaniel Coburn: *Pressure-volume relations and the geometry of the net of characteristics in two-dimensional supersonic flows.*

In a previous paper (Quarterly of Applied Mathematics vol. 3 (1945) pp. 106–116), the author showed that for the Karman-Tsien pressure-volume relation in the two-dimensional supersonic flow of a perfect, irrotational, compressible fluid, the net of characteristics (Mach lines) consists of a Tschebyscheff (fish) net. In the present paper, it is shown that any pressure-volume relation of the type $p=Av^{\alpha}$ (where A, $\alpha\neq 1$ are arbitrary constants) leads to one condition on the three metric coefficients of the net of characteristics. A second condition is furnished by the vanishing of the Riemann-Christoffel tensor for the plane. Various properties of the nets which satisfy these two conditions are studied. Thus, it is shown that the class of nets common to all pressure-volume relations of the above type consists of generalized Tschebyscheff nets. (Received February 2, 1946.)

175. W. F. Eberlein: *Wave equation in a stratified medium.*

The author considers the one-dimensional wave equation $U''(z)+[A+z+f(z)]\cdot U''(z)=0$ under the boundary conditions $U(0)=0$ and $U(z)\cdot\exp(i\omega t)$ an "outgoing wave" for large positive z. In the cases of interest $\lim_{z\to+\infty} f(z)=0$. A Laplace transform method of evaluating the characteristic values A is obtained that is applicable to the case $f(z)=\sum_{j=1}^{n} p_j(z)\exp(-\mu_j z)$, where $\mu_j>0$ and $p_j(z)$ is a polynomial in z. (Received March 22, 1946.)

176. Wilfred Kaplan: *Qualitative analysis of physical systems.*

The classical formulation of the problem of analysis of qualitative properties of physical systems defined by differential equations $dx_i/dt=X_i(x_1, \cdots, x_n)$ $(i=1, 2, \cdots, n)$ is criticized for the reason that the results may be extremely sensitive to small changes in the X_i, which can at best be approximately known from physical theory and observation. It is proposed that the inaccuracies of measurement be incorporated into the problem and that a mathematical description of a physical system be considered allowable for qualitative analysis only when the results are independent of the degree of inaccuracy (within prescribed limits). It is shown that this leads in a natural way to replacement of the above differential equations by a flow on the edges of an n-dimensional complex. Applications are made to relaxation oscillations and other problems. (Received March 16, 1946.)

177. Ida Roettinger: *On certain finite integral transformations.*

Finite integral transformations of the type $T\{F(x)\}=\int_0^{\pi} F(x)\phi(kx)dx=f(k)$, where the numbers k belong to a set $K=\{k\}$ of real numbers, are considered. Several properties of these transformations are obtained and specialized to the case where $\phi(kx)$, $k\in\{K\}$, are the characteristic functions of the Sturm-Liouville problem $y''(x)+k^2y(x)=0$, $L_i(y)\equiv a_{i1}y(0)+a_{i2}y'(0)+a_{i3}y(\pi)+a_{i4}y'(\pi)=0$, $i=1, 2$, $0\leqq x\leqq\pi$, and particular cases thereof. The constants a_{ij} are assumed to be such as to guarantee real characteristic values. These special transformations are useful in obtaining quickly a formal solution of the following type of boundary value problems: $\sum_{n=0}^{m} A_{2n}d^{2n}F(x)/dx^{2n}=G(x)$, $0\leqq x\leqq\pi$, where $G(x)$, A_{2n} and the quantities $L_i\{F^{(2n-2s-2)}\}$, $s=1, 2, \cdots, n-1$, $i=1, 2$, are assigned. The constants a_{ij} are allowed to be different for each s. The transformations are also useful in boundary value problems in partial differential equations, where one or more variables behave like x in the above problem. (See also Bull. Amer. Math. Soc. Abstracts 50-5-155 and 51-1-32.) (Received March 22, 1946.)

178. H. E. Salzer: *Coefficients for facilitating the use of the Gaussian quadrature formula.*

The n-point Gaussian quadrature formula, which requires the value of the function at x_i, the ith zero of the Legendre polynomial $P_n(x)$, involves an integrand at irregularly separated points. Thus the integrand is not amenable to an ordinary difference test for checking purposes. The nearest substitute is a divided difference test which is rather tedious when performed in the usual manner. Since the $(n-1)$th divided difference of $f(x_i)$, $i=1, \cdots, n$, can be written as $\sum_{i=1}^{n} C_i^n f(x_i)$ where C_i^n depend only upon $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$, C_i^n will be the same for each n (after the interval of integration is transformed to the usual $[-1, 1]$), and independent of $f(x)$. The coefficients C_i^n are given to $8D$ for $n=3, 4, 5$; to $7D$ for $n=6, 7, 8$; to $6D$ for $n=9, 10$. Limitations on the effectiveness of this check are due to (A) its inconclusive character because it is based upon a single divided difference and (B) the impossibility of a complete difference check whenever the Gaussian formula is employed with efficiency that appreciably exceeds that of the usual Cotes formula. (Received March 13, 1946.)

179. Seymour Sherman: *Stability calculations and time lag.*

Methods of Cauchy and Sturm have been applied to locate the zeros of transcendental expressions occurring in calculations for the stability of a one degree of freedom mechanical system with a retarded damping term. This is an example of what H. Bateman (*The control of an elastic fluid*, Bull. Amer. Math. Soc. vol. 51 (1945) especially pp. 618–626) called the transcendental problem. Previous analyses of this problem (F. Reinhardt, *Parallelbetreib von Synchronengeneratoren mit Kraftsmaschinenregeln konstanter Verzögerungszeit*, Wissenschaftlich Veröffentlichingen aus dem Siemens Werke vol. 18 (1939) pp. 24–44 and N. Minorsky, *Control problems*, Journal of the Franklin Institute, vol. 232 (1941) especially pp. 524–529) have not been consistent. Results here agree with Reinhardt at the point of the inconsistency, the possibility of stable oscillations in a system with time lag. The results are also in qualitative agreement with N. Minorsky (*Self-excited oscillations in dynamical systems possessing retarded actions*, Journal of Applied Mechanics vol. 64 (1942) pp. A65–71), as regards the fact that if the damping coefficient is larger than the absolute value of the retarded damping coefficient, then the oscillation is stable. (Received March 25, 1946.)

180. Andrew Sobczyk: *Stabilization of carrier-frequency servomechanisms.*

A servomechanism is any power amplifying device which employs a feedback arrangement, whereby the error, or difference of output and input quantities, controls a motor which drives the output in such a manner as to maintain the error as nearly zero as possible. The performance of such a device, as regards relative stability, power amplification, and fidelity of the output as a copy of the independent input, is greatly improved by adding to the control signal which is proportional to the error, a component proportional to the derivative of the error. In this paper, by means of a calculation of the steady-state effect of a linear transfer characteristic on the suppressed-carrier modulated error signal (compare Gardner-Barnes, *Transients in linear systems*, vol. 1, p. 248), the Nyquist-Bode stability criterion is extended to apply to carrier-frequency systems. A proportional-derivative effect on the modulation envelope is accomplished approximately by a band-rejection type of electrical network, which is resonant at carrier frequency. The effects on stability and fidelity of band width, and of the networks being off-tune, are shown graphically by so-called gain-phase margin diagrams, which may easily be constructed for various situations. New design formulae for the stabilizing networks, and the tolerances required on their components, in terms of stability and fidelity, are derived. (Received March 21, 1946.)

GEOMETRY

181. Reinhold Baer: *Projectivities with fixed points on every line of the plane.*

The class of projectivities under consideration comprises perspectivities and involutions. These two types exhaust our class in case the Theorem of Pappus holds. But already in the projective plane with coordinates from the field of real quaternions there may be found projectivities which are neither perspectivities nor involutions, and which still belong to our class. The properties of this class are found to be particularly striking, in case the projective plane under consideration is finite. (Received March 25, 1946.)

182. H. S. M. Coxeter: *The content of the general regular polytope.*

The regular solid $\{p, q\}$ has q p-gons at each vertex; that is, it has face $\{p\}$ and vertex figure $\{q\}$. Analogously, the four-dimensional polytope $\{p, q, r\}$ has cell $\{p, q\}$ and vertex figure $\{q, r\}$; and the n-dimensional polytope $\{p, q, \cdots, v, w\}$ has cell $\{p, q, \cdots, v\}$ and vertex figure $\{q, \cdots, v, w\}$. It is found that such a polytope of edge $2l$ has content $(g/n!)l^n b_1^{n/2}/b_1 b_2 \cdots b_{n-1} b_n^{1/2}$, wherein g is the order of the symmetry group and b_k the denominator of the kth convergent of the continued fraction $1/c_1 - 1/c_2 - 1/c_3 - \cdots$, $c_1 c_2 = \sec^2 \pi/p$, $c_2 c_3 = \sec^2 \pi/q$, $\cdots$, $c_{n-1} c_n = \sec^2 \pi/w$. In the case of the n-dimensional hypercube $\{4, 3, \cdots, 3, 3\}$, $g = 2^n n!$. Write $c_1 = 1$, $c_2 = c_3 = \cdots = 2$, whence every $b_k = 1$. (The kth convergent is simply $k/1$.) (Received March 6, 1946.)

183. John DeCicco: *Geodesic perspectivities upon a sphere.*

The following fundamental theorem is proved. If, under a perspective map of a surface Σ upon a sphere S, more than $3\infty^1$ geodesics of Σ correspond to the great circles of S, then every geodesic of Σ is mapped into a great circle of S, and Σ is a sphere or a plane. If Σ is a plane, the author obtains gnomic projection. Otherwise if Σ is a sphere, then both S and Σ are homothetic with respect to the point of perspectivity O and the corresponding points on S and Σ must be homothetic with respect to O. This may be contrasted with the corresponding result obtained by Kasner and the author, on conformal perspectivities of a sphere, where a point of S can correspond to either or both of the two perspective images on the sphere Σ. (Received March 7, 1946.)

184. Arnold Emch: *Dissection of two equivoluminal parallelotops into two finite series of equal numbers of congruent pieces in ordinary and higher Euclidean spaces.*

For the ordinary plane E_2 this problem has been known for a long time. Dehn has proved that in general it is not possible in E_3. By a new method based upon a particular affine relation between two equiareal rectangles it is shown how two equiareal polygons in E_2 can be dissected in the manner indicated above. In E_3 the designation "box" (Kiste by Schouten) is taken as a synonym for "rectangular parallelopipedon." The same term shall be used for rectangular parallelotop in $E_4, \cdots, E_n$, where the n edges on a vertex are mutually perpendicular. Based on an extension of the method for E_2 it is then successively shown how the problem can be solved for $E_3, E_4, \cdots, E_n$. The preliminary dissection of a parallelotop into an equivoluminal box in any E_n is not difficult. (Received February 19, 1946.)

185. Arnold Emch: *New properties of quadrics with umbilics based upon their stereographic projections.*

Nonsingular quadrics, with the exception of the hyperboloid of one sheet and the hyperbolic paraboloid, have real navel points or umbilics; that is, points such that the plane sections parallel to the tangent planes at these points are circles. Central quadrics have in general 12 such points of which four are real. An elliptical paraboloid has two real umbilics. In the case of quadrics of revolution they coincide with the vertices; that is, with the intersections of the axis of revolution. On a paraboloid of revolution, the infinite point of the axis must also be considered as an umbilic. The following theorem is proved: *Given a quadric Q with two or more umbilics (an infinite number in case of a sphere) $U, U', \cdots$, a real plane section S of Q and a plane of projection p, parallel to the tangent plane at one of the U's, intersecting Q in a circle E, the projection of S from U upon p is a circle C. If U' is a diametral umbilic of U, then from U', S is projected into a circle C', which is inverse to C with respect to E as a circle of inversion.* (Received February 20, 1946.)

186. G. B. Huff: *An arithmetic characterization of proper characteristics of linear systems.*

The characteristic $x=\{x_0;\ x_1,\ x_2,\ \cdots,\ x_\rho\}$ of a nondegenerate linear system of plane curves consists of the *order* x_0 and the *multiplicities* $x_1, x_2, \cdots, x_\rho$ at a set of ρ base points. If p, d are the *genus* and *dimension* of the system, x satisfies: (1) $x_0^2-x_1^2-x_2^2-\cdots-x_\rho^2=d+p-1$, and $3x_0-x_1-x_2-\cdots-x_\rho=d-p+1$; and a set of inequalities (2) $y_0x_0-y_1x_1-y_2x_2-\cdots-y_\rho x_\rho\geqq 0$, where y runs over the characteristics of all Bertini L-curves of order $y_0<x_0$ and defined at that base. A *proper* solution x of the Cremona equations (1) is one which is the characteristic of a nondegenerate linear system. It is shown for all ρ and certain values of p, d that a solution of (1) which also satisfies (2) is proper. (Received March 21, 1946.)

187. Edward Kasner and John DeCicco: *Comparison of union-preserving and contact transformations in space.*

In Bull. Amer. Math. Soc. vol. 50 (1944) pp. 98–107, the authors gave a general theory of union-preserving transformations in space from curve-elements of order n into lineal-elements. In the general case, these are defined by a directrix equation of the form: $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)})=0$, involving derivatives of order $(n-2)$. If $n=2$, the directrix equation $\Omega(X, Y, Z, x, y, z)=0$ involves no derivatives. Thus a single directrix equation in which a point in space corresponds to a surface gives rise to a union-preserving transformation, and also to a Lie contact transformation. In general, these transformations are distinct. A comparison of these two types of transformations is given. As an application the authors develop a new theory of parallel curves in space. In this theory, the parallel of a helix is a helix. Another example is that the parallel of a plane curve lies on the cylinder with elements orthogonal to the plane of the original curve and whose directorial curve is the evolute. The new curve is gauche in general. (Received March 7, 1946.)

188. Edward Kasner and John DeCicco: *Conformal perspectivities upon a sphere.*

Recently the authors proved that the only perspective conformalities upon a plane are Ptolemy's stereographic projection of a sphere and the obvious limiting case of a parallel plane. In the present work, all the prespective conformalities upon a sphere are obtained. If a surface Σ is conformally mapped upon a sphere S by a perspectivity with center at O, then Σ is a sphere inscribed in the cone (real or imaginary) whose vertex is at O and which is tangent to S. Thus a sphere S can be projected by a conformal perspectivity with center at O only into another sphere Σ; furthermore the centers of S and Σ are collinear with O and the distances of their centers from O are proportional to their radii. Two special cases are noteworthy. I. If a surface Σ admits a conformal perspectivity upon a sphere S from the center of S, then Σ is a sphere concentric with S. II. If a surface Σ admits a conformal perspectivity upon a sphere S from a point O on S, then Σ is a sphere tangent to S at O. Only in this last special case can Σ be a plane, in which event it is perpendicular to the diameter of S through O. (Received March 8, 1946.)

189. Edward Kasner and John DeCicco: *The distortion of angles in general cartography.*

Let a surface Σ be mapped in a point-to-point fashion (not conformal) upon a plane π. The azimuth ratio $\alpha = d\Theta/d\theta$ is defined as the rate of change of the inclination at a fixed point P on the surface Σ with respect to the inclination at the corresponding point p on the plane π. This is a function of (x, y, y'). An azimuth curve is the locus of a point along which the azimuth ratio does not vary. There are ∞^2 azimuth curves. These have many properties in common with the scale curves, which are defined as those along which the scale $\sigma = ds/dS$ is constant. Various geometrical properties of these azimuths are obtained. The case where the azimuths are all straight is studied in detail. (Received March 7, 1946.)

190. Harry Levy: *The projective geometry of Riemannian spaces of two dimensions.*

Cartan showed in 1937 that if K, the Gaussian curvature of a two-dimensional ds^2, is not constant, the orthogonal trajectories of the curves $K = \text{const.}$ are invariant under geodesic preserving transformations. The author shows the existence of a second invariant family of curves, the integral curves of $dx^1/L^1 = dx^2/L^2$, where $L^i = K^i{}_{,j}K^j - (4/3)K^j{}_{,j}K^i$. The coincidence of these two families (or the vanishing of L^i) is itself an invariant property; the families, when distinct, determine a projective connection which is the basis for the development of an invariant projective theory. In terms of projective invariants of the K and L net of curves, necessary and sufficient conditions are established for the existence of transitive groups of geodesic transformations; such groups contain but two parameters; the equivalence problem is reduced to the problem of the existence of functional relations among a set of at most 12 invariants. Geodesics possess an "arc" determined to within an affine transformation; there exists an absolute "parallelism" of geodesics which is invariant, and on parallel geodesics the ratio of "arc" is preserved. (Received March 21, 1946.)

191. Abraham Schwartz: *Higher normal spaces for spaces of class greater than one.*

Consider a $V_m \subset R_n$ with a first normal space of m_1 dimensions at a particular point P. Call a direction at P "omniconjugate" if it is conjugate to all directions. By elementary algebraic methods one can show from the Gauss-Codazzi-Ricci equations that the second normal space at P vanishes unless (1) there are $m - m_1$ linearly independent omniconjugate directions at P, or (2) the V_m is contained within some V_{m+r},

$1 \leqq r < m_1$, in such a way that there are $m-r$ linearly independent omniconjugate directions at P with respect to the V_{m+r}. (The local R_{m+r} of the V_{m+r} at P is contained within the local R_{m+n_1}.) If there is a second normal space of m_2 dimensions, then $m_2 \leqq s(s+1)(s+2)/6 + t(t+1)/2$, where s and t are integers determined by $s(s+1)/2 \leqq m_1 \leqq (s+1)(s+2)/2$, $t = m_1 - s(s+1)/2$. Similar statements can be made about the vanishing of the third and other higher normal spaces. (Received March 19, 1946.)

192. Y. C. Wong: *Contributions to the theory of surfaces in a 4-space of constant curvature.*

A Riemannian 4-space of constant curvature and a surface in it are denoted by S_4 and V_2, respectively. The method of studying V_2 in S_4 in this paper is invariant and is similar to those of G. Ricci (*Lezioni sulla teoria della superficie*, Verona-Padova, 1898) and W. Graustein (Bull. Amer. Math. Soc. vol. 36 (1930)) for their studies of surfaces in a Euclidean 3-space. In essence, the method consists of setting up a suitable system of invariant fundamental equations for a V_2 in S_4, and expressing the required imbedding conditions of V_2 in S_4 in terms of the intrinsic properties of V_2. Curvature properties, especially those about the curvature conic, of a general V_2 in S_4 are first discussed. Then the V_2's whose curvature conic is of certain particular nature are studied. These include the minimal V_2, with the R-surface of K. Kommerell (Math. Ann. vol. 60 (1905)) as a special case, ruled V_2, and V_2 with an orthogonal net of Voss. The paper concludes with a complete determination of those V_2's in S_4 whose first fundamental form and one of whose second fundamental forms are respectively identical with the first and second fundamental forms of a surface in a 3-space of constant curvature. (Received March 11, 1946.)

193. Y. C. Wong: *Scale hypersurfaces for conformal-Euclidean space.*

This paper contains generalizations to n-space of some of the results obtained recently by E. Kasner and J. DeCicco (Amer. J. Math. vol. 67 (1945)) for the scale curves in conformal maps of a surface on a plane. The fundamental form $ds^2 = e^{2\sigma}(dx_1^2 + \cdots + dx_n^2)$, with $\sigma = \sigma(x_1, \cdots, x_n)$, represents a conformal-Euclidean n-space C_n, conformally mappable on the Euclidean n-space R_n with rectangular Cartesian coordinates $x_1, \cdots, x_n$. The hypersurfaces $\sigma =$ constant in R_n are the scale hypersurfaces in the mapping of C_n on R_n, and any simple family of hypersurfaces in R_n is called quasi-isothermal if it represents the scale hypersurfaces of a conformal mapping of some C_n on R_n such that the scalar curvature of C_n is constant over each of the scale hypersurfaces. A few theorems are proved concerning the cases when a family of quasi-isothermal hypersurfaces is a family of (a) ∞^1 hyperplanes, (b) ∞^1 generalized cylinders of rotation. This subject is closely connected with the subject of the isoparametric hypersurfaces of T. Levi-Civita and B. Segre (Rendiconti della Reale Accademia Nazionale dei Lincei (6) vol. 26 (1937), vol. 27 (1938)) and incidentally connected with that of the subprojective Riemannian space of B. Kagan and H. Schapiro (*Abhandlung des Seminars für Vektor- und Tensoranalysis*, vol. 1, 1933). (Received March 11, 1946.)

Logic and Foundations

194. Ira Rosenbaum: *Hegel, mathematical logic, and the foundations of mathematics.*

Hegel, like Boole, DeMorgan, Pierce, Frege, Peano, Russell, Whitehead and others, sought a new logic more adequate than the traditional one to modern science and practice. Prior to Frege or Russell, Hegel possessed logistic intent which was determining factor in development of his logic. He was familiar with historic roots of modern symbolic logic—with work of Lully, Bruno, Leibniz, Ploucquet, Euler, and Bardili—rejecting their primitive efforts as inadequate. He knew the Sancho Panza dilemma, recognized by Church as closely related to Russell's paradox, and even proposes solution resembling theory of types. His denial of so-called "laws of thought" antedates denial of law $A \cdot A = A^2$ in Boole's logical algebra, of parallel-axiom in non-Euclidean geometry, of commutative law in quaternion theory. When mathematicians and mathematical philosophers were seeking to avoid the infinite, Hegel restored it, with new interpretation, to a place of central importance in foundations of analysis. He speaks of infinite as involving equality of whole and part, distinguishes "bad" infinite (Cantor's variable finite) from "good" infinite (recognized as reflexive) and objects to the phrase "and so on to infinity," shown eliminable by Frege. Once influential, he was known to Boole, DeMorgan, Bolzano, Pierce, G. Cantor, and Russell, while Frege refers to Fischer's Hegelian logic. (Received March 23, 1946.)

STATISTICS AND PROBABILITY

195. Will Feller: *A limit theorem for random variables with infinite moments.*

Let $\{X_k\}$ be an arbitrary sequence of mutually independent random variables with the same distribution function $V(x)$. It is assumed that some moment of order less than two is infinite; the first moment may be infinite, but if it is finite it should be normed to zero. Let $S_n = X_1 + \cdots + X_n$ and let $\{a_n\}$ be a monotonic positive numerical sequence. It is shown that the probability that the inequality $|S_n| > a_n$ takes place for infinitely many n is the same as the probability that $|X_n| > a_n$ for infinitely many n; it is one or zero according as the series $\sum \{V(-a_n) + 1 - V(a_n)\}$ diverges or converges. (Received March 21, 1946.)

196. Will Feller: *The law of the iterated logarithm for identically distributed random variables.*

Let $\{X_n\}$ be a sequence of mutually independent random variables with the same distribution function $V(x)$ with vanishing first moment and unit variance. Suppose that (*) $\int_{|t|>x} t^2 dV(t) = O((\log \log x)^{-1})$, and let $\{\phi_n\}$ be an arbitrary monotonic sequence, $\phi_n > 0$. The probability that the inequality $X_1 + \cdots + X_n > n^{1/2}\phi_n$ will be satisfied for infinitely many n is shown to be zero or one according as the series $\sum \phi_n n^{-1} \exp(-\phi_n^2/2)$ converges or diverges. The condition (*) is in a certain sense the best possible. If it is not satisfied, the above exact analogue to the strict law of the iterated logarithm does not hold, but slightly more complicated necessary and sufficient conditions are given in the paper. (Received March 21, 1946.)

197. Casper Goffman: *Measures of fluctuation of a variable mean.* Preliminary report.

Suppose a finite order of random variables $x_1, \cdots, x_n$ is given, all normally distributed, with the same known standard deviation, but with unknown means $a_1, \cdots, a_n$, not necessarily alike. A measure of fluctuation of the means is defined as a function $f(a_1, \cdots, a_n)$ such that (1) for every real h, $f(a_1+h, \cdots, a_n+h) = f(a_1, \cdots, a_n)$, and (2) for every real c, $f(ca_1, \cdots, ca_n) = c^2 f(a_1, \cdots, a_n)$. It is shown that, in an important sense, the only admissible measure of fluctuation, except for a constant factor, is the variance $f(a_1, \cdots, a_n)=\sum_{i=1}^{n}(a_i-n^{-1}\sum_{j=1}^{n}a_j)^2$. The method of maximum likelihood is applied to special functions to obtain tests for significant differences. These tests have applications to industrial problems. (Received March 22, 1946.)

198. Isaac Opatowski: *Average duration of transition in Markoff chains.*

Consider a chain of transitions $(i\rightarrow i+1)$, $(i+1\rightarrow i)$, where $i=0, 1, \cdots, n-1$. Let the usual conditional probabilities of these transitions within any time Δt be respectively $k_{i+1}\Delta t+o(\Delta t)$ and $g_i\Delta t+o(\Delta t)$, where the k_i's and g_i's are constant. Let the probability of any other transition during Δt be $o(\Delta t)$. The probability $P(t)$ of a transition $(0\rightarrow n)$ within a time t is an increasing function of t and $P(\infty)=\prod_{i=1}^{i=n}k_i/K_i$, where $s=-K_i$ are the n roots of the secular equation $\|a_{r,c}\|/s=0$ defined by $a_{r,r}=s+k_{r+1}+g_{r-1}$, $a_{r,r-1}=k_r$, $a_{r,r+1}=g_r$, $a_{r,c}=0$ for $|r-c|>1$; $r, c=0, 1, \cdots, n$; $g_{-1}=0$ (Bulletin of Mathematical Biophysics vol. 7 (1945) pp. 170–177). If $g_{n-1}=0$ then $P(\infty)=1$, if $g_{n-1}\neq 0$ then $P(\infty)<1$. Consequently, the average duration of the transition $(0\rightarrow n)$ is $E=\int_0^{P(\infty)}t\,dP/P(\infty)$. Its explicit expression is a simple symmetric function of the k_i's and K_i's. If all the g_i's are zero, $E=\sum_{i=1}^{i=n}k_i^{-1}$. By using the results of a previous paper (Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 411–414) simple expressions of E are obtained also in the cases in which the k_i's and g_i's are independent of i. The paper is a part of an article to appear in the Bulletin of Mathematical Biophysics vol. 8 (1946). (Received March 19, 1946.)

199. Isaac Opatowski: *Markoff chains with variable intensities: average duration of transition.*

Consider a simple Markoff chain. Let $k_{i+1}\Delta t+o(\Delta t)$ be the conditional probability of a transition $(i\rightarrow i+1)$ within any time Δt, where $i=0, 1, \cdots, n-1$ and $k_i=F(i)f(t)$. It is known that by changing t into a new time variable $T=\int_0^t f(t)dt$, the present chain may be treated as if its intensities k_i were constant and equal to $F(i)$. Let $t=\sum_m c_m T^m$ be a polynomial in T. Let $P(t)$ be the probability of a transition (0—n) within t. It is shown that $\int_0^1 t\,dP$, the average duration of a transition $(0\rightarrow n)$, equals $\sum_m m!c_m h_m$, where h_m is the complete homogeneous symmetric function of degree m of $1/F(1)$, $1/F(2), \cdots, 1/F(n)$. This formula is obtained by using a previous result on the moments of Markoff chains (Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 83–88). The paper is a part of an article to appear in the Bulletin of Mathematical Biophysics vol. 8 (1946). (Received March 20, 1946.)

TOPOLOGY

200. E. E. Floyd: *On the extensions of homeomorphisms on the interior of a two cell.*

Let $f(I)=R$ be a homeomorphism of the interior I of a two cell with boundary C onto a bounded plane region R. It is shown that if f is extensible to $\bar{I}$, then the extension is non-alternating on the boundary C. A condition is also derived which is equivalent to the existence of an extension g of f, where $g(\bar{I})=\bar{R}$, $g=f$ on I, and g is light and non-alternating on C. This is applied to conformal maps, yielding the following theorem: let $f(I)=R$ be a 1-1, conformal map of the interior I of the unit circle onto a plane bounded region R. A necessary and sufficient condition that f be extensible to $\bar{I}$ is that the boundary of R be locally connected. If the extension exists, it is light and non-alternating on the unit circle. (Received March 5, 1946.)

201. W. H. Gottschalk: *Almost periodicity, equi-continuity and total boundedness.*

Let $f\colon X\times T\to X$ be a transformation group, where X is a uniform space and T a topological group. A set E in T is called *relatively dense* if there is a compact set A in T such that each left translate of A meets E. A point x of X is called *almost periodic* if to each neighborhood U of x corresponds a relatively dense set E in T such that $f(x, E)\subset U$. The transformation group f is called *almost periodic* if to each index α corresponds a relatively dense set E in T such that $x\in X$ implies $f(x, E)\subset U_\alpha(x)$. Let G denote $[f^t | t\in T]$, provided with the strong uniformity. The following theorems are proved: (1) If the family G is equi-continuous at x, if f_x is continuous on T, and if x is almost periodic, then the orbit of x is totally bounded. Conversely, if the family G is equi-uniformly continuous and if the orbit of x is totally bounded, then x is almost periodic. (2) If X is compact and if f is continuous on $X\times T$, then the following statements are pairwise equivalent: (i) the transformation group G is almost periodic; (ii) the family G is equi-uniformly continuous; (iii) the space G is totally bounded. Bochner's characterization of almost periodic functions and a sharpened form of the Dirichlet-Kronecker theorem are corollaries of (1) and (2), respectively. (Received March 11, 1946.)

202. D. W. Hall and J. W. T. Youngs: *Comments on the cores of certain classes of spaces.*

This paper considers a class of spaces $\mathfrak{X}$, a class of spaces $\mathfrak{Y}$ containing $\mathfrak{X}$, and a class of transformations τ each of which maps certain elements of $\mathfrak{X}$ onto certain elements of $\mathfrak{Y}$. For any subclass $\mathfrak{A}$ of $\mathfrak{X}$ the closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ (with respect to τ) is defined as the subclass of $\mathfrak{Y}$ consisting of all possible images of spaces taken from $\mathfrak{A}$ under transformations taken from τ. Under a mild restriction on τ this closure satisfies the Kuratowski axioms for a topological space. Indeed, the stronger result holds that the closure of the union of *any* collection of subclasses of $\mathfrak{X}$ is the union of their closures. For a given subclass $\mathfrak{A}$ of $\mathfrak{X}$ the set $C(\mathfrak{A})$, called the core of $\mathfrak{A}$, denotes the largest closed subclass of $\mathfrak{A}$ with respect to τ. This paper studies the problem of characterizing the cores of certain classes of spaces under particular families of mappings. Examples are given showing that this approach provides a new setting for certain results of T. Radó, G. T. Whyburn, and R. H. Bing. (Received March 22, 1946.)

203. R. G. Helsel: *A theorem on surface area.*

J. W. T. Youngs (Ann. of Math. vol. 45 (1944) pp. 753–785) has proved that the Lebesgue area is Kerékjártó invariant in the 2-sphere case, that is, two Kerékjártó equivalent continuous mappings from a 2-sphere into 3-space determine Fréchet surfaces with equal Lebesgue area. The present paper proves the Kerékjártó invariance of both the Lebesgue area and the lower area in the 2-cell case (see Radó, Proc. Nat. Acad. Sci. U. S. A. vol. 31 (1945) pp. 102–106, for the definition of the lower area). The argument is based on recent topological results of Radó (Trans. Amer. Math. Soc. vol. 58 (1945) pp. 420–454). (Received March 13, 1946.)

204. Edwin Hewitt: *A characterization of rings of continuous functions.*

It is well known that the ring of all continuous real-valued and bounded functions on an arbitrary topological space X can be considered as a normed ring with real scalars, if $\|f\| = \sup |f(p)|$, where p runs through all points of X. The following algebraic characterization of rings of continuous functions, with this norm and the usual definitions of addition and multiplication, is established. A commutative normed ring $\mathfrak{R}$ with unit and with real scalars can be mapped onto the ring of all continuous real-valued functions defined on an appropriate bicompact Hausdorff space by a mapping which is both an isometry and an algebraic isomorphism if and only if $(x^2+e)^{-1}$ exists for all $x \in \mathfrak{R}$ and $(\|x\|e-X)^{-1}$ exists for no $x \in \mathfrak{R}$. The proof is carried out by topologizing the set of all maximal ideals in the ring R and is similar to corresponding proofs for rings with complex scalars given by Gelfand (Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 3–23). (Received February 19, 1946.)

205. Edwin Hewitt: *A generalization of the concept of complete regularity.* Preliminary report.

A generalization of complete regularity for topological spaces is introduced by the following definition. A topological space X is said to be completely regular with respect to the topological space T if: (1) for every pair of distinct points p and q in X, there is a continuous mapping f of X into T such that $f(p) \neq f(q)$; (2) for every $p \in X$ and every neighborhood U_p, there exist a point $s \in T$, a neighborhood V_t, and a continuous mapping g of X into T such that $g(p) = s$, and $g(q) \in V_t'$ for all $q \in U_p'$. If T is the closed interval $[0, 1]$, then complete regularity with respect to T is ordinary complete regularity. It is proved that X is homeomorphic to a subset of a certain Cartesian product of spaces T if X is completely regular with respect to T. It has been proved by M. H. Stone that every T_0-space is completely regular with respect to the T_0-space containing two points and having exactly one nonvoid proper subset which is open. It is shown that this result cannot be extended to T_1-spaces: there is no T_1-space such that all T_1-spaces are completely regular with respect to it. (Received February 19, 1946.)

206. Edwin Hewitt: *An effective construction of the Stone-Čech beta for a countable discrete space.*

Stone (Trans. Amer. Math. Soc. vol. 41 (1937) pp. 461–465) and Čech (Ann. of Math. vol. 38 (1937) pp. 831–837) have proved that for every completely regular space X there exists a unique bicompact Hausdorff space βX such that X is homeomorphic to a dense subset X^* of βX and such that every bounded continuous real-valued function defined on X^* can be extended over βX so as to remain continuous. Both proofs are nonconstructive, and very few spaces βX, for non-bicompact X, have been exhibited. By standard topological methods, an effective construction is given which exhibits β for a countable discrete space as a subset of the space of all characteristic functions on the closed unit interval. (Received February 19, 1946.)

207. Edwin Hewitt: *On rings of continuous functions.* Preliminary report.

It has been proved by Stone (Trans. Amer. Math. Soc. vol. 42 (1937) p. 475) and Gelfand and Kolmogoroff (C.R. (Doklady) Acad. Sci. URSS. vol. 22 (1939) pp. 11–15) that two bicompact Hausdorff spaces are homeomorphic if and only if the rings of real-valued continuous functions defined on the two spaces are algebraically isomorphic. Also, two completely regular spaces satisfying the first axiom of countability are homeomorphic if and only if the rings of bounded real-valued and continuous functions on the two spaces are algebraically isomorphic. It is to be expected that many topological properties of a space can be described in terms of the algebraic structure of the function ring. Algebraic characterizations are given of the dimension of an arbitrary normal space and of the Čech homology groups for an arbitrary completely regular space. The former depends upon an algebraic characterization of the ring of real continuous functions on the n-sphere, while the latter has a simple characterization in terms of elements in the function ring which can be represented as squares. (Received March 19, 1946.)

208. G. B. Huff: *Rational-distance sets in the plane.*

Erdös and Anning recently exhibited (Bull. Amer. Math. Soc. vol. 51 (1945) p. 598) examples of planar configurations with the property that the distance between any two was rational. In this note such a configuration is called a rational-distance set. It is shown that if S is a rational-distance set, and I_c is an inversion in a circle such that the square of the radius is rational, with center c at a point of S, then the transform of S by I_c is a rational-distance set. Examples of rational-distance sets are given, including one consisting of an infinite number of points on a line and two points off the line. (Received March 21, 1946.)

209. J. L. Kelley and Everett Pitcher: *Applications of natural homomorphism sequences.* II.

This is a continuation of the work reported in I of the same title (Bull. Amer. Math. Soc. Abstract 52-1-49). A principal theorem relates the homology groups of a complex to the homology groups of a finite set of subcomplexes which form a covering, the homology groups of intersections, and the homology groups of the nerve. This extends the Mayer-Vietoris formulas on coverings by two subcomplexes and the theorem of Helly on coverings by acyclic subcomplexes. (Received February 21, 1946.)

210. J. H. Roberts: *Open transformations and dimension.*

Suppose A and B are separable metric spaces and $f(A)=B$, where f is open; that is, if U is open in A then $f(U)$ is open in B. It is not assumed that f is continuous. If for every $y\in B$ the set $f^{-1}(y)$ is countable, then $\dim B \leqq \dim A$, a result obtained by P. Alexandroff in case f is continuous. For any A, if $\dim A=n$ and $-1<m<n$ then there exists a B and an open f such that $f(A)=B$, $\dim B=m$, and for every $y\in B$, $f^{-1}(y)$ is a single point. Thus open transformations cannot raise dimension if $f^{-1}(y)$ is countable, but open transformations exist which lower dimension at will, even with $f^{-1}(y)$ a single point. (Received March 30, 1946.)

211. G. E. Schweigert and G. S. Young: *Remarks concerning invariants for certain finite transformations.*

The transformations considered are either exactly k to 1, $T_1(A)=B$, or finite to one and interior, denoted by $T_2(A)=B$. The space A is compact and metric. It is assumed that A contains sets $M(i)$ converging to a set M and that each of these contains more than one point and is perfect. It is also assumed that these sets are pairwise disjoint. One theorem states that in the image of $M+\sum M(i)$ a similar situation of convergence of perfect sets prevails, the images being taken under T_1 or T_2. In the case in which the sets M and $M(i)$ are connected, the above shows that continua which are not hereditarily locally connected cannot be mapped onto continua which are; in other words, hereditarily locally connected continua are invariant under the inverses of T_1 and T_2. Other instances of invariance are also considered. (Received March 21, 1946.)

212. N. E. Steenrod: *Extensions of maps and products of cocycles.*

The Pontrjagin algebraic enumeration of the homotopy classes of maps of a 3-complex K^3 in a 2-sphere S^2 is generalized to a classification of maps of K^{n+1} in S^n. This is achieved using new products of cocycles. A product of order $i \geqq 0$ of a p-cochain and a q-cochain is defined which has dimension $p+q-i$ and is the ordinary product if i is zero. The square of order i of a p-cocycle is a cocycle (cocycle mod 2) if $p-i$ is odd (even). This leads to invariant squaring operations on cohomology classes. A square is always of order 2. Let f map the n-section K^n of a complex K in S^n and let z be the image in K of the basic n-cocycle of S_n under the cochain mapping induced by f. Then f can be extended to K^{n+2} if and only if z is a cocycle in K and its square of order $n-2$ is zero. The homotopy classification is obtained from this extension theorem. (Received March 21, 1946.)

213. Fred Supnick: *Rectilinear deformation.*

A linear graph in which each edge is a line segment is called a rectilinear graph. A finite planar nonsingular rectilinear graph G is called a simple graph. Two simple graphs G_1 and G_2 in the same plane, equivalent as linear graphs, are said to be strongly equivalent if the cyclic order of the edges in the stars of any two corresponding vertices, and of any two corresponding cycles of G_1 and G_2, is the same, and if corresponding vertices are inside corresponding cycles. A rectilinear deformation of a simple graph G is a "continuous deformation of G such that at each instant (during which it is being deformed) the graph is strongly equivalent to G." The theorem is proved: If G_1 is strongly equivalent to G_2, then G_1 can be deformed rectilinearly into G_2. This is done by giving a method for carrying out the deformation. The above theorem is based on the theorem: Let $[v_i, v_j]$ be an edge of a simple G which has no chain of less than three edges joining v_i to v_j besides $[v_i, v_j]$, then G can be deformed rectilinearly so that the vertices v_i and v_j coincide. Thus, suspended chains can be "straightened." Constructive methods are given throughout. (Received April 1, 1946.)

214. Fred Supnick: *Equivalent rectilinear graphs.*

Let G_1 be a rectilinear graph which is not disconnected by the removal of any one vertex. That simple closed cycle such that each vertex of the graph is either inside or on it is called the maximal cycle. The following theorem is established: There exists a graph G_2 strongly equivalent to G_1 with its maximal cycle convex. It is shown how to construct G_2. As a consequence of the deformation theorem G_1 can be rectilinearly deformed into G_2. Also it is shown that if two vertices u and v of a simple graph G_1 can be joined by a Jordan arc which does not otherwise intersect G_1, then a graph G_2 strongly equivalent to G_1 can be constructed such that the correspondents of u and v in G_2 can be joined by a line segment which does not otherwise intersect G_2. Again applying the deformation theorem G_1 can be rectilinearly deformed into G_2. Thus, the corollary at the end of the previous abstract has been established by a different method. Also, an example is given to show that the latter reasoning can not be directly extended to infinite graphs. (Received March 16, 1946.)

215. G. W. Whitehead: *A generalization of the Hopf invariant.*

A theorem of J. H. C. Whitehead (Ann. of Math. vol. 42 (1941) pp. 409–428) is generalized as follows: if S^p and S^q are spheres with one point in common, then $\pi_n(S^p \cup S^q) \approx \pi_n(S^p) + \pi_n(S^q) + \pi_n(S^{p+q-1})$ for $n < p+q+\min(p, q) - 3$. This theorem is used to construct a homomorphism H of $\pi_n(S^r)$ into $\pi_n(S^{2r-1})$ for $n < 3r-3$. If $\alpha \in \pi_{2r-1}(S^r)$, then $H(\alpha)$ has degree equal to the invariant of α defined by H. Hopf (Math. Ann. vol. 104 (1931) pp. 637–665). If $\alpha \in \pi_n(S^r)$ and $\beta \in \pi_r(X)$, an element $\beta \cdot \alpha \in \pi_n(X)$ is determined by superposition of representatives of α and β; it is shown that $(\beta_1 + \beta_2) \cdot \alpha = \beta_1 \cdot \alpha + \beta_2 \cdot \alpha + [\beta_1, \beta_2] \cdot H(\alpha)$, where $[\beta_1, \beta_2]$ is the product defined by J. H. C. Whitehead (ibid.). It is further shown that the homotopy groups $\pi_{8n}(S^{4n})$ and $\pi_{16n+2}(S^{8n})$ are different from zero; in fact, they contain elements α with $H(\alpha) \neq 0$. (Received March 8, 1946.)

216. R. L. Wilder: *A generalization of local co-connectedness and its applications.*

Let U be an open subset of a space S. Let r, k be non-negative integers such that if $\mathcal{E}$ is a fcos of S, then there exists a refinement $\mathcal{D}$ of $\mathcal{E}$ such that if an open set D meets U and lies in an element of $\mathcal{D}$, then there exists $E \in \mathcal{E}$ containing D such that exactly k r-cocycles in $U \cap D$ are linearly independent rel. cohomology in $U \cap E$. One then says that $p_r(U, x)$ is *uniformly* equal to k over U. In order that U be r-coulc, it is nas that $p_r(U, x)$ be uniformly equal to 0 over U. For open subsets of an n-gcm, the properties of being 0-ulc and of $p_n(U, x)$ being uniformly equal to 1 over U are dual; for $0 < r < n$, the r-ulc and $(n-r)$-coulc properties are dual. This renders much simpler the treatment of ulc properties in an orientable n-gcm. For example, a fundamental technique for dealing with an open ulc^k U is that of displacing a cycle Z^k on $\overline{U}$ into a "nearby" homologous cycle in U. This is easily accomplished by a co-realization Z_{n-k} in U, of a coordinate of Z^k based on the "co-properties" of U, the intersection of Z_{n-k} with the fundamental n-cycle of S being the desired cycle. Details are given in chapter 10 of the author's forthcoming book. (Received March 21, 1946.)

217. R. L. Wilder: *Certain topological properties in the large and their applications.*

A subset M of a space S is said to have Property $(P, Q)_r[(P, Q)^r]$ if for every pair of open sets P, Q such that $\overline{Q}$ is compact and $P \supset \overline{Q}$, at most a finite number of r-cocycles [compact r-cycles] of M in Q are linearly independent rel. cohomologies on M in P. The corresponding property when only cocycles that cobound in S are considered is denoted by $(P, Q, \sim)_r$. For subsets of compact spaces, Property $(P, Q)_r$ = Property S_r (Amer. J. Math. vol. 61 (1939) pp. 823–832). If a locally compact S has Property $(P, Q)_{r+1}$ and $p_r(x) \leqq \omega$ for all $x \in S$, then S has Property $(P, Q)_r$. An application of this shows that if S is n-dimensional, then a nasc that S be lc^n is that $p_r(x) \leqq \omega$ for all $x \in S$ and $r \leqq n$. (This shows, incidentally, condition (b) in the n-gm definition of Begle, Amer. J. Math. vol. 67 (1945) p. 63, to be unnecessary.) Properties $(P, Q, \sim)^{r-1}$ and $(P, Q, \sim)_r$ are dual; and if S has Properties $(P, Q, \sim)_r$ and $(P, Q, \sim)_{r+1}$ and M is a closed subset of S, then Property $(P, Q, \sim)_r$ of M is dual to Property $(P, Q, \sim)_{r+1}$ of $S-M$. When S is a spherelike n-gcm, Properties $(P, Q, \sim)_r$ and $(P, Q, \sim)^{n-r}$, $r \leqq n-2$, are dual for open sets, and if M is a closed subset of S, Property $(P, Q)_r$ of M is dual to Property $(P, Q)^{n-r-1}$ of $S-M$, $1 \leqq r \leqq n-1$. These properties and their applications are given in detail in chapter 11 of the author's forthcoming book *Topology of manifolds*. (Received March 21, 1946.)

218. R. L. Wilder: *Concerning generalized n-cells.*

Basic among the open n-gms and n-gms with boundary are the generalized cells: A *generalized n-cell* is a non-compact n-gm whose compact homology groups of dimension less than n reduce to the identity. A *generalized closed n-cell* is a compact space consisting of a spherelike $(n-1)$-gcm K and a generalized n-cell A such that (1) $K \cap A = 0$, (2) if Z^{n-1} is the fundamental cycle of K, then $Z^{n-1} \sim 0$ on $K \cup A$, and $K \cup A$ is an irreducible membrane rel. to this homology; and (3) $P_r(K \cup A, x) = 0$ for $r \leqq n$ and all $x \in K$. (In the separable metric case, the generalized 2-cell is an ordinary 2-cell and the generalized closed 2-cell is an ordinary closed 2-cell.) For a spherelike n-gcm S, the Jordan-Brouwer separation theorem may be stated thus: If M is a spherelike $(n-1)$-gcm in S, then $S-M$ is the union of two disjoint generalized n-cells A_i, $i=1, 2$; moreover, $\bar{A}_i = A_i \cup M$ is a generalized closed n-cell. However, of greater interest is the fact that if two generalized closed cells are joined along their boundaries, the result is a manifold; specifically, if a space S is the union of two generalized closed n-cells $C_i = K \cup A_i$, $i=1, 2$, where K and A_i satisfy the conditions relative to K and A of the above definition, and $A_1 \cap A_2 = 0$, then S is a spherelike n-gcm. It follows that if $K \cup A$ is a generalized closed n-cell as in the definition above, then the generalized n-cell A is a ulc^{n-1} subset of $K \cup A$. Details will be given in chapters 9 and 10 of the author's forthcoming book. (Received March 21, 1945.)

219. R. L. Wilder: *Homology groups of perfectly normal spaces.*

Let $\mathcal{J}$ be an arbitrary algebraic field. If S is a locally compact, lc^r space and M a compact G_δ subset of S, then the group $H^r(M, \mathcal{J})$ is a complete metric space with countable base $\{Z_i^r\}$ such that $\lim Z_i^r = 0$ and every coset is uniquely expressible in the form $\sum_{i=1}^{\infty} a^i Z_i^r$, $a^i \in \mathcal{J}$; the vector subspace generated by the elements Z_i^r is dense in $H^r(M, \mathcal{J})$ and uniquely determines the latter. (Compare L. Vietoris, Math. Ann. vol. 97 (1927) pp. 454–472, where an analogous result is given for the case where M is compact metric and hence may be considered a subset of the Hilbert fundamental parallelopiped.) When M has infinite Betti number $p^r(M, \mathcal{J})$ and is a subset of an orientable n-gcm S, the Alexander type of duality may be obtained via collections $\{Z_i^r\}$, $\{Z_r^j\}$, where the Z_r^j are cocycles mod $S-M$ such that $Z_i^r \cdot Z_r^j = \delta_i^j$, the $(n-r-1)$ cycles obtained from the intersections of δZ_r^i with the fundamental n-cycle of S forming a system whose elements are geometrically linked with the corresponding Z_i^r (cf. E. Begle, Amer. J. Math. vol. 67 (1945) pp. 59–70). For an open orientable n-gm which is the union of a countable collection of compact sets, a similar countable base of infinite cycles is obtainable and the Poincaré duality may be expressed as a homeomorphism between the Betti groups of infinite r-cycles and compact $(n-r)$-cycles. (Received March 21, 1946.)

220. G. S. Young: *Interior and border transformations on surfaces.*

Let M be a compact 2-manifold, with or without boundary. Let $f(M) = N$ be either an interior or border (G. E. Schweigert, Bull. Amer. Math. Soc. Abstract 47-1-104) transformation. Then either N is a 2-manifold and f is light, or N is a set obtained from a dendrite by a finite number of finite identifications, and f is not light. If for each point x in N, each component of the inverse of x is locally connected, and f is not light, then N is either an arc or a simple closed curve. This last result generalizes results of Whyburn and of Puckett. Extensions of these theorems are made to noncompact manifolds and to pseudo-manifolds. It is shown that each of these possible images of M can be realized by an interior monotone transformation. (Received March 21, 1946.)

221. J. W. T. Youngs: *The topological theory of Fréchet surfaces (2-cell case).*

The principal result of this paper is a solution of the representation problem: given one representation of a surface, obtain the totality of its representations. The solution is a consequence of a characterization of positive Fréchet equivalence in terms of a suitably modified Kerékjártó equivalence. If, for $i=1, 2$, X_i is an oriented 2-sphere containing a 2-cell H_i, then two mappings $f_1(H_1)=Y=f_2(H_2)$ are positively Kerékjártó equivalent if there exist: (1) a cactoid $\mathfrak{X}$ containing a hemicactoid $\mathfrak{H}$; (2) a pair of monotone mappings $m_1(X_1)=\mathfrak{X}=m_2(X_2)$ such that $m_1(H_1)=\mathfrak{H}=m_2(H_2)$, m_i is one-to-one on X_i-H_i, and each true cyclic element $\mathfrak{S}$ of $\mathfrak{X}$ can be oriented so that, simultaneously, Dgr $m_i(\mathfrak{S}; X_i)=1$, for $i=1, 2$ (the mapping $m_i(\mathfrak{S}; X_i)$ is m_i followed by the retraction of $\mathfrak{X}$ onto $\mathfrak{S}$); and (3) a light map $l(\mathfrak{H})=Y$, such that $f_i(x_i)=lm_i(x_i)$, $x_i\in H_i$, for $i=1, 2$. The mappings are positively Fréchet equivalent if for every $\epsilon>0$ there is a homeomorphism $h_\epsilon(H_1)=H_2$ such that $\rho\{f_1(x_1), f_2h_\epsilon(x_1)\}<\epsilon$. It is shown that positive Fréchet equivalence is the same as positive Kerékjártó equivalence. (Received March 21, 1946.)

NEW PUBLICATIONS

COOLIDGE, J. L. A history of the conic sections and quadric surfaces. Oxford University Press, 1945. 11+214 pp. $6.00.

FUETER, R. Analytische geometrie der ebene und des raumes. Basel, Birkhäuser, 1945. 180 pp. 22.50 s.fr.

MURNAGHAN, F. D. Analytic geometry. New York, Prentice-Hall, 1946. 8+402 pp. $3.25.

SOKOLNIKOFF, I. S. Mathematical theory of elasticity. New York, McGraw-Hill, 1946. 373 pp. $4.50.

VALIRON, G. Equations fonctionnelles applications. (Cours d'analyse mathématique, vol. 2.) Paris, Masson, 1945. 2+605 pp. 600 fr.

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PREFACE

When the last catalogue of the library of the American Mathematical Society was published in 1932, the Society had nearly 1,800 members, the number of books in the library was approximately 7,500. At the present time the library contains about 11,600 volumes and the members of the Society number 3,000. Many members who have joined the Society since 1932 may never have seen its catalogue; some may not be aware of the existence of the Society's library; of members of some years standing but very few have any means of knowing about the additions to the library's holdings. It is after these considerations that the Council of the Society recommended and the Board of Trustees authorized the publication of a new edition of the catalogue.

The Society's library is housed in the Low Memorial Library Building of Columbia University. Books may be consulted there. Loans to members outside New York City should be arranged through the library regularly used by the member. This library may request an interlibrary loan by addressing a letter to the American Mathematical Society, 531 West 116th Street, New York 27, New York.

Mention should be made of the fact that in recent years the library has acquired an almost complete collection of American doctoral dissertations in mathematics.

The abbreviations used in the present catalogue are explained on page v and should be consulted for its convenient use. A few general remarks follow.

1. The catalogue is divided into two parts. Section I contains serial publications; Section II all others. In cases in which listing in either section could be justified, the choice was determined by judgment as to the greatest convenience to the user of the catalogue. The precedent set by the Union List of Serial Publications was followed in most, but not in all, cases.
2. Bibliographical information is not generally included because of lack of space and lack of facilities for acquiring it in anything like completeness.
3. The library has greatly extended its holdings of dissertations. These are listed under the name of the author, with indication of the name of the degree conferring institution, and the year of award of the degree.
4. Well known series of books are listed in Section I under the general title of the series. Individual volumes are listed under the author's name in Section II.
5. Maiden names of women are used if their publications appeared under that name.
6. References from Section I to Section II, or vice-versa, without further amplification refer to an entry under the same principal name.
7. In many cases it is uncertain at the moment of going to press whether a journal whose publication was interrupted during the war years will be resumed.

The great amount of detailed work necessary for the publication of this catalogue was done by Mrs. A. K. Farwell, who was employed by the Society for this purpose, and by the staff of the New York office of the Society. Grateful acknowledgment is made of the valuable assistance received from members of the Columbia University Library Staff.

ARNOLD DRESDEN
Librarian

May 1946

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Pomeroy Ladue	1896–1901
David Eugene Smith	1902–1920
Raymond Clare Archibald	1921–1941
Arnold Dresden	1942–

STATISTICS

	Dissertations on film	Number of bound volumes
1902		121
1907		2,549
1912		3,871
1921		5,862
1925		6,798
1931		7,480
1946	528	11,632

EXPLANATION AND ABBREVIATIONS

The following general rules and abbreviations were used in preparation of Section I of this catalogue.

The journals, transactions, proceedings, etc., of a society are entered under the first word, not an article, of the latest form of the name of the society.

Learned societies and academies, other than English, with names beginning with an adjective denoting royal privilege are entered under the first word following the adjective. These adjectives, Kaiserlich, Königlich, Reale, Imperiale, etc., are abbreviated to K., R., I., etc., and are disregarded in the arrangement.

A serial published by a society, but having a distinctive title, is entered under the title with reference from the name of the society.

A serial not published by a society or public office is entered under the first word, not an article, of the latest form of the title.

Colleges and universities having a geographical designation are entered under the name of the city, state, or country contained in the title.

Volume numbers and dates are inclusive in all cases and the dates are for the period covered by the serial, not those of publication.

A dash between volume numbers or dates signifies "from and including the former to and including the latter."

A plus sign (see p. 1, line 10) indicates that the serial is currently received and that the set is complete from the last date or volume number given.

Superscripts designate numbers or parts of volumes.

Parallels indicate that publication ceased with the preceding date or volume (see p. 1, line 18).

Brackets (see p. 4, line 6) indicate that the volumes or years so enclosed are not complete.

volume	v	publication	publ.
number	no	translated	trans.
series	s	edited	ed.
new series	ns	January	Ja
edition	ed.	February	F
year	yr	March	Mr
second	2d	April	Ap
revised	rev.	May	My
enlarged	enl.	June	Je
fascicule	fasc.	July	Jl
section	sec.	August	Ag
film copy	fc	September	S
dissertation	diss.	October	O
no date	n.d.	November	N
no place	n.p.	December	D

SECTION I

SERIAL PUBLICATIONS

- Abhandlungen über den mathematischen unterricht in Deutschland. *See* Sec. II, under International commission on the teaching of mathematics
- Abhandlungen zur geschichte der mathematischen wissenschaften mit einschluss ihrer anwendungen. Leipzig. 1–10 issued as suppl. to Zeitschrift fur mathematik und physik
  - 1, 3, 6–7, 9, 12–13, 16^1, 17, 19–20, 22, 24–25, 28, 1877–1910
- Academia colombiana de ciencias exactas, físicas y naturales, Bogota. Supersedes Sociedad colombiana de ciencias naturales
  - Revista
    - 2+, 1938+
- Academia das ciencias de Lisboa. *See* Jornal de sciencias mathematicas, fysicas, e naturaes
- R. Academia de ciencias exactas, físicas, físico-quimicas et naturales, Madrid. Name varies slightly
  - Discursos leídos
    - 1922
  - Memorias
    - 14, 20, 22–30, 32, 1890–1929; s2 v1–2, 4–9, 1921–29||
    - Serie de ciencias exactas
      - 1, 1930
  - Revista. 16–29 also as s2 v1–14
    - 1–32, 1904–35
- Academia nacional de ciencias Antonio Alzate, Mexico. 1884–1929 as Sociedad científica "Antonio Alzate"
  - Memorias. Individual issues 4+ as Memorias y revista
    - 2–54, 1888–1934
    - Index: 1–27; 1–52
- Academia polytechnica do Porto. *See* Oporto
- Academia scientiarum fennica. *See* Suomalainen tiedeakatemia
- Académie des sciences, Cracow. *See* Akademija umiejętności
- Académie des sciences, Paris
  - Comptes rendus hebdomadaires des séances
    - 1–65, 100+, 1835+
    - Index: 122–151
- Académie des sciences d'Ukraine, Kief. *See* Akademii͡a nauk U.R.S.R., Kiev
- Académie des sciences de l'empereur François Joseph I, Prague. *See* Česká akademie vĕd a umĕní v Praze
- Académie des sciences de l'Union des républiques soviétiques socialistes. *See* Akademii͡a nauk S.S.S.R., Leningrad
- Académie des sciences et lettres de Montpellier
  - Bulletin. 1909–17 as Bulletin mensuel. Numbering irregular
    - 1911–19, 1921, 1925–37
    - Section des sciences
      - Mémoires
        - s2 v1–3, 1893–1907
- Académie des sciences, inscriptions et belles-lettres de Toulouse
  - Bulletin. Supersedes and is superseded by its Mémoires
    - 3, 1899/1900||
  - Mémoires
    - s10 v1–11, 1901–11
- Académie impériale des sciences, St. Petersburg. *See* Akademii͡a nauk S.S.S.R., Leningrad
- Académie royale des sciences de Serbie, Belgrade. *See* Srpska kral'evska akademija
- Académie royale des sciences, des lettres et des beaux-arts de Belgique, Brussels
  - Annuaire
    - 1–2, 5, 7–11, 14–15, 18–31, 68–85, 87+, 1835+

  - Bulletins
    - s3 v17–21, 23–30, 34, 36, 1889–98||
  - Classe des sciences
    - Bulletins. Suspended 1915–18
      - 1899–1900, 1902–14, 1919+ (incl. Annexe, 1915)
      - Index: 1899–1910; 1911–14
    - Mémoires. Collection in 4°
      - s2 v1+, 1907+
    - Mémoires. Collection in 8°
      - s2 v1+, 1904+
  - Mémoires couronnés et autres mémoires. Collection in 8°
    - 60–66, 1900–04||
    - Index: 1898–1914. This also indexes the other four Mémoires of this academy
  - Mémoires couronnés et mémoires des savants étrangers. Collection in 4°
    - 58–62, 1899–1904||
  - Mémoires de l'Académie. 1789–1845 as Nouveaux mémoires
    - s2 v10, 17, 19, 50–54, 1837–1904||
  - *See also* Sec. II
- ACADÉMIE tchèque des sciences, Prague. *See* ČESKÁ akademie věd a umění v Praze
- ACADEMY of science of St. Louis
  - Officers, Charter, Members, etc.
    - 1893
  - Transactions
    - 5–29, 1886–1938
    - Index: 1–10; 11–20
- ACCADEMIA d'Italia, Rome
  - Annuario
    - 1–9, 1929–37
  - Classe di scienze fisiche, matematiche e naturali
    - Memorie
      - 1–5, 1930–34
- R. ACCADEMIA dei Lincei, Rome. *See* R. ACCADEMIA nazionale dei Lincei
- R. ACCADEMIA delle scienze dell' istituto di Bologna
  - Classe di scienze fisiche
    - Memorie. To 1907 see the academy's Memorie
      - s6 v5–10, 1908/09–12/13; s7 v1–10, 1913/14–22/23; s8 v1–10, 1923/24–32/33 (incl. suppl. 1910)
    - Rendiconto. To 1907 see the academy's Rendiconto ns v12–37, 1907–33
  - Memorie. Superseded by Classe di scienze fisiche. Memorie
    - s5 v8–10, 1899–1904; s6 v1–4, 1904–07||
  - Rendiconto. Superseded by Classe di scienze fisiche. Rendiconto
    - ns v4–11, 1899–1907||
- R. ACCADEMIA delle scienze di Torino
  - Annuario
    - 1933/34||
  - Atti. Continued in sections
    - 30–48, 50–62, 1895–1926/27||
    - Index: Each 10th v includes index to preceding 10v
  - Classe di scienze fisiche, matematiche e naturali
    - Atti
      - 63–74, 1927/28–38/39
  - Memorie
    - s2 v48–65, 66¹, 67–69, 1889–1939
    - Index: Each 10th v in s2 includes index to preceding 10v
- R. ACCADEMIA delle scienze fisiche e matematiche, Naples. (Societa reale di Napoli)
  - Atti. Suspended 1917–26
    - s2 v8–14, 16–20, 1897–1935
  - Rendiconti
    - s3 v1–18, 20–36, 1895–1930; s4 v1–8, 1931–38
  - General index: 1737–1903
- R. ACCADEMIA di scienze, lettere e belle arti di Palermo
  - Atti
    - s3 v1–9, 1891–1912

- R. ACCADEMIA di scienze, lettere ed arti degli zelanti, Acireale
  - Classe di lettere
    - Memorie
      - s3 v10, 1917/18; s4 v2, 1927/29
  - Classe di scienze
    - Memorie
      - s3 v1–3, 10, 1901/02–18/22; s4 v2, 1927/31
- R. ACCADEMIA di scienze, lettere ed arti in Modena
  - Atti e memorie. To 1922 as its Memorie
    - s3 v1–14, 1898–1922; s4 v1–4, 1926–33/34; s5 v1–2, 1935/36–36/37
  - *See also* Sec. II
- R. ACCADEMIA nazionale dei Lincei, Rome. To Je1920 as R. Accademia dei Lincei
  - Atti
    - 1, 1847/48
  - Classe di scienze fisiche, matematiche e naturali
    - Rendiconti
      - s5 v1–33, 1892–1924; s6 v1–19, 23–25, 27–29, 1925–Je39||
  - Rendiconti. Continued as Rendiconti della Classe di scienze fisiche, and Rendiconti della Classe di scienze morali
    - s4, v1–7, 1884–91||
  - Rendiconti delle sedute solenni
    - 1892–97, 1904–12, 1914, 1916–28
  - General index: 1876–1910; 1911–24; 1925–34
- ACCADEMIA pontificia dei nuovi Lincei, Rome. *See* PONTIFICIA accademia delle scienze
- ACTA mathematica. Stockholm
  - 1+, 1882+
  - Index: 1–35
  - For appendices, 1884, 1885, *see* BIBLIOTHECA mathematica
- ACTUARIAL society of America, New York
  - Actuarial studies
    - no1, 4–5, 1919–32
  - List of members
    - 1911
  - Transactions. 1–6 as Papers and transactions
    - 1–35, 36^1, 37+, 1889+
- AIX-MARSEILLE, Université d'
  - Faculté des sciences
    - Annales
      - Ser. in 4°
        - 11–26, 1901–30|| (incl. suppl. to v16, 20)
      - Ser. in 8°
        - 1–12, 1921–39
- AKADEMIE der wissenschaften, Berlin
  - Abhandlungen [mathematical memoirs only]
    - 1892–94, 1897, 1901, 1903–04, 1906–07||
  - Monatsberichte. Supersedes its Bericht
    - 1857–81
    - Index: 1836–58; 1859–73; 1874–81
  - Physikalisch-mathematische klasse
    - Abhandlungen
      - 1909–38
    - Sitzungsberichte
      - 1922–37
  - Sitzungsberichte. Supersedes its Monatsberichte. Continued in classes
    - 1882–1921
- AKADEMIE der wissenschaften, Leipzig. 1846–1918 as K. Saechsische gesellschaft der wissenschaften. 1919+ full name: Saechsische akademie der wissenschaften
  - Mathematisch-physische klasse
    - Abhandlungen
      - 12–42, 1883–1938
    - Berichte
      - 26–89, 1874–1937
- AKADEMIE der wissenschaften, Munich
  - Almanach
    - 1909||

Mathematisch-naturwissenschaftliche abteilung. 1871–1923 as its Mathematisch-physikalische classe
Abhandlungen
14–25, 27–32, 1881–1928; ns v1–44, 1929–38 (incl. suppl. 1922–32)
Sitzungsberichte
1881–1912, [1913], 1914–37
Index: 1871–85; 1886–99

K. Akademie der wissenschaften, Vienna
Mathematisch-naturwissenschaftliche klasse
Sitzungsberichte
108–148, 1899–1939

K. Akademie van wetenschappen, Amsterdam
Afdeeling natuurkunde
Proceedings of the section of sciences
1–30, 1899–1927
Verhandelingen. Eerste sectie
13^4, 1928

Akademii͡a nauk S.S.S.R., Leningrad
Bulletin. s4 v1–2 also as s3 v33–34. 1928+ in sections
s3 v1, 5–32, 1860–88; s4 v1–2, 1890–92; s5 v3–25, 1895–1906; s6 v1–7, 8^1, 10–21, 1907–27||
Classe physico-mathématique
Mémoires
s8 v14^7, 15–17, 32^4, 33^{8-9}, 34^2, 1903–16
Comptes rendus (Doklady) de l'Académie des sciences de l'URSS. Supersedes its Doklady sA. 1934+ in 4v each year. 1934 v1–1935 v3 lack numbering, later supplied as v2-7. Published in two editions
International edition
ns v1–9, 11–17, 19–29, 32, 34+, 1933+
Russian edition
ns v22–29, 42+, 1939+
Fiziko-matematicheskiĭ institut imeni V. A. Steklova
Izvestii͡a
1–3, 1922–30||
Trudy. Title also in French: Travaux de l'Institut physico-mathématique Stekloff
1+, 1932+
Fiziko-matematicheskiĭ referativnyĭ zhurnal
1, 1939
Gruzinskiĭ filial
Matematicheskiĭ institut
Trudy
1+, 1937+
Soobshchenii͡a
1-2, 1940–41
Izvestii͡a. For earlier years see Academy's Bulletin
Otdelenie matematicheskikh i estestvennykh nauk. 1928–30 as its Fiziko-matematicheskoe otdelenie. 1936+ in series
s7 v1–7, 1928–34
Serii͡a fizicheskai͡a
1936–38, 1940
Serii͡a matematicheskai͡a
8+, 1944+

Akademii͡a nauk U.R.S.R., Kiev. Name varies: Ukraĭn'ska akademii͡a nauk, Vseukraĭn'ska akademii͡a nauk
Fizichno-matematichniĭ viddil
Trudy
1–11, 1923–29
Zapiski
1–5, 1923–31||
Institut budivel'noĭ mekhaniki
Kafedra matematichnoĭ fiziki
[Publications]
4–5, 8, 14, 1934–35

Zapiski
1–4, 1937-39
Institut matematiki
Zbirnik prats'
no1–6, 1938–41
Zhurnal
1–4, 5^1, 1934–38
Prirodnicho-tekhnichiĭ viddil. French name: Classe des sciences naturelles et techniques
Zhurnal Industri͡al'no-tekhnichnogo tsiklu
3^2, 1933
AKADEMIJA umiejętności, Krakow
Bulletin international. Comptes rendus des séances. Continued in classes
1898–1900||
Komisya bibliograficzna
Katalog literatury naukowej polskiej
1–10, 16–18, 1901–30
Wydział matematyczno-przyrodniczy . . .
Bulletin international
1901–09||
sA: Sciences mathématiques
1910–12, 1914–38
Comptes rendus mensuels des sciences mathématiques et naturelles
1929–38
AKTUARSKÉ vědy: pojistná matematika, matematiká, statistika. (Jednota československých matematiků a fysiků v Praze) Prague
2–7, 1930–38
ALLAHABAD, India
University
Mathematical association
Bulletin
1, 1927–28
"ALZATE, Antonio," Sociedad científica, Mexico. *See* ACADEMIA nacional de ciencias Antonio Alzate
AMERICAN academy of arts and sciences, Boston
Memoirs
ns v12+, 1902+
Proceedings. 9–31 also as ns v1–23
26+, 1890+
See also Sec. II, under BOWDITCH, N.
AMERICAN institute of actuaries, Chicago
Record. v7^2 never printed
1^{1-4}, 3–6, 7^1, 8+, 1909+
Index: 1–15; 16–25
Yearbook
1926–35, 1941
AMERICAN institute of physics, New York. *See* PHYSICAL review; REVIEW of scientific instruments
AMERICAN journal of mathematics. Baltimore. 1–2 as American journal of mathematics pure and applied
1+, 1878+
Index: 1–50
AMERICAN mathematical monthly.[1] Ithaca, N. Y.
1+, 1894+
See also CARUS mathematical monographs; and Sec. II, under MATHEMATICAL association of America
AMERICAN mathematical society, New York
Bulletin. s1 as Bulletin of the New York mathematical society
1–3, 1891–94; s2 v1+, 1894+
Index: 1–s2 v10; 11–20; 21–30; 31–40; 41–50

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[1] The AMERICAN mathematical monthly, founded by B. F. Finkel in 1894 and published by him until 1913, has been the official journal of the Mathematical Association of America, since its organization in December, 1915.

Colloquium publications[2]
1+, 1905+
List of members, etc.
1891, 1892, 1894, 1896–1918 (annually), 1920+ (biennially)
Transactions
1+, 1900+
Index: 1–5; 1–10; 11–20; 21–30; 31–40; 41–50
See also MATHEMATICAL reviews; MATHEMATICAL surveys; Sec. II, under AMERICAN mathematical society; KLEIN, F.
AMERICAN philosophical society, Philadelphia
Annual report of the library
1935–36
List of members
1892, 1933–34 (also in Proceedings 30–56)
Memoirs
1–20, 1935–43
Miscellanea
1^2, 1936||
Proceedings
29–87, 1891–1944
Index: 51–75
Transactions
ns v20–33, 1902–44
Yearbook
1937–42
General index: 1769–1940
AMERICAN physical society
Bulletin. Ithaca, N. Y.
1–3, 1899–1902||
See also PHYSICAL review
AMERICAN society of civil engineers, New York
Proceedings. Contain papers and discussions, separately paged, of all papers subsequently appearing in Transactions in revised form
32–34, 1906–08
Transactions
55+, 1905+
Index: 1–83; $84\text{–}99^2$
See also Sec. II
AMERICAN society of mechanical engineers
Transactions
46, 1924
Year book
1915–20, 1922, 1924–26
ANALYST. A journal of pure and applied mathematics. Des Moines, Ia. Superseded by Annals of mathematics
1–10, 1874–83||
Index: 1–10
ANNALES de mathématiques pures et appliquées. Paris. Superseded by Journal de mathématiques pures et appliquées
1–22, 1810–31||
ANNALES roumaines de mathématiques. (Institut mathématique roumain) Bucharest. v2 published in 1935
1–2, 1935–37
ANNALES scientifiques des institutions mathématiques savantes de l'Ukraine. *See* ZAPISKI Kharkivs'kogo matematichnogo tovaristva ta Ukrains'kogo institutu matematichnikh nauk
ANNALI delle scienze del regno lombardo-veneto. Padua
1–2, 1831–32
ANNALI delle università toscane. Pisa. v35–48 also as ns v1–14. Merged into Pisa. Scuola superiore. Annali
35–47, 48^2, 1916–29/30||

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[2] For the origin of the COLLOQUIUM publications and for their history see R. C. ARCHIBALD, A semicentennial history of the American Mathematical Society, 1888–1938, published by the Society in 1938.

ANNALI di matematica pura ed applicata. Milan, Bologna. Supersedes Annali di scienze matematiche e fisiche
s2 v3, 9, 1869–79; s3 v1–31, 1898–1922; s4 v1+, 1923+
Index: 1858–97; 1898–1922; 1923–37

ANNALI di scienze matematiche e fisiche. Superseded by Annali di matematica...
Index: 1850–57. Combined with index 1858–97 of Annali di matematica pura ed applicata

ANNALS of mathematical statistics. Ann Arbor, Mich.
1+, 1930+
Index: 1–10

ANNALS of mathematics. Princeton. Supersedes Analyst
1–12, 1884–99; s2 v1+, 1899+
Index: 1–12; s2 v1–12

ANNALS of mathematics studies. Princeton
1+, 1940+

ARCHIV der mathematik und physik. Leipzig, Berlin
s2 v1–17, 1884–1900; s3 v1–24, 1901–16
Index: s3 v1–10; 11–20
See also BERLINER mathematische gesellschaft

ARCHIV for mathematik og naturvidenskab. Oslo
1+, 1876+
Index: 1–25

ARCHIVES néerlandaises des sciences exactes et naturelles. (Société hollandaise des sciences à Harlem) The Hague
1–30, 1866–97; s2 v1–15, 1897–1910||
Index: 1–10
Serie 3A. Sciences exactes
1–14, 1911–33||
See also PHYSICA; Sec. II, under HOLLANDSCHE maatschappij der wetenschappen

ARCHIVO de matemáticas puras y aplicadas. Valencia
1, 2^{1-8}, 1896–Ag1897||

ARKIV för matematik, astronomi och fysik. (Svenska vetenskapsakademien) Stockholm
1+, 1903+

ASSOCIATION for symbolic logic. *See* JOURNAL of symbolic logic

ASSOCIATION for the improvement of geometrical teaching. Reorganized 1892 as Mathematical association, London
Report
8, 13, 1882–87
See also Sec. II

ASSOCIATION of mathematical teachers in New England
Joint bulletin. Boston
no1–2, 1907–08||

ASSOCIATION of teachers of mathematics for the middle states and Maryland
Bulletin. Continued in Association of mathematical teachers in New England. Joint bulletin
no1–2, 1904–05
See MATHEMATICS teacher

ASSOCIAZIONE "Mathesis" fra gli insegnanti di matematica delle scuole medie. *See* "MATHESIS" società italiana di matematica

ASTRONOMICAL society of the Pacific, San Francisco
Publications
1–36, 37^1, 1889–1925

ASTRONOMIE. (Société astronomique de France) Paris
46–47, 49, 1932–35

ASTRONOMY and astrophysics. Northfield, Minn. Superseded by Astrophysical journal
1–13, 1892–94||

ASTROPHYSICAL journal; an international review of spectroscopy and astronomical physics. Chicago. Supersedes Astronomy and astrophysics
1+, 1895+

AUSTRALIAN and New Zealand association for the advancement of science, Sydney. Through 1926 as Australasian association for the advancement of science
Reports
15–20, 1921–30
Index: 16–20

BELL system technical journal. New York
1+, 1922+ (incl. suppl. Jl1936)
Index: 1–10; 11–20
BERLINER mathematische gesellschaft
Sitzungsberichte. 1–18 as suppl. to Archiv der mathematik und physik
1–14, 1901–15
BIBLIOTHECA mathematica. Stockholm, Leipzig
1–3, 1884–86; s2 v1–13, 1887–99; s3 v1–14, 1900–15||
Index: s2 v1–10
K. BOEHMISCHE gesellschaft der wissenschaften, Prague. *See* ČESKÁ společnost nauk
BOLETÍN matemático. Buenos Aires
1+, 1928+
Suplemento informativo
1, 2^{1-8}, 1933–34||
BOLETÍN matemático elemental. Buenos Aires. Suppl. to Boletín matemático
1–4, 5^{1-4}, 1930–34||
BOLLETTINO di bibliografia e di storia delle scienze matematiche e fisiche. Rome
1–20, 1868–87||
Index: 1–20
BOLLETTINO di bibliografia e storia delle scienze matematiche. Turin. Merged into Bollettino di matematica
1–21, 1898–1919||
BOLLETTINO di matematica. Rome, Bologna
1–17, 1902–21; ns v1–5, 6^{1-3}, 7–8, 9^{3-4}, 10–17, 1922–38; s3 v1, 1939
BRITISH association for the advancement of science, London
Reports. No meetings held 1917–18
29–31, 33, 35, 93–103, 1859–1935
See also Sec. II
BRNO
Masarykova universita
Přírodovědecká fakulta
Spisy
1+, 1921+
BRUSSELS
Observatoire royal de Belgique
Annales
Annales astronomiques
ns v6, 1887
Annuaire astronomique
1906–07
BRYN Mawr college, Bryn Mawr, Pa.
Monographs
Reprint series
8, 1909
BUCHAREST. École polytechnique. *See* BULLETIN de mathématiques et de physique, pures et appliquées
BUENOS Aires
Instituto nacional del profesorado secundario
Revista del centro estudiantes
31–32, 1922
Universidad nacional
Facultad de ciencias exactas, físicas y naturales
Publicaciónes
sB. Científicas y técnicas. no1–3 as the university's Seminario matemático. Boletín
8–13, 15+, 1931+
Instituto matemático hispano-americano
Seminario matemático. 1928–32 as the university's Instituto matemático hispano-americano. Seminario matemático argentino
Boletín
1–3, 1928–33 (incl. suppl., v1–2)
BULLETIN de mathématiques élémentaires. Paris. 3–13 as Bulletin de sciences mathématiques et physiques élémentaires. Merged with Journal de mathématiques élémentaires (Vuibert)
1–9, 12^{12}, 1895–1906

- BULLETIN de mathématiques et de physique, pures et appliquées. (Bucharest. École polytechnique)
  - 1+, 1929+
- BULLETIN des sciences mathématiques. (Paris. École pratique des hautes études. Bibliothèque) 1870–76 as Bulletin des sciences mathématiques et astronomiques. 1884+ the astronomical section is continued in Bulletin astronomique
  - 1–11, 1870–76; s2 v1–2, 26^1, 27–63, 1877–1939
  - Index: s2 v1–30
- BULLETIN des sciences mathématiques, astronomiques, physiques et chimiques. *See* BULLETIN universel des sciences et de l'industrie
- BULLETIN universel des sciences et de l'industrie. (Société pour la propagation des connaissances scientifiques et industrielles) Paris
  - Section 1. Bulletin des sciences mathématiques, astronomiques, physiques et chimiques
    - 1–16, 1824–31||
- BULLETTINO. *See as if* BOLLETTINO

- CAGLIARI
  - Università
    - Facoltà di scienze matematiche, fisiche e naturali
      - Seminario scientifico
        - Rendiconti
          - 1–9, 1931–39
- CAHIERS scientifiques. Paris
  - 1–19, 1927–38
- CALCUTTA
  - University
    - Dept. of science
      - Journal
        - 1–10, 1919–33
    - University studies
      - s2 v1, 1918||
- CALCUTTA mathematical society
  - Bulletin
    - 1+, 1909+
- CALIFORNIA
  - University
    - Publications in mathematics
      - v1–2, 1912–34; ns v1+, 1943+
    - Scripps institution of oceanography, La Jolla
      - Bulletin: Technical series
        - 2^1, 1929
    - *See also* LICK observatory
- CALIFORNIA academy of sciences, San Francisco
  - Proceedings
    - Mathematics-physics
      - s3 $v1^{1-4}$, 1898
- CAMBRIDGE and Dublin mathematical journal. Cambridge, England. Superseded by Quarterly journal of pure and applied mathematics
  - ns v1–9, 1846–54||
  - Index: 1–9
- CAMBRIDGE philosophical society, Cambridge, England
  - List of fellows
    - 1901 (see also v12, 14, 16, 20 of Proceedings)
  - Proceedings
    - 10+, 1899+
  - Transactions
    - 17–23, 1899–1928||
- CAMBRIDGE tracts in mathematics and mathematical physics. (Cambridge, England. University)
  - 1–4, 6–7, 22–23, 25, 37+, 1905+
- CARNEGIE institution of Washington
  - Publications
    - 9, 78, 86, 105, 120, 151, 161, 165, 245–256, 318, 1905–30
- CARUS mathematical monographs. (Mathematical association of America)
  - 1+, 1925+

ČASOPIS pro pěstování matematiky a fysiky. (Jednota československých matematiků a fysiků) Prague
1-52, 53^{1-2}, 55-67, 1872-1938
Index: 1-30

CERNĂUTI, Roumania
Universitatea
Facultatea de ştiinte
Buletinul
4-10, 1930-36

ČESKÁ akademie císaře Františka Josefa pro vědy, slovesnost a umění v Praze. *See* ČESKÁ akademie věd a umění v Praze

ČESKÁ akademie věd a umění v Praze. 1890-1920 as Česká akademie císaře Františka Josefa pro vědy, slovesnost a umění v Praze
Bulletin international
7-38, 1903-37
Sborník přírodovědecký
1-4, 6-7, 1925-30
Třída II: Mathematicko-přírodnická
Rozpravy
1-46, 1891-1936
Věstník
34^{1-9}, 1925

ČESKÁ společnost nauk, Prague. German name: Boehmische gesellschaft der wissenschaften; French name: Société des lettres et des sciences de Bohême
Jahresbericht
1899-1902, 1910-12, 1915-17||
Třída mathematicko-přírodovedecka
Věstník. 1885 as its Zpravy . . . 1885-1917 have added title page Sitzungsberichte; 1918+ Mémoires
1899-1902, 1905-06, 1908-37
Výročni zpráva. Through 1917 also issued in German as its Jahresbericht
1920-37
General index: 1884-1904; 1905-35

CHICAGO
University
Decennial publications
s1 v8^{1-7}, $9^{1-5,7,9,11-12}$, 1902-04

CHINESE mathematical society
Journal
1^{1-2}, 2^{1-2}, 1936-40

CHRISTIAAN Huygens; internationaal mathematisch tijdschrift. Groningen, Amsterdam
1+, 1921+

CIRCOLO matematico, Palermo
Annuario
1898, 1900, 1905-12, 1914, 1928
List of members
1924
Rendiconti
1-61, 1884-1937
Index: 1-55
Supplemento
1-8, 14, 16, 1906-30||
Index: 1-8

COAST artillery journal. (United States coast artillery school, Fort Monroe, Va.) 1-56 as Journal of the United States artillery
1-70, 1892-1929

COLORADO mathematical society
Biennial calendar
no2, 1907-09

COLUMBIA university, New York
Ernest Kempton Adams fund for physical research
Publications
1, 3-8, 10, 1906-27||

COMMENTARII mathematici helvetici. (Schweizerische mathematische gesellschaft) Zurich
1-13, 16+, 1929+

Crelle's journal für mathematik. *See* Journal für die reine und angewandte mathematik
Culver-Stockton quarterly. Canton, Mo.
2^1, 1926
Czernowitz. *See* Cernăuti

K. Danske videnskabernes selskab, Copenhagen. French name: Académie royale des sciences et des lettres de Danemark
Mathematisk-fysiske meddelelser
1–16, 1917–39
Oversigt over . . . forhandlinger. Supersedes its Bekiendtgiörelse
1900–31
Selskabs skrifter
1. Historisk og filosofisk afdeling
s7 $v3^1$, 1914
2. Naturvidenskabelig og mathematisk afdeling
s4 v[3, 5–9, 12], 1828–45; s5 v[1, 4, 6], $8^{1,4-5}$, $9^{3,7}$, $10^{4,6,8-9}$, 1849–75; s6 $v1^6$, 2^{11}, 3^1, 5^2, 7^1, $8^{3,6}$, 10^1, 1882–99; s7 $v1^6$, $2^{2-3,5}$, 5^1, 6^{7-9}, 8^6, 10^3, 11^{2-4}, 12^{1-2}, 1904–14; s8 $v1^{1,4-5}$, $2^{3,5}$, 4^4, 7^2, 10^3, $11^{1,3}$, 12^{2-3}, 1914–29
Also reprints 1821–23 by C. F. Degen and H. C. Orsted
K. Deutsche akademie der naturforscher zu Halle. To 1922 as Leopoldinisch-Carolinische deutsche akademie der naturforscher
Nova acta
90–93, 1909–10; ns v1, 1932/34
Deutsche mathematik. (Deutsche forschungs-gemeinschaft) Leipzig
1–4, 1936–39
Beihefte
1–2, 1939–40
Deutsche mathematiker-vereinigung, Leipzig
Jahresbericht
1–49, 1890–1940
Index: 1–10
Ergänzungsbände
2^1, 3, 4^1, 6, 1900–30||
See also Zentralblatt für mathematik und ihre grenzgebiete; Sec. II, under Gutzmer, A.
Duke mathematical journal. (Duke university) Durham, N. C.
1+, 1935+

Edinburgh
Royal observatory
Annals
1–2, 1902–06
University
Mathematical dept.
Research papers
1914^{1-7}, 1915^{1-11}
Edinburgh mathematical society
Proceedings
1, 11–44, 1883–1925/26; s2 v1+, 1927+
Index: 1–20; 21–44
See also Mathematical notes
Edinburgh royal society. *See* Royal society of Edinburgh
Education mathématique. Paris
5–16, 1902–14
Education outlook. London. 1847–1923 as Educational times
50–68, 1897–1915
Educational times. *See* Education outlook
Elisha Mitchell scientific society, Chapel Hill, N. C.
Journal
9–12, 1892–95
Encyclopédie des sciences mathématiques pures et appliquées. *See* Sec. II
Encyklopädie der mathematischen wissenschaften. *See* Sec. II
Enseignement mathématique. (Commission internationale de l'enseignement mathématique) Paris, Geneva
1–37, 1899–1938

FINNISCHE akademie der wissenschaften. *See* SUOMALAINEN tiedeakatemia
FIZIKO-matematicheskiĭ institut imeni V. A. Steklova. *See* under AKADEMI͡IA nauk S.S.S.R., Leningrad
FIZIKO-matematicheskoe obshchestvo, Kazan. *See* under KAZAN
FORMULARIO matematico. Turin. 1–4 as Formulaire de mathématique. Suppl. to Rivista di matematica. There were two distinct forms of "Editio V, Tome V de editione completo," one published in 1905–06, as "proba de 100 exemplere," the other in 1908. The first of these contains: fasciculo 1 (304 pp., 1905), fasciculo 2 (pp. 305–391+8 plates, 1906), and indice et vocabulario (xlvii pp., 1906). The second consists of: prefatione, tabula de symbolos, bibliographia (xxxv pp.), fasciculo 1 (272 pp.), fasciculo 2 (pp. 273–463). Pages 1–272 in both forms seem to be identical, but otherwise the differences are considerable
1–5, 1895–1908||
See also Sec. II, under PEANO, G.
FORTSCHRITTE der mathematik. *See* JAHRBUCH über die fortschritte der mathematik
FRANCE
Bureau des longitudes, Paris
Annuaire
1903–08, 1910–19, 1923
FUKUOKA, Japan. Imperial university of Kyushu. *See* KYUSHU imperial university
FUNDAMENTA mathematicae. Warsaw
1+, 1920+
Index: 1–25

GAZETA matematica. Bucharest
1–18, 19^{1-11}, 20–21, 25+, 1895+ (also Jubilee v1895–1935)
Index: 1–40
GENTLEMAN'S diary; or, Mathematical repository. London. United with Ladies' diary in 1841 to form Lady's and gentleman's diary
75–78, 1815–18
GENTLEMAN'S mathematical companion. London
1–5, 1798–1827||
GESELLSCHAFT der wissenschaften zu Göttingen
Mathematisch-physikalische klasse
Nachrichten
1894–1933||
Nachrichten. Continued in three series: Nachrichten. Geschäftliche mitteilungen (later Jahresbericht); Mathematisch-physikalische klasse. Nachrichten; Philologisch-historische klasse. Nachrichten
1892–93
Geschäftliche mitteilungen. *See* its Jahresbericht
Jahresbericht
1894–1936
GIESSEN
Universität
Mathematisches seminar
Mitteilungen
1–2, 1921–31
GIORNALE di matematica finanziaria. Turin. 13+ also as s2 v1+
1–8, $10^{1-3,6}$, 11–12, 1919–30; s2 v1–6, 1931–36
GIORNALE di matematiche. (Battaglini) Naples. 1–31 as Giornale di matematiche ad uso degli studenti delle università italiane
1–3, 1863–65
GLASGOW philosophical society. *See* ROYAL philosophical society of Glasgow
GOETTINGEN
Universität (Georg-Augustus-universität)
Mathematischer verein
Berichte
no65–79, 1901–08
Mathematisch-naturwissenschaftliche fakultät
Jahrbuch. 1920–22 pt1 as a section of the university's Philosophische fakultät. Jahrbuch
1920–24||
See also Sec. II, under GÖTTINGEN university

GORKI universitet, Nizhniĭ-Novgorod
Uchenye zapiski. English title: Scientific records of the Gorky state university
1–2, 1935–36
GRENOBLE
Université
Annales. Continued in 2 sections: Section lettres-droit and Section sciences-médicine
13–34, 1901–23‖
Index: 1–20; 21–34
Section sciences-médicine
Annales
ns v1–14, 1924–37

HAMBURG
Universität. 1936+ as Hansische universität
Mathematisches seminar
Abhandlungen
1–13, 1921–40
See also HAMBURGER mathematische einzelschriften
HAMBURGER mathematische einzelschriften. (Hamburg. Universität. Mathematisches seminar)
1–11, 13–20, 26, 1923–39
HAMBURGISCHE gesellschaft zur verbreitung mathematischer kenntnisse
Jahresbericht
1859
HANSISCHE universität. *See* HAMBURG
HAVANA
Observatorio nacional
Boletín oficial
Mensual
s3 v1–2, 1936–37
HELLĒNIKĒ mathematikē hetaireia, Athens. French title: Société mathématique de Grèce
Deltion
1–19, 1919–38
HELSINGFORS. Universitet. *See* Sec. II, under HELSINGFORS
HIROSHIMA, Japan
University
Journal of science
sA: Mathematics, physics, chemistry
1–10, 1930–40
See also Sec. II
HOCHSCHUL-nachrichten. Munich
13, 1902/03
HOKKAIDO imperial university, Sapporo, Japan. *See* SAPPORO

ILLINOIS
University
Dept. of mathematics
Bulletins
18^{12}, 20^{42}, 22^{35}, 25^{43}, 1920–28
IMPERIAL academy, Tokyo
Proceedings
2^{1-6}, 13, 15–16, 1926–40
INDIAN academy of sciences
Proceedings. Each no in two parts: A and B
1+, 1934+ (incl. suppl. 1934)
INDIAN mathematical club. *See* INDIAN mathematical society
INDIAN mathematical society, Madras. 1909–10 as Indian mathematical club
Journal. In 1934 divided into two publications: Journal ns and Mathematics student
1–20, 1909–33; ns v1+, 1934+
List of members
1919, 1922, 1924, 1926 (also 1909, 1911 in society's publication for those years)
Report of the . . . conference. 1–8 in Journal, 9+ in Mathematics student
1–10, 1916–38
See also MATHEMATICS student

INDIAN physico-mathematical journal. Dacca
2–3, 1931–32
INSTITUT catholique. *See* under name of city
INSTITUT de France. *See* ACADÉMIE des sciences, Paris
INSTITUT grand-ducal de Luxembourg. 1853–68 as Société des sciences naturelles du grand-duché de Luxembourg; 1870–1904 Institut royal grand-ducal de Luxembourg
Section des sciences naturelles, physiques et mathématiques
Archives
8–9, 15, 17–21, 23–27, 1865–1904; ns v1–3, 1906–08
INSTITUT matematiki. *See* under AKADEMI͡IA nauk U.R.S.R., Kiev
INSTITUT mathématique roumain. *See* ANNALES roumaines de mathématiques
INSTITUT royal grand-ducal de Luxembourg. *See* INSTITUT grand-ducal de Luxembourg
INSTITUTE of actuaries, London
Journal
35+, 1901+
Index: 41–55; 56–65
Yearbook
1929+
INSTITUTO matemático hispano-americano. *See* BUENOS Aires. Universidad nacional
INTERMÉDIAIRE des mathématiciens. Paris
1–27, 1894–1920; s2 v1–4, 1922–25||
INTERNATIONAL association for promoting the study of quaternions and allied systems of mathematics
Bulletin. Lancaster, Pa. Not published 1902, 1904, 1906–07, 1911
1900–13||
INTERNATIONAL catalogue of scientific literature. London
A. Mathematics
1, 1902
ISIS. International review devoted to the history of science and civilization. Brussels, Bern. Suspended Jl1914–Ag1919
6–11, 12^1, 1924–29
I. R. ISTITUTO del regno lombardo-veneto. Supersedes Istituto nazionale italiano. Superseded by R. Istituto lombardo di scienze e lettere
Annali
1–2, 1831–32
ISTITUTO fisico-matematico G. Ferraris. *See* RASSEGNA di matematica e fisica
R. ISTITUTO lombardo di scienze e lettere, Milan. Supersedes I. R. Istituto del regno lombardo-veneto
Memorie. s3 in two classes
Classe di scienze matematiche e naturali
s3 v10–14, 1900–37
Rendiconti. s3 in three parts: Parte generale e atti officiale; Classe di lettere e scienze morali e storiche; Classe di scienze matematiche e naturali
s2 v33–69, 1900–36; s3 v1–2, 1937–38
R. ISTITUTO veneto di scienze, lettere ed arti, Venice
Memorie
5, 1855

JAHRBUCH über die fortschritte der mathematik. Berlin. 1932–34 united with Revue semestrielle des publications mathématiques
1–56, 59^2, 60^1, 61^1, 1868–1935
JAPANESE journal of astronomy and geophysics. (National research council of Japan) Tokyo
3–8, 10–17, 1925–40
JAPANESE journal of engineering; abstracts. (National research council of Japan) Tokyo
1–12, 1921–35
JAPANESE journal of mathematics. (National research council of Japan) Tokyo
1–16, 1924–40
JAPANESE journal of physics. (National research council of Japan) Tokyo
1–13, 1922–40
Index: 1–7
JASSY, Roumania
Universitatea
Annales scientifiques. Continued in two parts

3^{2-4}, 4^{1}, 5^{2-3}, 6^{4}, 7^{4}, 8^{1-2}, 11–22, 1906–36||
1. Mathématiques, physique, chemie
23+, 1937+
JEDNOTA československých matematiků a fysiků v Praze. 1862–1912? as Jednota českých matematiků a fysiků v Praze
Bibliografické zpravy
1–4, 5^{1-5}, 6, 7^{5}, 1927–33
Sbornik
1–11, 13–16, 18, 1898–1926
Výroční zpráva
1870/71, 1905/06, 1925/26, 1928/29
See also AKTUARSKÉ vědy; CASOPIS pro pěstováni matematiky a fysiky; ROZHLEDY matematicko přírodovědecké; Sec. II, under KAVÁN, J.
JEDNOTA českých matematiků a fysiků. *See* JEDNOTA československých matematiků a fysiků v Praze
JERUSALEM
Universitas
Scripta universitatis atque bibliothecae hierosolymitanarum
Mathematica et physica
1, 1923||
JOHNS Hopkins university, Baltimore, Md.
Circulars [mathematical circulars only]
3–5, 7–8, 10–11, 13, 15, 17, 20–22, 25, 27–30, 32, 35, 39, 45, 53, 57, 62, 66, 78, 85, 95, 115, 118, 123, 134, 151, 160, 167, 174, 191, 208, 232, 242, 257, 277, 1880–1915
JORNAL de sciencias mathematicas e astronomicas. (Academia polytechnica do Porto) Coimbra. Superseded by the academy's Annaes scientificos, later Oporto. Universidade. Faculdade des sciencias. Anais
9–15, 1889–1905||
JORNAL de sciencias mathematicas, fysicas, e naturaes. (Academia das ciencias de Lisboa) Lisbon
1–8, 1868–82
JOURNAL de mathématiques élémentaires. (de Longchamps) Paris. 1877–79 as Mathématiques élémentaires; 1880–81 Journal de mathématiques élémentaires et spéciales. 1882–1901 issued in 2 parts: Journal de mathématiques élémentaires and Journal de mathématiques spéciales. 1877–96 also numbered in four series of five v each
1–23, 25, 1877–1900
JOURNAL de mathématiques élémentaires. (Vuibert) Paris
1–3, 8–21, 25–27, 37–40, 44–54, 1877–1930
JOURNAL de mathématiques pures et appliquées. Paris. Supersedes Annales de mathématiques pures et appliquées
s1 $v2^{9}$, 1837; s3 v6, 1880; s5 v1–6, 7 [pp. 375–382 missing], 8–10, 1895–1904; s6 v1–10, 1905–14; s7 v1–4, 1915–18; s9 v4+, 1925+
JOURNAL de mathématiques spéciales. Paris. *See* JOURNAL de mathématiques élémentaires (Longchamps)
JOURNAL des géomètres-experts et topographes français. (Société des géomètres de France) Noyon, Paris. 1893–1927 as Journal des géomètres-experts français; v28+ also as no1+
1–2, 9–27, $28^{1-5,7,9-12}$, 29^{1-2}, 1893–1921
JOURNAL für die reine und angewandte mathematik. Berlin
1–13, 20–182, 1826–1940
Index: Each tenth v with the exception of 141–160
JOURNAL of mathematics and physics. (Massachusetts institute of technology) Cambridge, Mass.
1+, 1921+
JOURNAL of physical chemistry. Ithaca, N. Y.
1–16, 1896–1912
JOURNAL of symbolic logic. (Association for symbolic logic)
1+, 1936+
JOURNAL of the United States artillery. *See* COAST artillery journal
JUGOSLAVENSKA akademija znanosti i umjetnosti, Zagreb
[Catalogue of publications]
1867–1911
Ljetopis
11–23, 28–29, 34, 1896–1919
Index: 1867–1903

Matematičko-prirodoslovni razred
Bulletin international. 1–23 as its Izvješća o raspravama . . .
1–30, 1914–36
Prirodoslovna istraživanja kraljevine Jugoslavije. 1–15 as its Prirodoslovna istraživanja hrvatske i slavonije. 1–14 issued by the academy's Matematičko-prirodoslovni razred. Suspended 1925–29
2–3, 6–7, 1914–15
Rad
49–54, 56, 58, 1911–15
JUNTA para ampliación de estudios é investigaciónes científicas. Madrid
Anales
$17^{4a,5a}$, 1918
Laboratorio y seminario matemático
Publicaciones
1–4, 1916–20/29; 1933, 1935, 1936, 1937, 1938

KANSAS university quarterly. Lawrence. v6–10 in two series: A. Science and mathematics; B. Philology and history. Superseded by Kansas university science bulletin
1^2, 4^2, sA v9–10, 1892–1901||
Index: 1–10
KANSAS university science bulletin. Lawrence. Supersedes Kansas university quarterly. Also numbered in the university's Bulletin (1+, also called v11 no1+)
1–15, 1902–25
KAZAN
Universitet
Fiziko-matematicheskoe obshchestvo
Izvi͡estii͡a
s2 v1–18, 19^{1-2}, 23–25, 1891–1925; s3 v1–11, 1926–38
See also Sec. II, under KAZAN universitet
KHARKOV
Universitet
Naukovo-doslidniĭ institut matematiki i mekhaniki. Sektor geometriĭ
1–2, 1938–40
Ucheni zapiski
10, 1937
See ZAPISKI . . .
KIEV
Universitet. Through 1916 as Universitet Sv. Vladimira
Naukovi zapiski
Fizichno-matematichniĭ zbirnik
1, 3, 1935–37
Universitetskii͡a izvi͡estii͡a
43–53, 54^{1-5}, 1903–14
KOENIGSBERGER gelehrte gesellschaft
Naturwissenschaftliche klasse
Schriften [mathematical memoirs only]
1^{5-7}, 2^3, 3^3, $4^{1,3,5-6}$, 5^4, 6^6, $8^{1,3}$, $10^{3,5}$, 1924–33
KÖZÉPISKOLAI matematikai és fizikai lapok. Budapest
1^{3-5}, 2–14, 1925–38
KYUSHU imperial university, Fukuoka, Japan
Faculty of science
Memoirs
sA. Mathematics
1, 1940–41

LADIES' diary. *See* LADY's and gentleman's diary
LADY's and gentleman's diary. London. 1–137 as Ladies' diary (title varies); in 1841 united with Gentleman's diary
112–132, 134–137, 144, 147–151, 168, 1815–71||
LA PLATA
Universidad nacional
Facultad de ciencias físico-matemáticas
Anuario
1935–37, 1938–39, 1941

Catálogo y reglamento de la biblioteca
1922
Contribución al estudio de las ciencias físicas y matemáticas
Serie matemática. Continues in part its Serie matemático-física. United with other series to form its Publicaciones. s2. Revista
1^{1-5}, 1935-37||
Serie matemático-física. 1^{1-4} as Serie física. Continued in 2 series: Serie matemática and Serie física
1-5, 1914-31||
Index: 1-5
Serie técnica. United with other series to form its Publicaciones. s2. Revista
1-5, 1915-38
Publicaciones. All publications of the university listed in a single numerical series. Beginning no120, 1939+ they are also organized into three series
s2. Revista. Formed by the union of the university's Facultad de ciencias físico-matemáticas. Contribución al estudio de las ciencias físcias y matemáticas. Serie física, Serie matemática, Serie tecnica
1+, 1939+
Reunion anual de caminos
1937-40
LATVIJAS universitate. *See* RIGA
LENINGRAD
Leningradskiĭ industrial'nyĭ institut. Through 1934 as Leningradskiĭ ėlektro-mekhanicheskiĭ institut
Sbornik
3, 1934
Universitet
Uchenye zapiski
Seriîa fizicheskikh nauk
1-3, 1935-37
LENINGRADSKOE fiziko-matematicheskoe obshchestvo
Zhurnal
1-2, 1926/27-29
LEOPOLDINISCH-Carolinische deutsche akademie der naturforscher. *See* K. DEUTSCHE akademie der naturforscher zu Halle
LICK observatory, Mount Hamilton, Cal.
Contributions
2-5, 1891-95||
Publications
1-8, 9^{1-3}, 10-11, 1887-1913
LIOUVILLE, Journal de. *See* JOURNAL de mathématiques pures et appliquées
LONDON
University
University college
Dept. of statistics. Continues in part the college's Dept. of applied statistics
Statistical research memoirs
1+, 1936+
LONDON mathematical society
Journal
1+, 1926+
List of members
1865-99; 1897, 1901, 1903, 1908, 1911, 1913-15, 1921, 1929, 1933, 1935, 1937
Proceedings
1-35, 1865-1903; s2 v1+, 1903+
Index: 1-30; 31-35; s2 v1-30
LUND, Sweden
Universitet
Acta universitatis lundensis. Lunds universitets årsskrift
Afdelning 2. Medicin samt matematiska och naturvetenskapliga ämnen [mathematical memoirs only]
ns $v1^{5}$, 13^{8}, 14^{13-20}, $16^{3,11}$, $17^{2,7}$, 20^{1}, 22^{2}, 23^{2-4}, $24^{1,9}$, $25^{1,6}$, 1906-29
Matematiska seminarium
Meddelanden
1+, 1933+

- Lyons
  - Université
    - Annales
      - 6–7, 20, 1892–96
      - ns 1. Sciences, médecine
        - 12, 15–16, 21, 25, 31, 1903–12

- Madras. University. *See* Vizianagaram
- Magyar tudományos akadémia, Budapest
  - Almanach
    - 1903–06
  - *See also* Matematikai és physikai lapok; Mathematische und naturwissenschaftliche berichte aus Ungarn
- Manchester literary and philosophical society, Manchester, England
  - Memoirs and proceedings
    - 25–27, 37–56, 58+, 1876+
- Marseilles. Université d'Aix-Marseille. *See* Aix-Marseille
- Masarykova universita v Brne. *See* Brno
- Massachusetts institute of technology
  - Abstracts of scientific and technical publications
    - no4–9, 1929–32||
  - Dept. of mathematics
    - Contributions
      - s2 no1–50, 52–196, 1920–31
  - *See also* Journal of mathematics and physics; Technology quarterly
- Matematica elementare. Rome
  - 1–3, 1922–24||
- Matematiche pure ed applicate. Città di Castello
  - 1–2, 1901–03||
- Matematicheskiĭ listok. Reval
  - 1–18, 1915–16
- Matematicheskiĭ sbornik. (Moskovskoe matematicheskoe obshchestvo) Moscow. ns v1+ also as 43+
  - 1–42, 1866–1935; ns v1+, 1936+
  - Index: 1–15
  - *See also* Sec. II, under Moskovskoe matematicheskoe obshchestvo
- Matematikai és physikai lapok. (Magyar tudományos akadémia, Budapest. Matematikai és természettudományi osztály) Budapest
  - 8–13, 14^{2-8}, 15–45, 1899–1938
- Matematikai és természettudományi értesítö. (Magyar tudományos akadémia, Budapest. Matematikai és természettudományi osztály) Budapest
  - 4, $5^{1-2,5-9}$, 6, 9^{2-9}, $10^{1-4,8,9}$, 11, $12^{1-5,7-12}$, 18, 19^{2-5}, 21^{3-5}, 31^{3-5}, 32, 33^{2}, 34^{2}, $35^{1,2,5}$, 36^{1-4}, 37–57, 1885–1938
- Matematikai és természettudományi közlemények, vonatkozólag a hazai viszonyokra. (Magyar tudományos akadémia, Budapest. Matematikai és természettudományi bizottság) Budapest
  - 2–5, $6^{1-4,6}$,7–13, 15–19, 28, $29^{1,3-4}$, 30–36, 1863–1928
- Matematisk tidsskrift. (Matematisk forening i København) Copenhagen. Supersedes Nyt tidsskrift for matematik
  - A. (Elementary)
    - 1919+
  - B. (Advanced)
    - 1919+
- Mathematica. (Cluj. Universitatea. Seminarul de matematici)
  - 1+, 1929+
  - *See also* Sec. II, under Klausenburg university
- Mathematicae notae. (Rosario. Universidad nacional del Litoral. Instituto de matemáticas)
  - 1+, 1941+
- Mathematical association, London. *See* Mathematical gazette
- Mathematical association of America
  - List of members
    - 1916, 1918 (further lists in American mathematical monthly)
  - *See also* American mathematical monthly; Carus mathematical monographs
- Mathematical expositions. Toronto
  - 1+, 1940+

MATHEMATICAL gazette. (Mathematical association) London. Supersedes Association for the improvement of geometrical teaching. Report
no1–6, 1894–95; (ns) v1+, 1896+
Index: 1–6; (ns) v1–15
See also Sec. II, under MATHEMATICAL association, London

MATHEMATICAL magazine. (Martin) Washington
1^5, 2^{1-12}, 1883–1904

MATHEMATICAL messenger. Ada, La.; Tyler, Tex.
7^3, 1891

MATHEMATICAL monthly. (Runkle) Cambridge, Mass.
1–3, 1858–61||

MATHEMATICAL notes: a review of elementary mathematics and science. (Edinburgh mathematical society) Edinburgh. Suspended 1917–23, 1926–28
1–28, 1909–33

MATHEMATICAL repository. (Dodson, Leybourn) London
1–3, 1748–55; (s3) v1–5, 1806–30

MATHEMATICAL review. (Story) Worcester, Mass.
1, 1896–99||

MATHEMATICAL reviews. (American mathematical society, etc.) New York
1+, 1940+

MATHEMATICAL surveys. (American mathematical society) New York
1+, 1943+

MATHEMATICAL visitor. (Martin) Erie, Pa.
1, 2^2, 1878–83

MATHEMATICS news letter. *See* NATIONAL mathematics magazine

MATHEMATICS student; a quarterly dedicated to the service of students and teachers of mathematics in India. (Indian mathematical society) Madras. Issued in place of the society's Journal pt2
1+, 1933+

MATHEMATICS teacher. (Association of teachers of mathematics for the Middle States and Maryland; National council of teachers of mathematics) Syracuse, N. Y.; Lancaster, Pa.; etc.
1+, 1908+
Index: 14–22

MATHEMATISCH-naturwissenschaftliche mitteilungen. (Mathematisch-naturwissenschaftlicher verein in Wurttemburg) Stuttgart. Suspended 1893–Mr1899
1–5, 1884–92; s2 v1–16, 20, 1899–1922||

MATHEMATISCHE annalen. Leipzig, Berlin
14–15, 31–39, 41–92, 95–103, 112–117, 1879–1941
Index: 51–80; 81–100

MATHEMATISCHE gesellschaft in Hamburg
Mitteilungen
2–7, 1890–1939

MATHEMATISCHE und naturwissenschaftliche berichte aus Ungarn. (Magyar tudományos akadémia, Budapest) Berlin, Budapest, Leipzig
1–31, 34–38, 1882–1931||

MATHEMATISCHE zeitschrift. Berlin
1+, 1918+
Index: 1–25

MATHESIS polska; czasopismo poświęcone naukom ścisłym i ich metodologji
3–8, 1928–33
See also MINERWA; URANJA

MATHESIS: recueil mathématique. Ghent, Paris. Also numbered in series of 10v each. Suspended 1916–20. Since 1921 this periodical has been the official organ of Société mathématique de Belgique
1–34, 36–51, 1881–1937 (with suppl.)

"MATHESIS" società italiana di matematica
Bollettino. 1–12 as Bollettino dell' Associazione "Mathesis." Merged 1920 into Periodico di matematica per l'insegnamento secondario, later Periodico di matematiche: storia-didattica-filosofia
1^1, 2^2, 3^3, 9^{1-6} [pp. 67–91 missing], 1896–1906

MESSENGER of mathematics. Cambridge, London. 1–5 as Oxford, Cambridge, and Dublin messenger of mathematics. United 1930 with Quarterly journal of pure and applied mathematics to form Quarterly journal of mathematics, Oxford series
1–5, 1861–71; ns v1–58, 1871–1929||
Index: 26–58

MICHIGAN
University
Publications in mathematics
1, 1944
Studies
Humanistic series
11, 1915
Scientific series
2, 1916
MILAN
Seminario matematico e fisico
Rendiconti
5-7, 12, 1931-38
MINERWA; czasopismo poświęcone popularyzacji nauk przyrodniczych i techniki. Warsaw. Suppl. to Mathesis polska
1, 1933
MINUVSHĬE gody. St. Petersburg
1-12, Ja-D1908||
MONATSHEFTE für mathematik und physik. (Vienna. Universität. Mathematisches seminar) Vienna
1-36, 38-48, 1890-1939
Index: 1-10; 11-20
MONIST, a quarterly magazine devoted to the philosophy of science. Chicago
1-7, 9-27, 28^{1-3}, 30-46, 1890-1936||
Index: 1-17
MONOGRAFJE matematyczne. (Warsaw. Uniwersytet. Seminarjum matematyczne) Warsaw
1-10, 1932-38
MORAVSKA přírodovědecka společnost, Brünn. Latin title: Societas scientiarum naturalium Moravicae
Prace
1-9, 1924-35
Výroční zpráva
1+, 1925+
MOSCOW
Universitet
Bulletin
Série internationale. Sec. A. Mathématiques et mécanique
1^{1-10}, 1937-38
Fiziko-mekhanicheskiĭ matematicheskiĭ fakul'tet. Name varies: Otdi͡el fiziko-matematicheskiĭ; Fiziko-matematicheskiĭ fakul'tet
Nauchno-issledovatel' skiĭ institut matematiki i mekhaniki
1926-27
Geometricheskii kruzhka
Trudy
1930
Seminar po teorii veroiatnostei metamaticheskoi statistike
Trudy
1, 1930
Seminar po vektornomu i tenzornomu analizu s ikh prilozhenii͡ami k geometrii, mekhanike i fizike
Trudy
1, 1933
Uchenyi͡a zapiski
1-11, 13-27, 29, 1880-1916||
Uchenye zapiski
2, 1934
MOSKOVSKOE matematicheskoe obshchestvo. *See* MATEMATICHESKIĬ sbornik
MÜEGYETEMI lapok; haví folyóirat a mathematika . . . Budapest
1-3, 1876-78||
MYSORE
University
Science association
Journal
2^{2-4}, 1925-26

- NATIONAL academy of sciences, Washington
  - Memoirs
    - 14^2, $17^{[1]}$, 1919–24
  - Proceedings
    - 1–23, 25+, 1915+
    - Index: 1–10
- NATIONAL council of teachers of mathematics. *See* MATHEMATICS teacher
- NATIONAL mathematics magazine. 1–8, 1926–S1934 as Mathematics news letter
  - 10+, 1935+
- NATIONAL physical laboratory, Teddington, England. *See* TEDDINGTON, England
- NATIONAL research council of Japan, Tokyo
  - Report. no1 covers the period Ap1922–Mr1930 and contains a reissue of most of the material in its Proceedings 1922
    - 1–10, 1930–31
  - *See also* titles beginning JAPANESE journal of . . .
- NATIONAL Tsing Hua university. *See* PEIPING
- NATURE. London
  - 127+, 1931+
- NATURFORSCHENDE gesellschaft, Zurich
  - Vierteljahrsschrift
    - 34+, 1889+
    - Index: 31–40; 41–50; 51–60; 61–70; 71–80
- NAUCHNO-issledovatel'skie kafedry v Odesse. *See* ODESSA
- NAUKOVE tovaristvo ĭmeni Schevchenka, Lemberg. German title: Ukrainische Ševčenko-gesellschaft der wissenschaften; Polish: Naukowe towarzystwo imenia Szevczenka
  - Chronik. German edition
    - 17–44, 49–59, 1904–14||
  - Matematichno prirodopisno-lĭkars'ka sektsii͡a
    - Zbĭrnik
      - 9–29, 1903–30
- NAUKOVI zapiski Naukovo-doslidchikh-matematichnikh katedr Ukraini. *See* ZAPISKI Kharkivs'kogo matematichnogo tovaristva ta Ukrains'kogo institutu matematichnikh nauk
- NEW York mathematical society. *See* AMERICAN mathematical society
- NIEUW archief voor wiskunde. (Wiskundig genootschap te Amsterdam) Amsterdam. Supersedes Archief
  - 1–20, 1875–93; s2 v1+, 1894+
- NIEUW tijdschrift voor wiskunde. Groningen
  - 1+, 1913/14+
- NIPPON sugako-buturigakkwai, Tokyo. 1884–1918 as Tokyo sugaku buturigakkwai
  - Proceedings. 1–9 as the society's Kiji; s2 v1–2 no6 Hokoku; s2 v2 no7–v3 Kiiji-gaiyo. English title: Tokyo mathematico-physical society, later Physico-mathematical society of Japan. s2 v1–3 carry no series number
    - 1–9, 1884–1901; s2 v1–9, 1901–18; s3 v1–22, 1919–40
- NIZHNIĬ-Novgorod. *See* GORKI
- NORSK matematisk forening, Oslo
  - Skrifter
    - s1 no1–18, 1921–28
  - *See also* NORSK matematisk tidsskrift
- NORSK matematisk tidsskrift. (Norsk matematisk forening) Oslo
  - 1–22, 1919–40
- NORSKE videnskaps-akademi, Oslo. 1857–1924 as Videnskabs-selskabet i Kristiania
  - Årbok. Supersedes its Oversigt over videnskabs-selskabets møder
    - 1925+
  - Fordhandlinger. Superseded by Avhandlinger of two classes
    - $1885^{15,18,21}$, $1886^{1,16-17}$, 1888^9, 1889^7, 1892^{17}, 1898–99, 1901–24||
    - Index: 1858–1924
  - Matematisk-naturvidenskapelig klasse
    - Avhandlinger
      - 1925+
    - Skrifter
      - 1898+
  - Oversigt over videnskabs-selskabets møder. 1868–86 issued with and indexed in its Forhandlinger. Superseded by its Årbok
    - 1898–1924||

NOTRE Dame mathematical lectures. (University of Notre Dame) Notre Dame, Ind.
1+, 1942+
See also REPORTS of a mathematical colloquium
NOUVELLE correspondance mathématique. Brussels
1–6, 1874–80||
NOUVELLES annales de mathématiques, journal des candidats aux écoles polytechniques et normales. Paris. 1855–61 includes Bulletin de bibliographie, d'histoire et de biographie mathématiques
1–20, 1842–61; s2 v1–18, 1862–79; s4 v3–20, 1903–20
NOVA acta Regiae societatis scientiarum upsaliensis. *See* K. VETENSKAPS-societeten i Upsala
NOVA Scotian institute of science, Halifax
Proceedings and transactions. 8–10 also as s2 v1–3
8–9, 14–16, 18^{1-2}, 1890–1933
NOVOROSSIĬSKOE obshchestvo estestvoispytateleĭ. Odessa. *See* ODESSKOE obshchestvo estestvoispytateleĭ
NUMERUS; revista de matematici elementare pentru invatamantul secundar, normal, profesional şi militar. Bucharest
2, 1936
NYT tidsskrift for matematik. Copenhagen. In two sections: A, Elementary; B, Advanced. Supersedes Tidsskrift for mathematik, and Maanedsskrift for den elementaere mathematik. Superseded by Matematisk tidsskrift
sA, B, v11–29, 1900–19||
Index: 1859–1908

OBSHCHESTVO estestvoispytateleĭ, Warsaw. *See* WARSAW. Universtet
ODESSA
Naukova-doslidchi katedry. Through 1924 as Nauchno-issledovatel'skie kafedry v Odesse
Zhurnal
$1^{1,4,8-9}$, 2^{3}, 1923–26
ODESSKOE obshchestvo estesvoispytateleĭ, Odessa. Through 1918 as Novorossiĭskoe obshchestvo estesvoispytateleĭ
Zapiski
20, 25–29, 34–36, 1902–11
Index: 1–30
OPEN court, a quarterly magazine. Chicago. 1887–1933 as Open court, a monthly magazine
11–31, 32^{1-7}, 34–50, 1897–1936||
Index: 1–20
OPORTO, Portugal
Universidade
Faculdade de sciencias
Anais. Supersedes Jornal de sciencias mathematicas e astronomicas. 1–14 as Academia polytechnica do Porto, Coimbra. Annaes scientificos. Suspended 1922–26
1–24, 1905–39
OPUSCOLI matematici e fisici di diversi autori. Milan
1–2, 1832–34||
OREGON
University
Publications
$2^{2,8}$, 1923–25
Mathematics series
1^{1}, 1929
OSAKA, Japan
Imperial university
Faculty of science
Collected papers
sA. Mathematics
1–7, 1933–39
OXFORD, Cambridge, and Dublin messenger of mathematics. *See* MESSENGER of mathematics

PADUA
Università
Seminario matematico
Rendiconti
1–10, 1930–39
PALERMO rendiconti. *See* CIRCOLO matematico
PARIS
École polytechnique
Journal. 1 as Journal polytechnique, ou Bulletin du travail fait à l'École centrale des travaux publics
1–64, 1795–1894; s2 v1–34, 1895–1935; s3 v1+, 1937+
Index: 1–64
See also Sec. II, under CAUCHY, A. L.
Institut catholique
Bulletin. For 1896–1910 *see* its Revue
s2 v1–10, 1910–19
Revue. Supersedes and is superseded by its Bulletin
6^4, 7–14, 15^1, 1901–10
PEIPING
National Tsing Hua university. To 1928 as Tsing Hua college. Chinese name: Kuo-li Tsing Hua ta-hsueh
Science reports
sA: Mathematical, physical and engineering sciences
1–3, 1931/32–35/36
National university of Peking
Science quarterly. Superseded by its Science reports
4,5^{1-4}, 1934–35||
Science reports
1, 1936
PEKING. *See* PEIPING
PENNSYLVANIA
University
Publications
Series in mathematics
no1–3, 1897–1905||
PERIODICO di matematiche: storia-didattica-filosofia. (Associazione "Mathesis") Rome, Leghorn, Bologna. Supersedes Periodico di scienze matematiche e naturali per l'insegnamento secondario. Title varies: Periodico di matematica . . . ; etc. s1–3 also as yr1–33
6–13, 1891–98; s2 v1–5, 1899–1903; s3 v1–14, 15^1, 1904–18; s4 v1, 2^{2-5}, 3^{2-5}, 4–19, 1921–39
Index: s4 v1–10
Supplement
2–20, 1898–1917||
PHILOSOPHICAL society of Glasgow. *See* ROYAL philosophical society of Glasgow
PHYSICA. (Hollandsche maatschappij der wetenschappen, Haarlem) The Hague. A continuation in part of Physica; Nederlandsch tijdschrift voor natuurkunde. Numbered as Archives néerlandaises des sciences exactes et naturelles. s4A
1–6, 1933–39
PHYSICAL review, a journal of experimental and theoretical physics. (American physical society; American institute of physics) New York
16–35, 1903–12; s2 v1+, 1913+
PHYSICO-mathematical society of Japan. *See* NIPPON sugaku-buturigakkwai
PHYSIKALISCH-medicinische societät, Erlangen
Sitzungsberichte
25–70, 1893–1938
PHYSIKALISCHE zeitschrift der sowjetunion. Kharkov
1^{1-2}, 1932
PINAR del Rio, Cuba
Escuela normal
Seminario de estudios superiores de matematicas
Publicaciones
1+, 1941+
PISA
Scuola normale superiore

Annali
Scienze fisiche e matematiche
1–16, 1871–1930; s2 v1–9, 1932–40
Index: 1–16
PITAGORA; giornale di matematica . . . Palermo
2–21, 1896–1915; s2 v1–2, 3^1, 1917–18||
POLSKA akademija umiejętności. *See* AKADEMIJA umiejętności, Krakow
POLSKIE towarzystwo matematyczne, Krakow
Roszprawy
1, 1921||
Rocznik. 1921 as Rozprawy
1–16, 1922–37
Dodatek
[1], 2–4, 1922–29
POLYTECHNIC engineer. (Engineering society of the college of arts and engineering of the Polytechnic institute of Brooklyn)
1901–05
PONTIFICIA accademia delle scienze, Rome. 1847-1901 as Accademia pontificia dei nuovi Lincei; 1902–22 Pontificia accademia romana dei nuovo Lincei; 1923–35 Pontificia accademia delle scienze, nuovi Lincei
Annuario
1929/30–30/31, 1933/34–34/35; ns v1, 1936/37
Atti. Superseded by its Acta
1, 48–88, 1847–1934/35||
Memorie
16–18, 1899–1901
Indice generale: 1913–27
See also SCIENTIARUM nuncius radiophonicus
POPULAR astronomy. A review of astronomy and allied sciences. (Carleton college, Northfield, Minn. Goodsell observatory; American association of variable star observers) Northfield, Minn.
1–5, 7+, 1899+
PORTO. *See* OPORTO
PRACE matematyczno-fizyczne. Warsaw
1–44, 1888–1937
Index: 1–10
K. PREUSSISCHE akademie der wissenschaften, Berlin. *See* AKADEMIE der wissenschaften, Berlin
K. PREUSSISCHE geodätische institut
Veröffentlichungen
8, 1886; ns v10–11, 52, 108, 1902–38
PRIRODOSLOVNA istraživanja hrvatske i slavonije. *See* JUGOSLAVENSKA akademija znanosti i umjetnosti, Zagreb. Prirodoslovna istraživanja kraljevine Jugoslavije
PRÓGRESO matemático. Saragossa
1–5, 1891–95; s2 v1–2, 1899–1900||
PROVINCIAAL Utrechtsch genootschap van kunsten en wetenschappen
Verslag van het verhandelde in de algemeene vergadering
1938–39/40

QUARTERLY journal of pure and applied mathematics. London. Supersedes Cambridge and Dublin mathematical journal. United 1930 with Messenger of mathematics to form Quarterly journal of mathematics, Oxford series
1–50, 1857–1927||
Index: 16–30; 31–45
QUARTERLY of applied mathematics. (Brown university) Providence
1+, 1943+

RADIO review; a monthly record of scientific progress in radiotelegraphy and telephony. London
1–2, 3^{1-3}, 1919–22||
RASSEGNA di matematica e fisica. (Istituto fisico-matematico G. Ferraris) Rome
2–6, $7^{1-6,9-?}$, 1921–27||
RENNES
Université
Travaux scientifiques
1–17, 1901/02–24|| (incl. 2 suppl. to v4)

Repertorium der literarischen arbeiten aus dem gebiete der reinen und angewandten mathematik. Leipzig
1–2, 1877–79||

Reports of a mathematical colloquium. Notre Dame, Ind. s1 as Ergebnisse eines mathematischen kolloquiums
s2 v1+, 1939+

Review of scientific instruments with physics news and views. (American institute of physics)
4+, 1933+

Revista de ciencias. (Lima. Escuela de ingenieros; Lima. Universidad mayor de San Marcos. Facultad de ciencias) Lima
1–16, 34+, 1897+

Revista de matematica. Turin. *See* Rivista di matematica

Revista de matemáticas y físicas elementales. Buenos Aires
1–5, 1919–24||

Revista matematica din Timişoara
1–2, $3^{1-9,11-12}$, 1921–24

Revista matemática hispano-americana. (Sociedad matemática española) Madrid. Supersedes Revista de matemáticas, Buenos Aires, and Revista socieded matemática española
1–2, $3^{1-4,6-10}$, 4-7, 1919–25; s2 v1–11, 1926–36; s4 v2+, 1942+

Revista trimestral de matemáticas. Saragossa. Suspended Ja–Ag1906
1–6, 1901–06||

Revue de mathématiques. *See* Rivista di matematica

Revue de mathématiques spéciales. Paris. 25–29 never published
1–20, 22–24, 30–40, 1890–1930

Revue des questions scientifiques. (Société scientifique de Bruxelles) Louvain. s4 v1–30 also as 81–110; s5 v1+ as 111+
s4 $v3^1$, 7–30, 1923–36; s5 v1+, 1937+

Revue des revues littéraires et scientifiques hongroises. (Association nationale des sociétés et des institutions scientifiques de Hongrie; Centre d'études hongroises en France) Paris. 1935–37 in 2pts: 1. . . . 2. Sciences, sciences appliquées. Suppl. to Revue des études hongroises, later Études hongroises
1936–37

Revue générale des sciences pures et appliquées. Paris
13+, 1902+
Index: 1–25

Revue scientifique. (Revue rose) Paris
77, 1939

Revue semestrielle des publications mathématiques. (Wiskundig genootschap te Amsterdam). In 1932 merged with and since 1934 absorbed by Jahrbuch über die fortschritte der mathematik
1–38, 39^{1-6}, 1893–1934||
Index: 1–5; 6–10; 11–15; 16–20; 21–25; 26–30; 31–36

Rice institute of liberal and technical learning, Houston, Tex.
Rice institute pamphlets
1^{1-2}, $4^{1,4}$, 5^4, $7^{2,4}$, $8^{1,3}$, 9^3, 14^4, 16^4, 21^4, 23^{1-2}, 25^4, 1915–38
See also Sec. II

Riga
Latvijas universitate
Matēmatikas un dabas zinātnu fakultāte
Raksti
1–2, 1929/31–32/36 (incl. 2 suppl. to v1)

Rio de Janeiro
Observatorio
Annuario
54–56, 1938–40
Taboas das marés
1938

Rivista di fisica, matematica e scienze naturali. (Società cattolica italiana per gli studi scientifici) Florence, Naples, Pisa, Pavia. Publication suspended 1913–01926
1–26, 1900–12; s2 v2–3, 1927–29

Rivista di matematica. Turin. v6–7 as Revue de mathématiques
1–8, 1891–1906||
Index: 1–5
Supplement. *See* Formulario matematico

Rome
Università
Seminario matematico
Rendiconti. 3 never published
1-2, 4-6, 1913/14-19/20; s2 v1-6, 1922/23-28/29; s4 v1, 1936/37
Rosario, Argentina
Universidad nacional del Litoral
Facultad de ciencias matemáticas, fisico-químicas y naturales aplicadas à la industria
Instituto de matemáticas
Publicaciones
1+, 1939+
See also Mathematicae notae
Royal Dublin society
Scientific proceedings
ns v1+, 1877+
Index: 1-8; 9-11
Scientific transactions
s2 v1-9, 1877-1909||
Index: 1-6; 7-9
See also Sec. II
Royal Irish academy, Dublin
Proceedings
sA v24-30, 32+, 1902+
General index: 1786-1906; 1907-32
Royal philosophical society of Glasgow. Also Philosophical society of Glasgow
Proceedings
30, 1898/99
Royal society of Edinburgh
Proceedings
3-4, 10-12, 15+, 1850+
Transactions
37+, 1891+
Royal society of London
Philosophical transactions
sA v192+, 1899+
Index: 230-239
Proceedings. 76+ in two series: A, containing papers of a mathematical or physical character; B, containing papers of a biological character
62-75, 1897-1905; sA v76+, 1905+
Index: 130-177
Yearbook
1-2, 1896/97-97/98
See also International catalogue of scientific literature; Sec. II
Royal society of Queensland
Proceedings
15-19, 21-22, 1900-10
Royal statistical society, London
Journal. 73+ also as ns v1+
103+, 1940+
Rozhledy matematicko přírodovědecké. (Jednota československých matematiků a fysiků) Prague. Appeared as pt of Časopis pro pěstování matematiky a fysiky v22-50; v30, nsv1+ as suppl. (přílohy)
ns v1$^{2-4}$, 2, 3$^{2-4}$, 4-10, 12-15, 1922-36
Runkle's magazine. *See* Mathematical monthly

K. Saechsische akademie der wissenschaften, Leipzig. *See* Akademie der wissenschaften, Leipzig
St. Gallische naturwissenschaftliche gesellschaft
Jahrbuch. 1858/60-1900/01 as its Bericht über die thätigkeit
1900/01-01/02
Sapporo, Japan
Hokkaido imperial university
Faculty of science

Journal
s1. Mathematics
1–8, 1930–40
SARAGOSSA
Universidad
Facultad de ciencias
Anales
1–2, 1907/08
SCHOOL mathematics. Chicago. Ap–O1903 as suppl. to School science
1^{1-3}, 1903; (ns) v1^{1-2}, 1904||
SCHOOL science. *See* SCHOOL science and mathematics
SCHOOL science and mathematics. (Central association of science and mathematics teachers) Chicago. 1–4 as School science
3+, 1903+
Supplement. *See* SCHOOL mathematics
SCHOOLDAY magazine, for all homes and schools. Philadelphia
1^{1}, 1871
SCHWEIZERISCHE mathematische gesellschaft. *See* COMMENTARII mathematici helvetici
SCIENCE. Cambridge, Mass.; New York; etc. 1901+ includes the official proceedings and most of the papers of the American association for the advancement of science
ns v1+, 1895+
SCIENTIARUM nuncius radiophonicus. (Pontificia accademia delle scienze, Rome) Vatican City
1–40, 1931–35
SCRIPPS institution of oceanography. *See* CALIFORNIA. University
SCRIPTA mathematica; a quarterly journal devoted to the philosophy, history, and expository treatment of mathematics. (Yeshiva College) New York
1+, 1932+
SCRIPTA mathematica library. New York
1+, 1934+
SEMINARIO matemático argentino. *See* BUENOS Aires. Universidad nacional. Instituto matemático hispano-americano
SENDAI, Japan
Tôhoku teikoku daigaku
Science reports
s1. Mathematics, physics, chemistry
1–29, 1912–41 (incl. 1936 suppl.)
Index: 1–10
ŠEVČENKO-gesellschaft der wissenschaften. *See* NAUKOVE tovaristvo imeni Shevchenka
SMITHSONIAN institution, Washington, D. C.
Annual report of the board of regents
1885/86–86/87, 1889/90, 1895/96–1903/04, 1905/06+
Smithsonian miscellaneous collections
51^{4}, 71^{1}, 74^{1}, 1910–22
See also Sec. II, under BECKER, G. F.
SOCIEDAD científica "Antonio Alzate." *See* ACADEMIA nacional de ciencias Antonio Alzate
SOCIEDAD científica argentina, Buenos Aires
Anales
55+, 1903+
See also Sec. II
SOCIEDAD cubana de ingenieros, Havana
Revista
24^{5-6}, 25–35, 1932–40
SOCIEDAD matemática española, Madrid
Revista. Continued in Revista matemática hispano-americana
1–6 no52, 1911–17|| (incl. suppl. 1–3, 1917)
SOCIETÀ cattolica italiana per gli studi scientifici. *See* RIVISTA di fisica, matematica e scienze naturali
SOCIETÀ delle XL. *See* SOCIETÀ italiana delle scienze
SOCIETÀ italiana delle scienze, Rome. More commonly known as Società delle XL [dei quaranta]
Annuario
1922, 1927
Memorie di matematica e di fisica
s3 v1–2, 12–24, 1867–1938

SOCIETÀ italiana di matematica. *See* "MATHESIS" società italiana di matematica
SOCIETÀ italiana per il progresso delle scienze, Rome
  Atti
    1, 1907
SOCIETAS scientiarum naturalium Moravicae. *See* MORAVSKA přírodovědecka společnost, Brünn
SOCIETATEA română de ştiinţe, Bucharest
  Secţia matematică
    Bulletin mathématique
      37–40, 1935–38
SOCIÉTÉ astronomique de France, Paris. *See* ASTRONOMIE
SOCIÉTÉ des lettres et des sciences de Bohême. *See* ČESKÁ společnost nauk
SOCIÉTÉ des sciences physiques et naturelles de Bordeaux
  Mémoires
    s4 v1–5, 1893–95; s5 v1–5, 1896–1901; s6 v1–4, 5^{2}, 1901–14; s7 $v1^{1}$, 2, 3^{1}, 5, 1917–36
  Procès verbaux. 1850/51–93/94 in its Mémoires
    1894/95–1920/21, 1927/28, 1932/33–34/35
  General index: 1850–1900
  See also Sec. II
SOCIÉTÉ hollandaise des sciences à Harlem. *See* ARCHIVES néerlandaises des sciences exactes et naturelles
SOCIÉTÉ mathématique de Belgique. *See* MATHESIS
SOCIÉTÉ mathématique de France, Paris
  Bulletin
    1–63, 1872–1935
  Comptes-rendus. Earlier yrs in its Bulletin. 1915–19, 1922–25 for members only
    1912–14, 1916–21, 1923–34
  See also Sec. II
SOCIÉTÉ mathématique de Grèce. *See* HELLĒNIKĒ mathematikē hetaireia
SOCIÉTÉ nationale des sciences naturelles et mathématiques de Cherbourg
  Mémoires. Also numbered in series of 10v each
    25–27, 29–38, 1885–1912
    Index: 21–30
SOCIÉTÉ philomathique de Paris
  Bulletin
    s9 v1–10, 1898–1908; s10 v1–11, 13^{2-3}, 14^{1-3}, 15^{1-2}, 16^{1}, 17^{1-2}, 18^{1-2}, 20, 1909–31; 118–119, 1935–36
SOCIÉTÉ physico-mathématique de Kasan. *See* KAZAN. Universitet. Fiziko-matematicheskoe obshchestvo
SOCIÉTÉ polonaise de mathématique, Cracow. *See* POLSKIE towarzystwo matematyczne
SOCIÉTÉ polytechnique de Varsovie. *See* WARSZAWSKIE towarzystwo politechniczne
SOCIÉTÉ pour la propagation des connaissances scientifiques et industrielles. *See* BULLETIN universel des sciences et de l'industrie
SOCIÉTÉ royale des sciences de Liége
  Bulletin
    1+, 1932+
  Mémoires
    s2 v19–20, 1897–98; s3 v1–20, 1899–1935; s4 v1+, 1936+
    Index: s2 v1–20
SOCIÉTÉ scientifique de Bruxelles
  Annales. Suspended 1914–19. Continued in series
    26–46, 1902–26||
    Index: 1–25; 26–46
    sA. Sciences mathématiques
      47–56, 1927–36||
    sB. Sciences physiques et naturelles
      47–56, 1927–36||
    s1. Sciences mathématiques et physiques. Formed by the union of its sA and part of sB
      57+, 1937+
  See also REVUE des questions scientifiques
SOFIA
  Universitet
    Fiziko-matematicheski fakul'tet

Godishnik
35, 1938/39
Fiziko-matematichesko druzhestvo
Spisanie
20–21, 1934–36
SPHINX-Oedipe. Nancy
1–11, 12^{1-8}, 14^{10-12}, 15^{1-10}, 18^{5-7}, 1906–23 (incl. suppls. to v15)
SRPSKA kral'evska akademija, Belgrade. French title: Académie royale des sciences de Serbie
Akademija prirodnih nauka
Bulletin de l'Académie des sciences mathématiques et naturelles
A. Sciences mathématiques et physiques
1–4, 1933–38
STATISTICA. (Warsaw. Szkola główna gospodarstwa weijskiego)
2, 6, 1932–38
STOCKHOLM
Astronomiska observatoriet
Annalen. 1–11 as Astronomiska iakttagelser och undersökningar på Stockholms observatorium
10^{4-12}, 11–13, 1916–41
STRASBOURG
Université
Institut de mathématiques
Publications. Continued as occasional numbers in Actualités scientifiques et industrielles
3–4, 6–10, 1922–32
STUDIA mathematica. Lemberg
1–8, 1929–39
SUOMALAINEN tiedeakatemia, Helsingfors. Latin name: Academia scientiarum fennica
Esitelmiä ja poytäkirjoja. Suspended 1915–19
1908–25
Sitzungsberichte. German edition of Esitelmiä . . .
1908–33
Toimituksia . . . Annales
sA v1–49, 51, 1909–39
General index: 1909–31
SUOMEN tiedeakatemia, Helsingfors. *See* SUOMALAINEN tiedeakatemia
SVENSKA vetenskapsakademien, Stockholm
Bihang til Handlingar
Afdelning I. Continued as Arkiv för matematik, astronomi och fysik
$8^{[20]}$, $9^{[4,6-7]}$, 12–28, 1884–1903‖
Handlingar [mathematical parts only]
ns (s4) 24^{5}, $55^{1-2,6}$, 57^{8-9}, $58^{1,3-4,8,10}$, $60^{1,8}$, $61^{2,15-16}$, $63^{1,4,12}$, 1891–1922; ns (s5) 1^{1-2}, $2^{2,5}$, $4^{2-3,5,7-8}$, $6^{2,4-6}$, 7^{1-2}, $9^{1,6}$, 13+, 1934+
Öfversegt af . . . förhandlingar
$22^{[8]}$, $26^{[2]}$, $28^{[6]}$, $29^{[8]}$, $30^{[10]}$, $31^{[5]}$, $33^{[3,8]}$, $34^{[4]}$, $39^{[8]}$, $40^{[2,5,9-10]}$, $41^{[1,5,9]}$, $42^{[1]}$, $43^{[1,4,10]}$, $44^{[3,5-6,9-10]}$, $45^{[2,5]}$, $46^{[3,5,7,10]}$, $47^{[3,5,7-9]}$, $48^{[3,8-10]}$, $49^{[3,5-6,10]}$, $50^{[1,7,9]}$, $51^{[4]}$, $52^{[2,4-5,7,9-10]}$, $53^{[1]}$, $54^{[3]}$, 1865–97
See also ARKIV för matematik, astronomi och fysik; STOCKHOLM, Astronomiska observatoriet
SZEGED
Egyetem
Acta universitatis szegediensis
Acta scientiarum mathematicarum. 1–9 as Acta litterarum ac scientiarum: Sectio scientiarum mathematicarum
1–8, 9^{1-4}, 1922–40

TAIHOKU, Formosa
University
Faculty of science and agriculture
Memoirs (mathematical series)
1–47, 1929–40
TASHKENT
Universitet. Full name: Sredne-aziatskiĭ gosudarstvennyĭ universitet. Title also as Turkestanskiĭ gosudarstvennyĭ universitet. French name: L'Université de l'Asie centrale
Bi͡ulleten'
10–16, 18–21, 1925–35

TAVRICHESKII universitet
Matematicheskai͡a laboratorii͡a
Protokoly (Proceedings of the mathematical laboratory of the Crimean university)
1–3, 1919–21||
TBILISSKIĬ . . . institut. *See* under AKADEMII͡A nauk S.S.S.R., Leningrad. Gruzinskiĭ filial
TECHNOLOGY quarterly and proceedings of the society of arts. (Massachusetts institute of technology) Boston. 1887–D1891 as Technology quarterly. 8–14 include Review of American chemical research, 1–7, Ap1895–D1901 (separately paged except in v8)
4–21, 1891–1908||
TEDDINGTON, England
National physical laboratory
Report. London
1915/16–23
TEMESZVAR
Şcoala politechnică
Bulletin scientifique . . . Comptes rendus des séances de la Société scientifique de Timişoara
4–5, 1931–34
TEXAS
University
Publications
[1891], 1915^4
Scientific series
no27, 1914
TIMIŞOARA. *See* TEMESZVAR
TÔHOKU imperial university, Sendai. *See* under SENDAI
TÔHOKU mathematical journal. Sendai
1–47, 1911–40
Index: 1–10; 11–20; 21–30
TOKYO
Imperial university
Faculty of science. Through 1886 as the university's Science dept.; 1887–1925 as its College of science
Journal. Continued in series
19^5, 32^{11}, 36^{1-2}, 37^5, $41^{4,9}$, 44^5, 1903–22
1: Mathematics, astronomy, physics, chemistry
$1^{1,3,6,9,11}$, $2^{1-3,5-7,9-10}$, 1925–34
University of literature and science
Science reports
Section A
3, 1935–40
TOKYO mathematico-physical society. *See* NIPPON sugako-buturigakkwai
TOMSK
Tomskiĭ industrial'nyĭ institut. Through 1924 as Tomskiĭ tekhnologicheskiĭ institut; 1925–32 Sibirskiĭ tekhnologischeskiĭ institut
Izvestii͡a
$46^{1,3}$, 1924–25
Universitet
Nauchno-issledovatel'skiĭ institut matematiki i mekhaniki
Izvestii͡a
1–2, 1935–38
TORONTO
University
Mathematical and physical society
Papers read . . .
1890/91–91/92
TOULOUSE
Université
Faculté des sciences
Annales
1–12, 1887–98; s2 v1–10, 1899–1908; s3 v1–28, 1909–36; s4 v1–2, 1937–38
TOWARZYSTWO naukowe warszawskie. French name: Société scientifique de Varsovie
Wydział III: Nauk matematyczno-fizycznych
Prace
33–34, 1930–33||

Sprawozdania z posiedzeń
24–30, 1931–37
TOWARZYSTWO przyjaciół nauk w Wilnie. French name: Société des sciences et des lettres de Vilno
Wydział 2: Matematycznych i przyrodniczych
Prace
12–13, 1938–39
TUCUMÁN, Argentina
Universidad
Revista
sA. Matemáticas y física teórica
1+, 1940+
TURIN
R. Politecnico. Name varies: R. Scuola di ingegneria; etc.
Conferenze di fisica e di matematica
1–4, 1929/30–33/34
Università
Osservatorio astronomico
Osservazioni meteorologiche
1900, 1908–09, 1914–15||
TURKESTANSKIĬ gosudarstvennyĭ universitet. *See* TASHKENT

UKRAÏN'SKA akademii͡a nauk. *See* AKADEMII͡A nauk U.R.S.R., Kiev
UNIÓN matemática argentina, Buenos Aires
Revista
8+, 1942+
UNIONE matematica italiana
Bollettino. Bologna
1–17, 1922–38; s2 v1–2, 1939–40
Index: 1–17
UNITED States bureau of education, Washington, D. C.
Bulletin
no6–9, 12–13, 16, 1911; no2, 4, 13–14, 1912; no45, 1914; no39, 1915; no27, 1917; no9, 1918; no58, 1919; no6, 32, 1921; no18, 1924
UNITED States coast and geodetic survey, Washington, D. C.
Special publications
38, 52–53, 57, 59–60, 67–68, 74–80, 84, 86, 91–93, 95, 99–101, 103–104, 106, 109–110, 112, 114, 120, 122, 129–130, 134, 137–138, 140, 145, 151, 153, 156, 158–161, 164, 169, 171–173, 179, 186, 189, 192–195, 198–200, 202–204, 207, 209–210, 212, 214, 217–227, 229, 1915–41
UNITED States navy department
Hydrographic office, Washington, D. C.
Publications
171, 1916
Naval observatory, Washington, D. C.
Nautical almanac office
Astronomical papers
1–8, 9^{2-3}, 10^{1}, 1882–1925
Ephemeris and nautical almanac
1917–41 (incl. suppl. 1900, 1917, 1918, 1925, 1932)
Publications
s2 v1–7, 9–14, 1900–38 (incl appendices $4^{1,3}$, 9, 10^{2}, 13^{1})
UNTERRICHTSBLATTER fur mathematik und naturwissenschaften (Verein zur förderung des mathematischen und naturwissenschaftlichen unterrichts) Berlin
1–33, 1895–1927
UPSALA
Universitet
Årsskrift
1867–68, 1869^{2}, 1872^{2}, 1874^{3}, 1875^{5}, 1876^{2}, 1879^{1}, 1901–11, 1923^{3}
See also Sec. II, under SÖDERBLOM, A.
URANJA. (Polskie towarzystwo przyjaciół astronomji) Warsaw. Suppl. to Mathesis polska
10–11, 1931–32

VEREIN für naturwissenschaft und mathematik, Ulm. 1888–1912/15 as Verein für mathematik und naturwissenschaften

Mitteilungen
9–14, 1899–1909
K. VETENSKAPS-societeten i Upsala
Nova acta Regiae societatis scientiarum upsaliensis
s3 v7^7, 11^1, $12^{4,7,12}$, 13^7, $14^{4,13}$, $15^{5,10}$, 16^4, 17^3, 1870–95; s4 v$1^{3-4,6,8}$, 1905–07
VIDENSKABS-selskabet i Kristiania. *See* NORSKE videnskaps-akademi, Oslo
VI͡ESTNIK opytnoi fiziki i elementarnoi matematiki. Kiev
24–25, 26^{1-4}, 27^{12}, 28^1, 30^5, 33^9, 36^{10}, 39^{9-10}, 1901–08
VIZIANAGARAM, India
Maharajah's college (affiliated with the University at Madras)
Science association
Journal
2^{2-4}, 1925–26
VLADIVOSTOK
Gosudarstvennyĭ dal'nevostochnyĭ universitet
Trudy. 1925/26–30 issued in series
s7. Fizika, matematika
1–2, 1926
VORONEZH
Universitet
Trudy . . . Acta universitatis voronegiensis (olim jurievensis-dorpatensis). Text in Russian with résumés in German
1–3, 5, 8–10, 1925–39
VRIEND der wiskunde. Culemborg, Schoonhoven, Haarlem
1–31, 1886–1916||
Supplement
1–5, 25, 1887–1913||
VSEUKRAÏN'SKA akademii͡a nauk, Kiev. *See* AKADEMII͡A nauk U.R.S.R., Kiev

WARSAW
Universitet
Koło matematyczno-fizyczne
Wydawnictwa
19, 21, 26, 1929–33
Obshchestvo estestvoispytateleĭ
Trudy
1–3, 1891–95||
See also MONOGRAFJE matematyczne
WARSZAWSKIE towarzystwo politechniczne. French name: Société polytechnique de Varsovie
Sprawozdania i prace. 1–13 also as v1–6^2
1–16, 1921–33
WASHINGTON (State)
University
Publications in mathematics. v1^1 as its Publications in mathematics and physical science
1^{1-3}, 3^1, 1915–30
WESTERN Australian astronomical society
Proceedings
1, 1913/14||
WIADOMOŚCI matematyczne. Warsaw
1–23, 1897–1919
WISKUNDIG genootschap onder de zinspreuk: Een onvermoeide arbeid komt alles teboven, Amsterdam
Verslag
1927
See also REVUE semestrielle des publications mathématiques; WISKUNDIGE opgaven met de oplossingen
WISKUNDIG tijdschrift. Haarlem
1–3, 7–10, 12^1, 15–16, 1904–20
WISKUNDIGE opgaven met de oplossingen. (Genootschap der mathematische wetenschappen; Wiskundig genootschap te Amsterdam) Amsterdam, Groningen. 1–4 as Wiskunstige opgaven met hare ontbindingen

4, 1870–74; s2 v1–16, 1875–1937
Index: 1–10

ZAPISKI Kharkivs'kogo matematichnogo tovaristva ta Ukrains'kogo institutu matematichnikh nauk. Kharkov. 1–s2 v16 as Khar'kov. Universitet. Khar'kovskoe matematicheskoe obshchestvo. Soobshchenii͡a (s1 as its Soobshchenii͡a i protokoly); s3 v1 Uchenye zapiski Nauchno-issledovatel'skikh kafedr Ukraini. Otdel matematicheskiĭ; s3 v2–3 Naukovi zapiski Naukovo-doslidchikh-matematichnikh katedr Ukraini
1–9, 1879–87; s2 v1–15, 16^{1-2}, 1888–1918; s3 v1–3, 1924–28; s4 1–18, 1927–40

ZEITSCHRIFT für angewandte mathematik und mechanik. Berlin
1–20, 1921–40

ZEITSCHRIFT für mathematik und physik. Leipzig
3–13, 18–21, 30, 32–64, 1858–1917||
Supplement. *See* ABHANDLUNGEN zur geschichte der mathematischen wissenschaften . . .

ZEITSCHRIFT für mathematischen und naturwissenschaftlichen unterricht. (Verein zur forderung des unterrichts in der mathematik und den naturwissenschaften) Leipzig. Subtitle varies. Includes the society's Mitteilungen, 1892–1909
13–31, 1882–1900

ZEITSCHRIFT für vermessungswesen. (Deutscher verein für vermessungswesen) Stuttgart
31–44, 45^{1-2}, 49–68, 1902–39

ZENTRALBLATT fur geophysik, meteorologie und geodäsie. Berlin
1–4, 1937–40

ZENTRALBLATT für mathematik und ihre grenzgebiete. (Deutsche mathematiker-vereinigung) Berlin
1–20, 1931–39
Index: 1–5

ZENTRALBLATT für mechanik. Göttingen. Formerly issued as a section of Zentralblatt für mathematik und ihre grenzgebiete
1–8, 1933–39
Index: 1–5

SECTION II

NONSERIAL PUBLICATIONS

Aachen. *See* Aix-la-Chapelle.

Abason, E. Contributions à l'étude du coefficient de déformation des fonctions non sinusoïdales; applications d'une nouvelle méthode pour la décomposition d'une fonction périodique dans ses harmoniques. Bucarest, 1927.

Abbe, C. The mechanics of the earth's atmosphere. 3d collection. Washington, 1910. (Smithsonian miscellaneous collection, v51^4)

Abbe, E. Ueber die gesetzmässigkeit in vertheilung der fehler bei beobachtungsreiche. Habilitationsschrift, Jena, 1863.

Abbott, J. C. The projective theory of non-Euclidean geometry. Diss., Notre Dame, 1941.

Abel, J. J. Chemistry in relation to biology and medicine with especial reference to insulin and other hormones. Baltimore, 1939.

Abel, N. H. Oeuvres complètes. 2v, Christiania, 1881.

Niels Henrik Abel. Mémorial publié à l'occasion du centenaire de sa naissance. Christiania, 1902. [Apart from correspondence, etc., of Abel, the volume contains contributions by B. Bjørnson, E. Holst, C. Størmer, and L. Sylow.]

Abenbeder. *See* Ibn Badr.

Abernathy, J. R. On the application of divided differences to approximation. Diss., Michigan, 1932. fc.

Abraham, M. Die electrischen schwingungen um einen stabförmigen leiter, behandelt nach der Maxwell'schen theorie. Diss., Berlin, 1897.

Academia nacional de ciencias Antonio Alzate, Mexico. Réponses au questionnaire d'histoire naturelle systématique, distribué par la Société scientifique "Antonio Alzate." Mexico, 1896.

Académie royale des sciences, des lettres et des beaux-arts de Belgique. Aperçu historique et travaux (1772–1904). Brussels, 1904.

Catalogue onomastique des accroissements de la bibliothèque, sciences 1883–1914. By F. Leuridant and J. Perrée. Brussels, 1919.

Fondations académiques. Brussels, 1919.

L'académie royale de Belgique depuis sa fondation (1772–1922). Brussels, 1922.

See also Le Nain, L.; Sec. I.

R. Accademia delle scienze di Torino. Onoranze centenarie internazionali ad Amedeo Avogadro. Torino, 1911.

See also Sec. I.

R. Accademia di scienze, lettere ed arti in Modena. Cenno storico. Modena, 1912.

See also Sec. I.

Achelis, E. The world calendar. 2d ed, New York, n. d.

Ackermann, R. Böschungsstrahlen und böschungsflächen. Diss., Halle, 1913.

Actuarial society of America, and Association of life insurance medical directors. Medico-actuarial mortality investigations. New York, 1912.

Joint occupation study, 1928. New York, 1929.

Occupational mortality ratings. New York, 1929

Medical impairment study, 1929. New York, 1931. Suppl., 1932.

Impairment study, 1936. New York, 1936.

Occupational study, 1937. New York, 1938.

Impairment study, 1938. New York, 1939.

Blood pressure study, 1939. New York, 1940. Suppl., 1941.

See also Sec. I.

Adam, P. E. Sur les systèmes triples orthogonaux. Diss., Paris, 1887.

Adam, W. Methodische anweisung zum ausziehen der quadrat- und kubikwurzel. Wittstock, 1869.

Adams, D. Arithmetic, in which the principles of operating by numbers are analytically explained and synthetically applied. Keene, New Hampshire, 1827.

Adams, E. P. The quantum theory. 2d ed., Washington, 1923. (National research council, Bulletin, no 39)

Adams, E. P., Greenhill, G., and Hippisley, R. L. Smithsonian mathematial formulae and tables of elliptic functions. Washington, 1922. (Smithsonian miscellaneous collection, v74^1)

Adams, J. L. The milky way. Sydney, 1905.
    The infinity of the starry universe. Sydney, 1906.

Adams, O. S. Application of the theory of least squares to the adjustment of triangulation. 1915. (U.S. coast and geodetic survey, Special publ. no 28)
    General theory of the Lambert conformal conic projection. 1918. (U.S. coast and geodetic survey, Special publ. no 53)
    Lambert projection tables for the United States coast and geodetic survey. 1918. (U.S. coast and geodetic survey, Special publ no 52)
    A study of map projections in general. 1919. (U.S. coast and geodetic survey, Special publ. no 60)
    General theory of polyconic projections. 1919. (U S. coast and geodetic survey, Special publ. no 57)
    Latitude developments connected with geodesy and cartography. 1921. (U.S. coast and geodetic survey, Special publ. no 67)
    Radio-compass bearings. 1921. (U.S. coast and geodetic survey, Special publ. no 75)
    Elliptic functions applied to conformal world maps. 1925. (U.S. coast and geodetic survey, Special publ. no 112)
    Tables for Albers projection. 1927. (U.S. coast and geodetic survey, Special publ. no 130)
    Conformal projection of the sphere within a square. 1929. (U.S. coast and geodetic survey, Special publ. no 153)
    The Bowie method of triangulation adjustment as applied to the first-order net in the western part of the United States 1930. (U.S. coast and geodetic survey, Special publ. no 159)
    Triangulation in Colorado [1927 datum]. 1930. (U.S. coast and geodetic survey, Special publ. no 160)
    First and second order triangulation and traverse in North Carolina (1927 datum). 1935. (U S. coast and geodetic survey, Special publ. no 192)
    Azimuths from plane coordinates. 1936. (U.S coast and geodetic survey, Serial no 584)
    First and second order triangulation and traverse in North Carolina (1927 datum). v2, 1940. (U.S. coast and geodetic survey, Special publ. no 218)
    See also Bowie, W ; Deetz, C. H.

Adams, O. S., and Claire, C. N. Manual of plane-coordinate computation. 1935. (U.S. coast and geodetic survey, Special publ. no 193)
    Manual of traverse computation on the Lambert grid. 1935. (U.S. coast and geodetic survey, Special publ. no 194)
    Manual of traverse computation on the transverse mercator grid. 1935. (U.S. coast and geodetic survey, Special publ. no 195)

Ader, O. B. Concerning an affine invariant of convex regions. Diss., Kentucky, 1937. fc.

Adhémar, R. d'. Sur une classe d'équations aux dérivées partielles du second ordre, du type hyperbolique, à trois ou quatre variables indépendantes. Diss., Paris, 1904.
    See also Borel, E. F. E. J.

Adkisson, V. W. Cyclicly connected continous curves whose complementary domain boundaries are homeomorphic, preserving branch points. Diss., Pennsylvania, 1930.

Adrian, T. Ueber projektivitäts- und dualitats-beziehungen im gebiete mehrfach unendlicher kegelschnittschaaren. Diss., Berlin, 1882.

Agard, H. L. The extension of some theorems in the theory of sets of points to *m*-dimensional space. Diss., Yale, 1911. fc.

Ahl, F. Untersuchungen uber geodatische linien. Diss., Kiel, 1901.

Ahrendt, A. Untersuchungen über die parallelflachen der flachen zweiten grades. Diss., Rostock, 1888.

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AMERICAN MATHEMATICAL SOCIETY

CATALOGUE OF THE LIBRARY

EDITED BY
ARNOLD DRESDEN
LIBRARIAN

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1946

PREFACE

When the last catalogue of the library of the American Mathematical Society was published in 1932, the Society had nearly 1,800 members, the number of books in the library was approximately 7,500. At the present time the library contains about 11,600 volumes and the members of the Society number 3,000. Many members who have joined the Society since 1932 may never have seen its catalogue; some may not be aware of the existence of the Society's library; of members of some years standing but very few have any means of knowing about the additions to the library's holdings. It is after these considerations that the Council of the Society recommended and the Board of Trustees authorized the publication of a new edition of the catalogue.

The Society's library is housed in the Low Memorial Library Building of Columbia University. Books may be consulted there. Loans to members outside New York City should be arranged through the library regularly used by the member. This library may request an interlibrary loan by addressing a letter to the American Mathematical Society, 531 West 116th Street, New York 27, New York.

Mention should be made of the fact that in recent years the library has acquired an almost complete collection of American doctoral dissertations in mathematics.

The abbreviations used in the present catalogue are explained on page v and should be consulted for its convenient use. A few general remarks follow.

1. The catalogue is divided into two parts. Section I contains serial publications; Section II all others. In cases in which listing in either section could be justified, the choice was determined by judgment as to the greatest convenience to the user of the catalogue. The precedent set by the Union List of Serial Publications was followed in most, but not in all, cases.
2. Bibliographical information is not generally included because of lack of space and lack of facilities for acquiring it in anything like completeness.
3. The library has greatly extended its holdings of dissertations. These are listed under the name of the author, with indication of the name of the degree conferring institution, and the year of award of the degree.
4. Well known series of books are listed in Section I under the general title of the series. Individual volumes are listed under the author's name in Section II.
5. Maiden names of women are used if their publications appeared under that name.
6. References from Section I to Section II, or vice-versa, without further amplification refer to an entry under the same principal name.
7. In many cases it is uncertain at the moment of going to press whether a journal whose publication was interrupted during the war years will be resumed.

The great amount of detailed work necessary for the publication of this catalogue was done by Mrs. A. K. Farwell, who was employed by the Society for this purpose, and by the staff of the New York office of the Society. Grateful acknowledgment is made of the valuable assistance received from members of the Columbia University Library Staff.

Arnold Dresden
Librarian

May 1946

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Edward Lincoln Stabler	1895
Pomeroy Ladue	1896–1901
David Eugene Smith	1902–1920
Raymond Clare Archibald	1921–1941
Arnold Dresden	1942–

STATISTICS

	Dissertations on film	Number of bound volumes
1902		121
1907		2,549
1912		3,871
1921		5,862
1925		6,798
1931		7,480
1946	528	11,632

EXPLANATION AND ABBREVIATIONS

The following general rules and abbreviations were used in preparation of Section I of this catalogue.

The journals, transactions, proceedings, etc., of a society are entered under the first word, not an article, of the latest form of the name of the society.

Learned societies and academies, other than English, with names beginning with an adjective denoting royal privilege are entered under the first word following the adjective. These adjectives, Kaiserlich, Koniglich, Reale, Imperiale, etc., are abbreviated to K., R., I., etc., and are disregarded in the arrangement.

A serial published by a society, but having a distinctive title, is entered under the title with reference from the name of the society.

A serial not published by a society or public office is entered under the first word, not an article, of the latest form of the title.

Colleges and universities having a geographical designation are entered under the name of the city, state, or country contained in the title.

Volume numbers and dates are inclusive in all cases and the dates are for the period covered by the serial, not those of publication.

A dash between volume numbers or dates signifies "from and including the former to and including the latter."

A plus sign (see p. 1, line 10) indicates that the serial is currently received and that the set is complete from the last date or volume number given.

Superscripts designate numbers or parts of volumes.

Parallels indicate that publication ceased with the preceding date or volume (see p. 1, line 18).

Brackets (see p. 4, line 6) indicate that the volumes or years so enclosed are not complete.

volume	v	publication	publ.
number	no	translated	trans.
series	s	edited	ed.
new series	ns	January	Ja
edition	ed.	February	F
year	yr	March	Mr
second	2d	April	Ap
revised	rev.	May	My
enlarged	enl.	June	Je
fascicule	fasc.	July	Jl
section	sec.	August	Ag
film copy	fc	September	S
dissertation	diss.	October	O
no date	n.d.	November	N
no place	n.p.	December	D

SECTION I

SERIAL PUBLICATIONS

- ABHANDLUNGEN über den mathematischen unterricht in Deutschland. *See* Sec. II, under INTERNATIONAL commission on the teaching of mathematics
- ABHANDLUNGEN zur geschichte der mathematischen wissenschaften mit einschluss ihrer anwendungen. Leipzig. 1–10 issued as suppl. to Zeitschrift für mathematik und physik
  - 1, 3, 6–7, 9, 12–13, 16^1, 17, 19–20, 22, 24–25, 28, 1877–1910
- ACADEMIA colombiana de ciencias exactas, físicas y naturales, Bogota. Supersedes Sociedad colombiana de ciencias naturales
  - Revista
    - 2+, 1938+
- ACADEMIA das ciencias de Lisboa. *See* JORNAL de sciencias mathematicas, fysicas, e naturaes
- R. ACADEMIA de ciencias exactas, físicas, físico-quimicas et naturales, Madrid. Name varies slightly
  - Discursos leídos
    - 1922
  - Memorias
    - 14, 20, 22–30, 32, 1890–1929; s2 v1–2, 4–9, 1921–29||
    - Serie de ciencias exactas
      - 1, 1930
  - Revista. 16–29 also as s2 v1–14
    - 1–32, 1904–35
- ACADEMIA nacional de ciencias Antonio Alzate, Mexico. 1884–1929 as Sociedad científica "Antonio Alzate"
  - Memorias. Individual issues 4+ as Memorias y revista
    - 2–54, 1888–1934
    - Index: 1–27; 1–52
- ACADEMIA polytechnica do Porto. *See* OPORTO
- ACADEMIA scientiarum fennica. *See* SUOMALAINEN tiedeakatemia
- ACADÉMIE des sciences, Cracow. *See* AKADEMIJA umiejętności
- ACADÉMIE des sciences, Paris
  - Comptes rendus hebdomadaires des séances
    - 1–65, 100+, 1835+
    - Index: 122–151
- ACADÉMIE des sciences d'Ukraine, Kief. *See* AKADEMII͡A nauk U.R.S.R., Kiev
- ACADÉMIE des sciences de l'empereur François Joseph I, Prague. *See* ČESKÁ akademie věd a umění v Praze
- ACADÉMIE des sciences de l'Union des républiques soviétiques socialistes. *See* AKADEMII͡A nauk S.S.S.R., Leningrad
- ACADÉMIE des sciences et lettres de Montpellier
  - Bulletin. 1909–17 as Bulletin mensuel. Numbering irregular
    - 1911–19, 1921, 1925–37
    - Section des sciences
      - Mémoires
        - s2 v1–3, 1893–1907
- ACADÉMIE des sciences, inscriptions et belles-lettres de Toulouse
  - Bulletin. Supersedes and is superseded by its Mémoires
    - 3, 1899/1900||
  - Mémoires
    - s10 v1–11, 1901–11
- ACADÉMIE impériale des sciences, St. Petersburg. *See* AKADEMII͡A nauk S.S.S.R., Leningrad
- ACADÉMIE royale des sciences de Serbie, Belgrade. *See* SRPSKA kral'evska akademija
- ACADÉMIE royale des sciences, des lettres et des beaux-arts de Belgique, Brussels
  - Annuaire
    - 1–2, 5, 7–11, 14–15, 18–31, 68–85, 87+, 1835+

Bulletins
s3 v17–21, 23–30, 34, 36, 1889–98||
Classe des sciences
Bulletins. Suspended 1915–18
1899–1900, 1902–14, 1919+ (incl. Annexe, 1915)
Index: 1899–1910; 1911–14
Mémoires. Collection in 4°
s2 v1+, 1907+
Mémoires. Collection in 8°
s2 v1+, 1904+
Mémoires couronnés et autres mémoires. Collection in 8°
60–66, 1900–04||
Index: 1898–1914. This also indexes the other four Mémoires of this academy
Mémoires couronnés et mémoires des savants étrangers. Collection in 4°
58–62, 1899–1904||
Mémoires de l'Académie. 1789–1845 as Nouveaux mémoires
s2 v10, 17, 19, 50–54, 1837–1904||
See also Sec. II

Académie tchèque des sciences, Prague. *See* Česká akademie véd a umění v Praze

Academy of science of St. Louis
Officers, Charter, Members, etc.
1893
Transactions
5–29, 1886–1938
Index: 1–10; 11–20

Accademia d'Italia, Rome
Annuario
1–9, 1929–37
Classe di scienze fisiche, matematiche e naturali
Memorie
1–5, 1930–34

R. Accademia dei Lincei, Rome. *See* R. Accademia nazionale dei Lincei

R. Accademia delle scienze dell' istituto di Bologna
Classe di scienze fisiche
Memorie. To 1907 see the academy's Memorie
s6 v5–10, 1908/09–12/13; s7 v1–10, 1913/14–22/23; s8 v1–10, 1923/24–32/33 (incl. suppl. 1910)
Rendiconto. To 1907 see the academy's Rendiconto ns v12–37, 1907–33
Memorie. Superseded by Classe di scienze fisiche. Memorie
s5 v8–10, 1899–1904; s6 v1–4, 1904–07||
Rendiconto. Superseded by Classe di scienze fisiche. Rendiconto
ns v4–11, 1899–1907||

R. Accademia delle scienze di Torino
Annuario
1933/34||
Atti. Continued in sections
30–48, 50–62, 1895–1926/27||
Index: Each 10th v includes index to preceding 10v
Classe di scienze fisiche, matematiche e naturali
Atti
63–74, 1927/28–38/39
Memorie
s2 v48–65, 66^1, 67–69, 1889–1939
Index: Each 10th v in s2 includes index to preceding 10v

R. Accademia delle scienze fisiche e matematiche, Naples. (Societa reale di Napoli)
Atti. Suspended 1917–26
s2 v8–14, 16–20, 1897–1935
Rendiconti
s3 v1–18, 20–36, 1895–1930; s4 v1–8, 1931–38
General index: 1737–1903

R. Accademia di scienze, lettere e belle arti di Palermo
Atti
s3 v1–9, 1891–1912

- R. ACCADEMIA di scienze, lettere ed arti degli zelanti, Acireale
  - Classe di lettere
    - Memorie
      - s3 v10, 1917/18; s4 v2, 1927/29
  - Classe di scienze
    - Memorie
      - s3 v1–3, 10, 1901/02–18/22; s4 v2, 1927/31
- R. ACCADEMIA di scienze, lettere ed arti in Modena
  - Atti e memorie. To 1922 as its Memorie
    - s3 v1–14, 1898–1922; s4 v1–4, 1926–33/34; s5 v1–2, 1935/36–36/37
  - *See also* Sec. II
- R. ACCADEMIA nazionale dei Lincei, Rome. To Je1920 as R. Accademia dei Lincei
  - Atti
    - 1, 1847/48
  - Classe di scienze fisiche, matematiche e naturali
    - Rendiconti
      - s5 v1–33, 1892–1924; s6 v1–19, 23–25, 27–29, 1925–Je39||
  - Rendiconti. Continued as Rendiconti della Classe di scienze fisiche, and Rendiconti della Classe di scienze morali
    - s4, v1–7, 1884–91||
  - Rendiconti delle sedute solenni
    - 1892–97, 1904–12, 1914, 1916–28
  - General index: 1876–1910; 1911–24; 1925–34
- ACCADEMIA pontificia dei nuovi Lincei, Rome. *See* PONTIFICIA accademia delle scienze
- ACTA mathematica. Stockholm
  - 1+, 1882+
  - Index: 1–35
  - For appendices, 1884, 1885, *see* BIBLIOTHECA mathematica
- ACTUARIAL society of America, New York
  - Actuarial studies
    - no1, 4–5, 1919–32
  - List of members
    - 1911
  - Transactions. 1–6 as Papers and transactions
    - 1–35, 36^1, 37+, 1889+
- AIX-MARSEILLE, Université d'
  - Faculté des sciences
    - Annales
      - Ser. in 4°
        - 11–26, 1901–30|| (incl. suppl. to v16, 20)
      - Ser. in 8°
        - 1–12, 1921–39
- AKADEMIE der wissenschaften, Berlin
  - Abhandlungen [mathematical memoirs only]
    - 1892–94, 1897, 1901, 1903–04, 1906–07||
  - Monatsberichte. Supersedes its Bericht
    - 1857–81
    - Index: 1836–58; 1859–73; 1874–81
  - Physikalisch-mathematische klasse
    - Abhandlungen
      - 1909–38
    - Sitzungsberichte
      - 1922–37
  - Sitzungsberichte. Supersedes its Monatsberichte. Continued in classes
    - 1882–1921
- AKADEMIE der wissenschaften, Leipzig. 1846–1918 as K. Saechsische gesellschaft der wissenschaften. 1919+ full name: Saechsische akademie der wissenschaften
  - Mathematisch-physische klasse
    - Abhandlungen
      - 12–42, 1883–1938
    - Berichte
      - 26–89, 1874–1937
- AKADEMIE der wissenschaften, Munich
  - Almanach
    - 1909||

Mathematisch-naturwissenschaftliche abteilung. 1871–1923 as its Mathematisch-physikalische classe
Abhandlungen
14–25, 27–32, 1881–1928; ns v1–44, 1929–38 (incl. suppl. 1922–32)
Sitzungsberichte
1881–1912, [1913], 1914–37
Index: 1871–85; 1886–99

K. Akademie der wissenschaften, Vienna
Mathematisch-naturwissenschaftliche klasse
Sitzungsberichte
108–148, 1899–1939

K. Akademie van wetenschappen, Amsterdam
Afdeeling natuurkunde
Proceedings of the section of sciences
1–30, 1899–1927
Verhandelingen. Eerste sectie
13^4, 1928

Akademii͡a nauk S.S.S.R., Leningrad
Bulletin. s4 v1–2 also as s3 v33–34. 1928+ in sections
s3 v1, 5–32, 1860–88; s4 v1–2, 1890–92; s5 v3–25, 1895–1906; s6 v1–7, 8^1, 10–21, 1907–27||
Classe physico-mathématique
Mémoires
s8 v14^7, 15–17, 32^4, 33^{8-9}, 34^2, 1903–16
Comptes rendus (Doklady) de l'Académie des sciences de l'URSS. Supersedes its Doklady sA. 1934+ in 4v each year. 1934 v1–1935 v3 lack numbering, later supplied as v2-7. Published in two editions
International edition
ns v1–9, 11–17, 19–29, 32, 34+, 1933+
Russian edition
ns v22–29, 42+, 1939+
Fiziko-matematicheskiĭ institut imeni V. A. Steklova
Izvesti͡a
1–3, 1922–30||
Trudy. Title also in French: Travaux de l'Institut physico-mathématique Stekloff
1+, 1932+
Fiziko-matematicheskiĭ referativnyĭ zhurnal
1, 1939
Gruzinskiĭ filial
Matematicheskiĭ institut
Trudy
1+, 1937+
Soobshcheni͡a
1-2, 1940–41
Izvesti͡a. For earlier years see Academy's Bulletin
Otdelenie matematicheskikh i estestvennykh nauk. 1928–30 as its Fiziko-matematicheskoe otdelenie. 1936+ in series
s7 v1–7, 1928–34
Serii͡a fizicheskai͡a
1936–38, 1940
Serii͡a matematicheskai͡a
8+, 1944+

Akademii͡a nauk U.R.S.R., Kiev. Name varies: Ukraïn'ska akademii͡a nauk, Vseukraïn'ska akademii͡a nauk
Fizichno-matematichniĭ viddil
Trudy
1–11, 1923–29
Zapiski
1–5, 1923–31||
Institut budivel'noï mekhaniki
Kafedra matematichnoï fiziki
[Publications]
4–5, 8, 14, 1934–35

- Zapiski
  - 1–4, 1937–39
- Institut matematiki
  - Zbirnik prats'
    - no1–6, 1938–41
  - Zhurnal
    - 1–4, 5^1, 1934–38
- Prirodnicho-tekhnichiĭ viddil. French name: Classe des sciences naturelles et techniques
  - Zhurnal Industrii͡al'no-tekhnichnogo tsiklu
    - 3^2, 1933
- AKADEMIJA umiejętności, Krakow
  - Bulletin international. Comptes rendus des séances. Continued in classes
    - 1898–1900||
  - Komisya bibliograficzna
    - Katalog literatury naukowej polskiej
      - 1–10, 16–18, 1901–30
  - Wydział matematyczno-przyrodniczy . . .
    - Bulletin international
      - 1901–09||
      - sA: Sciences mathématiques
        - 1910–12, 1914–38
    - Comptes rendus mensuels des sciences mathématiques et naturelles
      - 1929–38
- AKTUARSKÉ vědy: pojistná matematika, matematiká, statistika. (Jednota československých matematiků a fysiků v Praze) Prague
  - 2–7, 1930–38
- ALLAHABAD, India
  - University
    - Mathematical association
      - Bulletin
        - 1, 1927–28
- "ALZATE, Antonio," Sociedad científica, Mexico. *See* ACADEMIA nacional de ciencias Antonio Alzate
- AMERICAN academy of arts and sciences, Boston
  - Memoirs
    - ns v12+, 1902+
  - Proceedings. 9–31 also as ns v1–23
    - 26+, 1890+
  - *See also* Sec. II, under BOWDITCH, N.
- AMERICAN institute of actuaries, Chicago
  - Record. v7^2 never printed
    - 1^{1-4}, 3–6, 7^1, 8+, 1909+
    - Index: 1–15; 16–25
  - Yearbook
    - 1926–35, 1941
- AMERICAN institute of physics, New York. *See* PHYSICAL review; REVIEW of scientific instruments
- AMERICAN journal of mathematics. Baltimore. 1–2 as American journal of mathematics pure and applied
  - 1+, 1878+
  - Index: 1–50
- AMERICAN mathematical monthly.[1] Ithaca, N. Y.
  - 1+, 1894+
  - *See also* CARUS mathematical monographs; and Sec. II, under MATHEMATICAL association of America
- AMERICAN mathematical society, New York
  - Bulletin. s1 as Bulletin of the New York mathematical society
    - 1–3, 1891–94; s2 v1+, 1894+
    - Index: 1–s2 v10; 11–20; 21–30; 31–40; 41–50

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[1] The AMERICAN mathematical monthly, founded by B. F. Finkel in 1894 and published by him until 1913, has been the official journal of the Mathematical Association of America, since its organization in December, 1915.

Colloquium publications[2]
1+, 1905+
List of members, etc.
1891, 1892, 1894, 1896–1918 (annually), 1920+ (biennially)
Transactions
1+, 1900+
Index: 1–5; 1–10; 11–20; 21–30; 31–40; 41–50
See also Mathematical reviews; Mathematical surveys; Sec. II, under American mathematical society; Klein, F.
American philosophical society, Philadelphia
Annual report of the library
1935–36
List of members
1892, 1933–34 (also in Proceedings 30–56)
Memoirs
1–20, 1935–43
Miscellanea
1^2, 1936||
Proceedings
29–87, 1891–1944
Index: 51–75
Transactions
ns v20–33, 1902–44
Yearbook
1937–42
General index: 1769–1940
American physical society
Bulletin. Ithaca, N. Y.
1–3, 1899–1902||
See also Physical review
American society of civil engineers, New York
Proceedings. Contain papers and discussions, separately paged, of all papers subsequently appearing in Transactions in revised form
32–34, 1906–08
Transactions
55+, 1905+
Index: 1–83; 84–99^2
See also Sec. II
American society of mechanical engineers
Transactions
46, 1924
Year book
1915–20, 1922, 1924–26
Analyst. A journal of pure and applied mathematics. Des Moines, Ia. Superseded by Annals of mathematics
1–10, 1874–83||
Index: 1–10
Annales de mathématiques pures et appliquées. Paris. Superseded by Journal de mathématiques pures et appliquées
1–22, 1810–31||
Annales roumaines de mathématiques. (Institut mathématique roumain) Bucharest. v2 published in 1935
1–2, 1935–37
Annales scientifiques des institutions mathématiques savantes de l'Ukraine. *See* Zapiski Kharkivs'kogo matematichnogo tovaristva ta Ukrains'kogo institutu matematichnikh nauk
Annali delle scienze del regno lombardo-veneto. Padua
1–2, 1831–32
Annali delle università toscane. Pisa. v35–48 also as ns v1–14. Merged into Pisa. Scuola superiore. Annali
35–47, 48^2, 1916–29/30||

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[2] For the origin of the Colloquium publications and for their history see R. C. Archibald, A semicentennial history of the American Mathematical Society, 1888–1938, published by the Society in 1938.

ANNALI di matematica pura ed applicata. Milan, Bologna. Supersedes Annali di scienze matematiche e fisiche
s2 v3, 9, 1869–79; s3 v1–31, 1898–1922; s4 v1+, 1923+
Index: 1858–97; 1898–1922; 1923–37
ANNALI di scienze matematiche e fisiche. Superseded by Annali di matematica...
Index: 1850–57. Combined with index 1858–97 of Annali di matematica pura ed applicata
ANNALS of mathematical statistics. Ann Arbor, Mich.
1+, 1930+
Index: 1–10
ANNALS of mathematics. Princeton. Supersedes Analyst
1–12, 1884–99; s2 v1+, 1899+
Index: 1–12; s2 v1–12
ANNALS of mathematics studies. Princeton
1+, 1940+
ARCHIV der mathematik und physik. Leipzig, Berlin
s2 v1–17, 1884–1900; s3 v1–24, 1901–16
Index: s3 v1–10; 11–20
See also BERLINER mathematische gesellschaft
ARCHIV för mathematik og naturvidenskab. Oslo
1+, 1876+
Index: 1–25
ARCHIVES néerlandaises des sciences exactes et naturelles. (Société hollandaise des sciences à Harlem) The Hague
1–30, 1866–97; s2 v1–15, 1897–1910||
Index: 1–10
Serie 3A. Sciences exactes
1–14, 1911–33||
See also PHYSICA; Sec. II, under HOLLANDSCHE maatschappij der wetenschappen
ARCHIVO de matemáticas puras y aplicadas. Valencia
1, 2^{1-8}, 1896–Ag1897||
ARKIV för matematik, astronomi och fysik. (Svenska vetenskapsakademien) Stockholm
1+, 1903+
ASSOCIATION for symbolic logic. *See* JOURNAL of symbolic logic
ASSOCIATION for the improvement of geometrical teaching. Reorganized 1892 as Mathematical association, London
Report
8, 13, 1882–87
See also Sec. II
ASSOCIATION of mathematical teachers in New England
Joint bulletin. Boston
no1–2, 1907–08||
ASSOCIATION of teachers of mathematics for the middle states and Maryland
Bulletin. Continued in Association of mathematical teachers in New England. Joint bulletin
no1–2, 1904–05
See MATHEMATICS teacher
ASSOCIAZIONE "Mathesis" fra gli insegnanti di matematica delle scuole medie. *See* "MATHESIS" società italiana di matematica
ASTRONOMICAL society of the Pacific, San Francisco
Publications
1–36, 37^1, 1889–1925
ASTRONOMIE. (Société astronomique de France) Paris
46–47, 49, 1932–35
ASTRONOMY and astrophysics. Northfield, Minn. Superseded by Astrophysical journal
1–13, 1892–94||
ASTROPHYSICAL journal; an international review of spectroscopy and astronomical physics. Chicago. Supersedes Astronomy and astrophysics
1+, 1895+
AUSTRALIAN and New Zealand association for the advancement of science, Sydney. Through 1926 as Australasian association for the advancement of science
Reports
15–20, 1921–30
Index: 16–20

Bell system technical journal. New York
1+, 1922+ (incl. suppl. Jl1936)
Index: 1–10; 11–20
Berliner mathematische gesellschaft
Sitzungsberichte. 1–18 as suppl. to Archiv der mathematik und physik
1–14, 1901–15
Bibliotheca mathematica. Stockholm, Leipzig
1–3, 1884–86; s2 v1–13, 1887–99; s3 v1–14, 1900–15||
Index: s2 v1–10
K. Boehmische gesellschaft der wissenschaften, Prague. *See* Česká společnost nauk
Boletín matemático. Buenos Aires
1+, 1928+
Suplemento informativo
1, 2^{1-8}, 1933–34||
Boletín matemático elemental. Buenos Aires. Suppl. to Boletín matemático
1–4, 5^{1-4}, 1930–34||
Bollettino di bibliografia e di storia delle scienze matematiche e fisiche. Rome
1–20, 1868–87||
Index: 1–20
Bollettino di bibliografia e storia delle scienze matematiche. Turin. Merged into Bollettino di matematica
1–21, 1898–1919||
Bollettino di matematica. Rome, Bologna
1–17, 1902–21; ns v1–5, 6^{1-3}, 7–8, 9^{3-4}, 10–17, 1922–38; s3 v1, 1939
British association for the advancement of science, London
Reports. No meetings held 1917–18
29–31, 33, 35, 93–103, 1859–1935
See also Sec. II
Brno
Masarykova universita
Přírodovědecká fakulta
Spisy
1+, 1921+
Brussels
Observatoire royal de Belgique
Annales
Annales astronomiques
ns v6, 1887
Annuaire astronomique
1906–07
Bryn Mawr college, Bryn Mawr, Pa.
Monographs
Reprint series
8, 1909
Bucharest. École polytechnique. *See* Bulletin de mathématiques et de physique, pures et appliquées
Buenos Aires
Instituto nacional del profesorado secundario
Revista del centro estudiantes
31–32, 1922
Universidad nacional
Facultad de ciencias exactas, físicas y naturales
Publicaciónes
sB. Científicas y técnicas. no1–3 as the university's Seminario matemático. Boletín
8–13, 15+, 1931+
Instituto matemático hispano-americano
Seminario matemático. 1928–32 as the university's Instituto matemático hispano-americano. Seminario matemático argentino
Boletín
1–3, 1928–33 (incl. suppl., v1–2)
Bulletin de mathématiques élémentaires. Paris. 3–13 as Bulletin de sciences mathématiques et physiques élémentaires. Merged with Journal de mathématiques élémentaires (Vuibert)
1–9, 12^{12}, 1895–1906

BULLETIN de mathématiques et de physique, pures et appliquées. (Bucharest. École polytechnique)
- 1+, 1929+

BULLETIN des sciences mathématiques. (Paris. École pratique des hautes études. Bibliothèque) 1870–76 as Bulletin des sciences mathématiques et astronomiques. 1884+ the astronomical section is continued in Bulletin astronomique
- 1–11, 1870–76; s2 v1–2, 26^1, 27–63, 1877–1939
- Index: s2 v1–30

BULLETIN des sciences mathématiques, astronomiques, physiques et chimiques. *See* BULLETIN universel des sciences et de l'industrie

BULLETIN universel des sciences et de l'industrie. (Société pour la propagation des connaissances scientifiques et industrielles) Paris
- Section 1. Bulletin des sciences mathématiques, astronomiques, physiques et chimiques
  - 1–16, 1824–31||

BULLETTINO. *See as if* BOLLETTINO

CAGLIARI
- Università
  - Facoltà di scienze matematiche, fisiche e naturali
    - Seminario scientifico
      - Rendiconti
        - 1–9, 1931–39

CAHIERS scientifiques. Paris
- 1–19, 1927–38

CALCUTTA
- University
  - Dept. of science
    - Journal
      - 1–10, 1919–33
  - University studies
    - s2 v1, 1918||

CALCUTTA mathematical society
- Bulletin
  - 1+, 1909+

CALIFORNIA
- University
  - Publications in mathematics
    - v1–2, 1912–34; ns v1+, 1943+
  - Scripps institution of oceanography, La Jolla
    - Bulletin: Technical series
      - 2^1, 1929
- *See also* LICK observatory

CALIFORNIA academy of sciences, San Francisco
- Proceedings
  - Mathematics-physics
    - s3 $v1^{1-4}$, 1898

CAMBRIDGE and Dublin mathematical journal. Cambridge, England. Superseded by Quarterly journal of pure and applied mathematics
- ns v1–9, 1846–54||
- Index: 1–9

CAMBRIDGE philosophical society, Cambridge, England
- List of fellows
  - 1901 (see also v12, 14, 16, 20 of Proceedings)
- Proceedings
  - 10+, 1899+
- Transactions
  - 17–23, 1899–1928||

CAMBRIDGE tracts in mathematics and mathematical physics. (Cambridge, England. University)
- 1–4, 6–7, 22–23, 25, 37+, 1905+

CARNEGIE institution of Washington
- Publications
  - 9, 78, 86, 105, 120, 151, 161, 165, 245–256, 318, 1905–30

CARUS mathematical monographs. (Mathematical association of America)
- 1+, 1925+

Časopis pro pěstování matematiky a fysiky. (Jednota československých matematiků a fysiků) Prague
1–52, 53^{1-2}, 55–67, 1872–1938
Index: 1–30
Cernăuti, Roumania
Universitatea
Facultatea de ştiinte
Buletinul
4–10, 1930–36
Česká akademie císaře Františka Josefa pro vědy, slovesnost a umění v Praze. *See* Česká akademie věd a umění v Praze
Česká akademie věd a umění v Praze. 1890–1920 as Česká akademie císaře Františka Josefa pro vědy, slovesnost a umění v Praze
Bulletin international
7–38, 1903–37
Sborník přírodovědecký
1–4, 6–7, 1925–30
Třída II: Mathematicko-přírodnická
Rozpravy
1–46, 1891–1936
Věstník
34^{1-9}, 1925
Česká společnost nauk, Prague. German name: Boehmische gesellschaft der wissenschaften; French name: Société des lettres et des sciences de Bohême
Jahresbericht
1899–1902, 1910–12, 1915–17||
Třída mathematicko-přírodovedecka.
Věstník. 1885 as its Zpravy . . . 1885–1917 have added title page Sitzungsberichte; 1918+ Mémoires
1899–1902, 1905–06, 1908–37
Výročni zpráva. Through 1917 also issued in German as its Jahresbericht
1920–37
General index: 1884–1904; 1905–35
Chicago
University
Decennial publications
s1 $v8^{1-7}$, $9^{1-5,7,9,11-12}$, 1902–04
Chinese mathematical society
Journal
1^{1-3}, 2^{1-2}, 1936–40
Christiaan Huygens; internationaal mathematisch tijdschrift. Groningen, Amsterdam
1+, 1921+
Circolo matematico, Palermo
Annuario
1898, 1900, 1905–12, 1914, 1928
List of members
1924
Rendiconti
1–61, 1884–1937
Index: 1–55
Supplemento
1–8, 14, 16, 1906–30||
Index: 1–8
Coast artillery journal. (United States coast artillery school, Fort Monroe, Va.) 1–56 as Journal of the United States artillery
1–70, 1892–1929
Colorado mathematical society
Biennial calendar
no2, 1907–09
Columbia university, New York
Ernest Kempton Adams fund for physical research
Publications
1, 3–8, 10, 1906–27||
Commentarii mathematici helvetici. (Schweizerische mathematische gesellschaft) Zurich
1–13, 16+, 1929+

CRELLE'S journal für mathematik. *See* JOURNAL für die reine und angewandte mathematik
CULVER-Stockton quarterly. Canton, Mo.
2^1, 1926
CZERNOWITZ. *See* CERNĂUTI

K. DANSKE videnskabernes selskab, Copenhagen. French name: Académie royale des sciences et des lettres de Danemark
Mathematisk-fysiske meddelelser
1–16, 1917–39
Oversigt over . . . forhandlinger. Supersedes its Bekiendtgiörelse
1900–31
Selskabs skrifter
1. Historisk og filosofisk afdeling
s7 v3^1, 1914
2. Naturvidenskabelig og mathematisk afdeling
s4 v[3, 5–9, 12], 1828–45; s5 v[1, 4, 6], $8^{1,4-5}$, $9^{3,7}$, $10^{4,6,8-9}$, 1849–75; s6 v1^6, 2^{11}, 3^1, 5^2, 7^1, $8^{3,6}$, 10^1, 1882–99; s7 v1^6, $2^{2-3,5}$, 5^1, 6^{7-9}, 8^6, 10^3, 11^{2-4}, 12^{1-2}, 1904–14; s8 v$1^{1,4-5}$, $2^{3,5}$, 4^4, 7^2, 10^3, $11^{1,3}$, 12^{2-3}, 1914–29
Also reprints 1821–23 by C. F. Degen and H. C. Orsted
K. DEUTSCHE akademie der naturforscher zu Halle. To 1922 as Leopoldinisch-Carolinische deutsche akademie der naturforscher
Nova acta
90–93, 1909–10; ns v1, 1932/34
DEUTSCHE mathematik. (Deutsche forschungs-gemeinschaft) Leipzig
1–4, 1936–39
Beihefte
1–2, 1939–40
DEUTSCHE mathematiker-vereinigung, Leipzig
Jahresbericht
1–49, 1890–1940
Index: 1–10
Ergänzungsbände
2^1, 3, 4^1, 6, 1900–30||
See also ZENTRALBLATT für mathematik und ihre grenzgebiete; Sec. II, under GUTZMER, A.
DUKE mathematical journal. (Duke university) Durham, N. C.
1+, 1935+

EDINBURGH
Royal observatory
Annals
1–2, 1902–06
University
Mathematical dept.
Research papers
1914^{1-7}, 1915^{1-11}
EDINBURGH mathematical society
Proceedings
1, 11–44, 1883–1925/26; s2 v1+, 1927+
Index: 1–20; 21–44
See also MATHEMATICAL notes
EDINBURGH royal society. *See* ROYAL society of Edinburgh
EDUCATION mathématique. Paris
5–16, 1902–14
EDUCATION outlook. London. 1847–1923 as Educational times
50–68, 1897–1915
EDUCATIONAL times. *See* EDUCATION outlook
ELISHA Mitchell scientific society, Chapel Hill, N. C.
Journal
9–12, 1892–95
ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. *See* Sec. II
ENCYKLOPÄDIE der mathematischen wissenschaften. *See* Sec. II
ENSEIGNEMENT mathématique. (Commission internationale de l'enseignement mathématique) Paris, Geneva
1–37, 1899–1938

FINNISCHE akademie der wissenschaften. *See* SUOMALAINEN tiedeakatemia
FIZIKO-matematicheskiĭ institut imeni V. A. Steklova. *See* under AKADEMIÎA nauk S.S.S.R., Leningrad
FIZIKO-matematicheskoe obshchestvo, Kazan. *See* under KAZAN
FORMULARIO matematico. Turin. 1–4 as Formulaire de mathématique. Suppl. to Rivista di matematica. There were two distinct forms of "Editio V, Tome V de editione completo," one published in 1905–06, as "proba de 100 exemplere," the other in 1908. The first of these contains: fasciculo 1 (304 pp., 1905), fasciculo 2 (pp. 305–391+8 plates, 1906), and indice et vocabulario (xlvii pp., 1906). The second consists of: prefatione, tabula de symbolos, bibliographia (xxxv pp.), fasciculo 1 (272 pp.), fasciculo 2 (pp. 273–463). Pages 1–272 in both forms seem to be identical, but otherwise the differences are considerable
1–5, 1895–1908||
See also Sec. II, under PEANO, G.
FORTSCHRITTE der mathematik. *See* JAHRBUCH über die fortschritte der mathematik
FRANCE
Bureau des longitudes, Paris
Annuaire
1903–08, 1910–19, 1923
FUKUOKA, Japan. Imperial university of Kyushu. *See* KYUSHU imperial university
FUNDAMENTA mathematicae. Warsaw
1+, 1920+
Index: 1–25

GAZETA matematica. Bucharest
1–18, 19^{1-11}, 20–21, 25+, 1895+ (also Jubilee v1895–1935)
Index: 1–40
GENTLEMAN'S diary; or, Mathematical repository. London. United with Ladies' diary in 1841 to form Lady's and gentleman's diary
75–78, 1815–18
GENTLEMAN'S mathematical companion. London
1–5, 1798–1827||
GESELLSCHAFT der wissenschaften zu Göttingen
Mathematisch-physikalische klasse
Nachrichten
1894–1933||
Nachrichten. Continued in three series: Nachrichten. Geschäftliche mitteilungen (later Jahresbericht); Mathematisch-physikalische klasse. Nachrichten; Philologisch-historische klasse. Nachrichten
1892–93
Geschäftliche mitteilungen. *See* its Jahresbericht
Jahresbericht
1894–1936
GIESSEN
Universität
Mathematisches seminar
Mitteilungen
1–2, 1921–31
GIORNALE di matematica finanziaria. Turin. 13+ also as s2 v1+
1–8, $10^{1-3,6}$, 11–12, 1919–30; s2 v1–6, 1931–36
GIORNALE di matematiche. (Battaglini) Naples. 1–31 as Giornale di matematiche ad uso degli studenti delle università italiane
1–3, 1863–65
GLASGOW philosophical society. *See* ROYAL philosophical society of Glasgow
GOETTINGEN
Universität (Georg-Augustus-universität)
Mathematischer verein
Berichte
no65–79, 1901–08
Mathematisch-naturwissenschaftliche fakultät
Jahrbuch. 1920–22 pt1 as a section of the university's Philosophische fakultät.
Jahrbuch
1920–24||
See also Sec. II, under GÖTTINGEN university

GORKI universitet, Nizhniĭ-Novgorod
Uchenye zapiski. English title: Scientific records of the Gorky state university
1-2, 1935-36
GRENOBLE
Université
Annales. Continued in 2 sections: Section lettres-droit and Section sciences-médicine
13-34, 1901-23||
Index: 1-20; 21-34
Section sciences-médicine
Annales
ns v1-14, 1924-37

HAMBURG
Universität. 1936+ as Hansische universität
Mathematisches seminar
Abhandlungen
1-13, 1921-40
See also HAMBURGER mathematische einzelschriften
HAMBURGER mathematische einzelschriften. (Hamburg. Universität. Mathematisches seminar)
1-11, 13-20, 26, 1923-39
HAMBURGISCHE gesellschaft zur verbreitung mathematischer kenntnisse
Jahresbericht
1859
HANSISCHE universität. *See* HAMBURG
HAVANA
Observatorio nacional
Boletín oficial
Mensual
s3 v1-2, 1936-37
HELLĒNIKĒ mathematikē hetaireia, Athens. French title: Société mathématique de Grèce
Deltion
1-19, 1919-38
HELSINGFORS. Universitet. *See* Sec. II, under HELSINGFORS
HIROSHIMA, Japan
University
Journal of science
sA: Mathematics, physics, chemistry
1-10, 1930-40
See also Sec. II
HOCHSCHUL-nachrichten. Munich
13, 1902/03
HOKKAIDO imperial university, Sapporo, Japan. *See* SAPPORO

ILLINOIS
University
Dept. of mathematics
Bulletins
18^{12}, 20^{42}, 22^{35}, 25^{43}, 1920-28
IMPERIAL academy, Tokyo
Proceedings
2^{1-6}, 13, 15-16, 1926-40
INDIAN academy of sciences
Proceedings.-Each no in two parts: A and B
1+, 1934+ (incl. suppl. 1934)
INDIAN mathematical club. *See* INDIAN mathematical society
INDIAN mathematical society, Madras. 1909-10 as Indian mathematical club
Journal. In 1934 divided into two publications: Journal ns and Mathematics student
1-20, 1909-33; ns v1+, 1934+
List of members
1919, 1922, 1924, 1926 (also 1909, 1911 in society's publication for those years)
Report of the . . . conference. 1-8 in Journal, 9+ in Mathematics student
1-10, 1916-38
See also MATHEMATICS student

Indian physico-mathematical journal. Dacca
2–3, 1931–32
Institut catholique. *See* under name of city
Institut de France. *See* Académie des sciences, Paris
Institut grand-ducal de Luxembourg. 1853–68 as Société des sciences naturelles du grand-duché de Luxembourg; 1870–1904 Institut royal grand-ducal de Luxembourg
Section des sciences naturelles, physiques et mathématiques
Archives
8–9, 15, 17–21, 23–27, 1865–1904; ns v1–3, 1906–08
Institut matematiki. *See* under Akademiia nauk U.R.S.R., Kiev
Institut mathématique roumain. *See* Annales roumaines de mathématiques
Institut royal grand-ducal de Luxembourg. *See* Institut grand-ducal de Luxembourg
Institute of actuaries, London
Journal
35+, 1901+
Index: 41–55; 56–65
Yearbook
1929+
Instituto matemático hispano-americano. *See* Buenos Aires. Universidad nacional
Intermédiaire des mathématiciens. Paris
1–27, 1894–1920; s2 v1–4, 1922–25||
International association for promoting the study of quaternions and allied systems of mathematics
Bulletin. Lancaster, Pa. Not published 1902, 1904, 1906–07, 1911
1900–13||
International catalogue of scientific literature. London
A. Mathematics
1, 1902
Isis. International review devoted to the history of science and civilization. Brussels, Bern. Suspended Jl1914–Ag1919
6–11, 12^1, 1924–29
I. R. Istituto del regno lombardo-veneto. Supersedes Istituto nazionale italiano. Superseded by R. Istituto lombardo di scienze e lettere
Annali
1–2, 1831–32
Istituto fisico-matematico G. Ferraris. *See* Rassegna di matematica e fisica
R. Istituto lombardo di scienze e lettere, Milan. Supersedes I. R. Istituto del regno lombardo-veneto
Memorie. s3 in two classes
Classe di scienze matematiche e naturali
s3 v10–14, 1900–37
Rendiconti. s3 in three parts: Parte generale e atti officiale; Classe di lettere e scienze morali e storiche; Classe di scienze matematiche e naturali
s2 v33–69, 1900–36; s3 v1–2, 1937–38
R. Istituto veneto di scienze, lettere ed arti, Venice
Memorie
5, 1855

Jahrbuch über die fortschritte der mathematik. Berlin. 1932–34 united with Revue semestrielle des publications mathématiques
1–56, 59^2, 60^1, 61^1, 1868–1935
Japanese journal of astronomy and geophysics. (National research council of Japan) Tokyo
3–8, 10–17, 1925–40
Japanese journal of engineering; abstracts. (National research council of Japan) Tokyo
1–12, 1921–35
Japanese journal of mathematics. (National research council of Japan) Tokyo
1–16, 1924–40
Japanese journal of physics. (National research council of Japan) Tokyo
1–13, 1922–40
Index: 1–7
Jassy, Roumania
Universitatea
Annales scientifiques. Continued in two parts

3^{2-4}, 4^{1}, 5^{2-3}, 6^{4}, 7^{4}, 8^{1-2}, 11–22, 1906–36‖
1. Mathématiques, physique, chemie
23+, 1937+
JEDNOTA československých matematiků a fysiků v Praze. 1862–1912? as Jednota českých matematiků a fysiků v Praze
Bibliografické zpravy
1–4, 5^{1-5}, 6, 7^{5}, 1927–33
Sbornik
1–11, 13–16, 18, 1898–1926
Výroční zpráva
1870/71, 1905/06, 1925/26, 1928/29
See also AKTUARSKÉ vědy; ČASOPIS pro pěstováni matematiky a fysiky; ROZHLEDY matematicko přírodovědecké; Sec. II, under KAVÁN, J.
JEDNOTA českých matematiků a fysiků. *See* JEDNOTA československých matematiků a fysiků v Praze
JERUSALEM
Universitas
Scripta universitatis atque bibliothecae hierosolymitanarum
Mathematica et physica
1, 1923‖
JOHNS Hopkins university, Baltimore, Md.
Circulars [mathematical circulars only]
3–5, 7–8, 10–11, 13, 15, 17, 20–22, 25, 27–30, 32, 35, 39, 45, 53, 57, 62, 66, 78, 85, 95, 115, 118, 123, 134, 151, 160, 167, 174, 191, 208, 232, 242, 257, 277, 1880–1915
JORNAL de sciencias mathematicas e astronomicas. (Academia polytechnica do Porto) Coimbra. Superseded by the academy's Annaes scientificos, later Oporto. Universidade. Faculdade des sciencias. Anais
9–15, 1889–1905‖
JORNAL de sciencias mathematicas, fysicas, e naturaes. (Academia das ciencias de Lisboa) Lisbon
1–8, 1868–82
JOURNAL de mathématiques élémentaires. (de Longchamps) Paris. 1877–79 as Mathématiques élémentaires; 1880–81 Journal de mathématiques élémentaires et spéciales. 1882–1901 issued in 2 parts: Journal de mathématiques élémentaires and Journal de mathématiques spéciales. 1877–96 also numbered in four series of five v. each
1–23, 25, 1877–1900
JOURNAL de mathématiques élémentaires. (Vuibert) Paris
1–3, 8–21, 25–27, 37–40, 44–54, 1877–1930
JOURNAL de mathématiques pures et appliquées. Paris. Supersedes Annales de mathématiques pures et appliquées
s1 $v2^{9}$, 1837; s3 v6, 1880; s5 v1–6, 7 [pp. 375–382 missing], 8–10, 1895–1904; s6 v1–10, 1905–14; s7 v1–4, 1915–18; s9 v4+, 1925+
JOURNAL de mathématiques spéciales. Paris. *See* JOURNAL de mathématiques élémentaires (Longchamps)
JOURNAL des géomètres-experts et topographes français. (Société des géomètres de France) Noyon, Paris. 1893–1927 as Journal des géomètres-experts français; v28+ also as no1+
1–2, 9–27, $28^{1-5,7,9-12}$, 29^{1-2}, 1893–1921
JOURNAL für die reine und angewandte mathematik. Berlin
1–13, 20–182, 1826–1940
Index: Each tenth v with the exception of 141–160
JOURNAL of mathematics and physics. (Massachusetts institute of technology) Cambridge, Mass.
1+, 1921+
JOURNAL of physical chemistry. Ithaca, N. Y.
1–16, 1896–1912
JOURNAL of symbolic logic. (Association for symbolic logic)
1+, 1936+
JOURNAL of the United States artillery. *See* COAST artillery journal
JUGOSLAVENSKA akademija znanosti i umjetnosti, Zagreb
[Catalogue of publications]
1867–1911
Ljetopis
11–23, 28–29, 34, 1896–1919
Index: 1867–1903

Matematičko-prirodoslovni razred
Bulletin international. 1–23 as its Izvješća o raspravama . . .
1–30, 1914–36
Prirodoslovna istraživanja kraljevine Jugoslavije. 1–15 as its Prirodoslovna istraživanja hrvatske i slavonije. 1–14 issued by the academy's Matematičko-prirodoslovni razred. Suspended 1925–29
2–3, 6–7, 1914–15
Rad
49–54, 56, 58, 1911–15
JUNTA para ampliación de estudios é investigaciónes científicas. Madrid
Anales
$17^{4a,6a}$, 1918
Laboratorio y seminario matemático
Publicaciones
1–4, 1916–20/29; 1933, 1935, 1936, 1937, 1938

KANSAS university quarterly. Lawrence. v6–10 in two series: A. Science and mathematics; B. Philology and history. Superseded by Kansas university science bulletin
1^2, 4^2, sA v9–10, 1892–1901||
Index: 1–10
KANSAS university science bulletin. Lawrence. Supersedes Kansas university quarterly. Also numbered in the university's Bulletin (1+ also called v11 no1+)
1–15, 1902–25
KAZAN
Universitet
Fiziko-matematicheskoe obshchestvo
Izvi͡estii͡a
s2 v1–18, 19^{1-2}, 23–25, 1891–1925; s3 v1–11, 1926–38
See also Sec. II, under KAZAN universitet
KHARKOV
Universitet
Naukovo-doslidniĭ institut matematiki i mekhaniki. Sektor geometriĭ
1–2, 1938–40
Ucheni zapiski
10, 1937
See ZAPISKI . . .
KIEV
Universitet. Through 1916 as Universitet Sv. Vladimira
Naukovi zapiski
Fizichno-matematichniĭ zbirnik
1, 3, 1935–37
Universitetskii͡a izvi͡estii͡a
43–53, 54^{1-5}, 1903–14
KOENIGSBERGER gelehrte gesellschaft
Naturwissenschaftliche klasse
Schriften [mathematical memoirs only]
1^{5-7}, 2^3, 3^3, $4^{1,3,5-6}$, 5^4, 6^6, $8^{1,3}$, $10^{3,5}$, 1924–33
KÖZÉPISKOLAI matematikai és fizikai lapok. Budapest
1^{3-6}, 2–14, 1925–38
KYUSHU imperial university, Fukuoka, Japan
Faculty of science
Memoirs
sA. Mathematics
1, 1940–41

LADIES' diary. *See* LADY's and gentleman's diary
LADY's and gentleman's diary. London. 1–137 as Ladies' diary (title varies); in 1841 united with Gentleman's diary
112–132, 134–137, 144, 147–151, 168, 1815–71||
LA PLATA
Universidad nacional
Facultad de ciencias físico-matemáticas
Anuario
1935–37, 1938–39, 1941

Catálogo y reglamento de la biblioteca
1922
Contribución al estudio de las ciencias físicas y matemáticas
Serie matemática. Continues in part its Serie matemático-física. United with other series to form its Publicaciones. s2. Revista
1^{1-5}, 1935-37||
Serie matemático-física. 1^{1-4} as Serie física. Continued in 2 series: Serie matemática and Serie física
1-5, 1914-31||
Index: 1-5
Serie técnica. United with other series to form its Publicaciones. s2. Revista
1-5, 1915-38
Publicaciones. All publications of the university listed in a single numerical series. Beginning no120, 1939+ they are also organized into three series
s2. Revista. Formed by the union of the university's Facultad de ciencias físico-matemáticas. Contribución al estudio de las ciencias físcias y matemáticas. Serie física, Serie matemática, Serie tecnica
1+, 1939+
Reunion anual de caminos
1937-40
LATVIJAS universitate. *See* RIGA
LENINGRAD
Leningradskiĭ industrial'nyĭ institut. Through 1934 as Leningradskiĭ èlektro-mekhanicheskiĭ institut
Sbornik
3, 1934
Universitet
Uchenye zapiski
Serii͡a fizicheskikh nauk
1-3, 1935-37
LENINGRADSKOE fiziko-matematicheskoe obshchestvo
Zhurnal
1-2, 1926/27-29
LEOPOLDINISCH-Carolinische deutsche akademie der naturforscher. *See* K. DEUTSCHE akademie der naturforscher zu Halle
LICK observatory, Mount Hamilton, Cal.
Contributions
2-5, 1891-95||
Publications
1-8, 9^{1-2}, 10-11, 1887-1913
LIOUVILLE, Journal de. *See* JOURNAL de mathématiques pures et appliquées
LONDON
University
University college
Dept. of statistics. Continues in part the college's Dept. of applied statistics
Statistical research memoirs
1+, 1936+
LONDON mathematical society
Journal
1+, 1926+
List of members
1865-99; 1897, 1901, 1903, 1908, 1911, 1913-15, 1921, 1929, 1933, 1935, 1937
Proceedings
1-35, 1865-1903; s2 v1+, 1903+
Index: 1-30; 31-35; s2 v1-30
LUND, Sweden
Universitet
Acta universitatis lundensis. Lunds universitets årsskrift
Afdelning 2. Medicin samt matematiska och naturvetenskapliga ämnen [mathematical memoirs only]
ns $v1^{5}$, 13^{8}, 14^{13-20}, $16^{3,11}$, $17^{2,7}$, 20^{1}, 22^{2}, 23^{2-4}, $24^{1,9}$, $25^{1,6}$, 1906-29
Matematiska seminarium
Meddelanden
1+, 1933+

Lyons
Université
Annales
6–7, 20, 1892–96
ns 1. Sciences, médecine
12, 15–16, 21, 25, 31, 1903–12

Madras. University. *See* Vizianagaram
Magyar tudományos akadémia, Budapest
Almanach
1903–06
See also Matematikai és physikai lapok; Mathematische und naturwissenschaftliche berichte aus Ungarn
Manchester literary and philosophical society, Manchester, England
Memoirs and proceedings
25–27, 37–56, 58+, 1876+
Marseilles. Université d'Aix-Marseille. *See* Aix-Marseille
Masarykova universita v Brne. *See* Brno
Massachusetts institute of technology
Abstracts of scientific and technical publications
no4–9, 1929–32||
Dept. of mathematics
Contributions
s2 no1-50, 52–196, 1920–31
See also Journal of mathematics and physics; Technology quarterly
Matematica elementare. Rome
1–3, 1922–24||
Matematiche pure ed applicate. Città di Castello
1–2, 1901–03||
Matematicheskiĭ listok. Reval
1–18, 1915–16
Matematicheskiĭ sbornik. (Moskovskoe matematicheskoe obshchestvo) Moscow. ns v1+ also as 43+
1–42, 1866–1935; ns v1+, 1936+
Index: 1–15
See also Sec. II, under Moskovskoe matematicheskoe obshchestvo
Matematikai és physikai lapok. (Magyar tudományos akadémia, Budapest. Matematikai és természettudományi osztály) Budapest
8–13, 14^{2-8}, 15–45, 1899–1938
Matematikai és természettudományi értesítö. (Magyar tudományos akadémia, Budapest. Matematikai és természettudományi osztály) Budapest
4, $5^{1-2,5-9}$, 6, 9^{2-9}, $10^{1-4,8,9}$, 11, $12^{1-5,7-12}$, 18, 19^{2-5}, 21^{3-5}, 31^{3-5}, 32, 33^{2}, 34^{2}, $35^{1,2,5}$, 36^{1-4}, 37–57, 1885–1938
Matematikai és természettudományi közlemények, vonatkozólag a hazai viszonyokra. (Magyar tudományos akadémia, Budapest. Matematikai és természettudományi bizottság) Budapest
2–5, $6^{1-4,6}$,7–13, 15–19, 28, $29^{1,3-4}$, 30–36, 1863–1928
Matematisk tidsskrift. (Matematisk forening i København) Copenhagen. Supersedes Nyt tidsskrift for matematik
A. (Elementary)
1919+
B. (Advanced)
1919+
Mathematica. (Cluj. Universitatea. Seminarul de matematici)
1+, 1929+
See also Sec. II, under Klausenburg university
Mathematicae notae. (Rosario. Universidad nacional del Litoral. Instituto de matemáticas)
1+, 1941+
Mathematical association, London. *See* Mathematical gazette
Mathematical association of America
List of members
1916, 1918 (further lists in American mathematical monthly)
See also American mathematical monthly; Carus mathematical monographs
Mathematical expositions. Toronto
1+, 1940+

MATHEMATICAL gazette. (Mathematical association) London. Supersedes Association for the improvement of geometrical teaching. Report
no1–6, 1894–95; (ns) v1+, 1896+
Index: 1–6; (ns) v1–15
See also Sec. II, under MATHEMATICAL association, London
MATHEMATICAL magazine. (Martin) Washington
1^5, 2^{1-12}, 1883–1904
MATHEMATICAL messenger. Ada, La.; Tyler, Tex.
7^3, 1891
MATHEMATICAL monthly. (Runkle) Cambridge, Mass.
1–3, 1858–61||
MATHEMATICAL notes: a review of elementary mathematics and science. (Edinburgh mathematical society) Edinburgh. Suspended 1917–23, 1926–28
1–28, 1909–33
MATHEMATICAL repository. (Dodson, Leybourn) London
1–3, 1748–55; (s3) v1–5, 1806–30
MATHEMATICAL review. (Story) Worcester, Mass.
1, 1896–99||
MATHEMATICAL reviews. (American mathematical society, etc.) New York
1+, 1940+
MATHEMATICAL surveys. (American mathematical society) New York
1+, 1943+
MATHEMATICAL visitor. (Martin) Erie, Pa.
1, 2^2, 1878–83
MATHEMATICS news letter. *See* NATIONAL mathematics magazine
MATHEMATICS student; a quarterly dedicated to the service of students and teachers of mathematics in India. (Indian mathematical society) Madras. Issued in place of the society's Journal pt2
1+, 1933+
MATHEMATICS teacher. (Association of teachers of mathematics for the Middle States and Maryland; National council of teachers of mathematics) Syracuse, N. Y.; Lancaster, Pa.; etc.
1+, 1908+
Index: 14–22
MATHEMATISCH-naturwissenschaftliche mitteilungen. (Mathematisch-naturwissenschaftlicher verein in Wurttemburg) Stuttgart. Suspended 1893–Mr1899
1–5, 1884–92; s2 v1–16, 20, 1899–1922||
MATHEMATISCHE annalen. Leipzig, Berlin
14–15, 31–39, 41–92, 95–103, 112–117, 1879–1941
Index: 51–80; 81–100
MATHEMATISCHE gesellschaft in Hamburg
Mitteilungen
2–7, 1890–1939
MATHEMATISCHE und naturwissenschaftliche berichte aus Ungarn. (Magyar tudományos akadémia, Budapest) Berlin, Budapest, Leipzig
1–31, 34–38, 1882–1931||
MATHEMATISCHE zeitschrift. Berlin
1+, 1918+
Index: 1–25
MATHESIS polska; czasopismo poświęcone naukom ścisłym i ich metodologji
3–8, 1928–33
See also MINERWA; URANJA
MATHESIS: recueil mathématique. Ghent, Paris. Also numbered in series of 10v each. Suspended 1916–20. Since 1921 this periodical has been the official organ of Société mathématique de Belgique
1–34, 36–51, 1881–1937 (with suppl.)
"MATHESIS" società italiana di matematica
Bollettino. 1–12 as Bollettino dell' Associazione "Mathesis." Merged 1920 into Periodico di matematica per l'insegnamento secondario, later Periodico di matematiche: storia-didattica-filosofia
1^1, 2^2, 3^3, 9^{1-6} [pp. 67–91 missing], 1896–1906
MESSENGER of mathematics. Cambridge, London. 1–5 as Oxford, Cambridge, and Dublin messenger of mathematics. United 1930 with Quarterly journal of pure and applied mathematics to form Quarterly journal of mathematics, Oxford series
1–5, 1861–71; ns v1–58, 1871–1929||
Index: 26–58

MICHIGAN
University
Publications in mathematics
1, 1944
Studies
Humanistic series
11, 1915
Scientific series
2, 1916
MILAN
Seminario matematico e fisico
Rendiconti
5-7, 12, 1931-38
MINERWA; czasopismo poświęcone popularyzacji nauk przyrodniczych i techniki. Warsaw. Suppl. to Mathesis polska
1, 1933
MINUVSHĪE gody. St. Petersburg
1-12, Ja-D1908||
MONATSHEFTE für mathematik und physik. (Vienna. Universität. Mathematisches seminar) Vienna
1-36, 38-48, 1890-1939
Index: 1-10; 11-20
MONIST, a quarterly magazine devoted to the philosophy of science. Chicago
1-7, 9-27, 28^{1-3}, 30-46, 1890-1936||
Index: 1-17
MONOGRAFJE matematyczne. (Warsaw. Uniwersytet. Seminarjum matematyczne) Warsaw
1-10, 1932-38
MORAVSKA přírodovědecka společnost, Brünn. Latin title: Societas scientiarum naturalium Moravicae
Prace
1-9, 1924-35
Výroční zpráva
1+, 1925+
MOSCOW
Universitet
Bulletin
Série internationale. Sec. A. Mathématiques et mécanique
1^{1-10}, 1937-38
Fiziko-mekhanicheskiĭ matematicheskiĭ fakul'tet. Name varies: Otdi͡el fiziko-matematicheskiĭ; Fiziko-matematicheskiĭ fakul'tet
Nauchno-issledovatel' skiĭ institut matematiki i mekhaniki
1926-27
Geometricheskii kruzhka
Trudy
1930
Seminar po teorii veroiatnostei metamaticheskoi statistike
Trudy
1, 1930
Seminar po vektornomu i tenzornomu analizu s ikh prilozhenii͡ami k geometrii, mekhanike i fizike
Trudy
1, 1933
Uchenyi͡a zapiski
1-11, 13-27, 29, 1880-1916||
Uchenye zapiski
2, 1934
MOSKOVSKOE matematicheskoe obshchestvo. *See* MATEMATICHESKIĬ sbornik
MÜEGYETEMI lapok; haví folyóirat a mathematika . . . Budapest
1-3, 1876-78||
MYSORE
University
Science association
Journal
2^{2-4}, 1925-26

NATIONAL academy of sciences, Washington
Memoirs
14^{2}, $17^{[1]}$, 1919–24
Proceedings
1–23, 25+, 1915+
Index: 1–10
NATIONAL council of teachers of mathematics. *See* MATHEMATICS teacher
NATIONAL mathematics magazine. 1–8, 1926-S1934 as Mathematics news letter
10+, 1935+
NATIONAL physical laboratory, Teddington, England. *See* TEDDINGTON, England
NATIONAL research council of Japan, Tokyo
Report. no1 covers the period Ap1922–Mr1930 and contains a reissue of most of the material in its Proceedings 1922
1–10, 1930–31
See also titles beginning JAPANESE journal of . . .
NATIONAL Tsing Hua university. *See* PEIPING
NATURE. London
127+, 1931+
NATURFORSCHENDE gesellschaft, Zurich
Vierteljahrsschrift
34+, 1889+
Index: 31–40; 41–50; 51–60; 61–70; 71–80
NAUCHNO-issledovatel'skie kafedry v Odesse. *See* ODESSA
NAUKOVE tovaristvo īmeni Schevchenka, Lemberg. German title: Ukrainische Ševčenko-gesellschaft der wissenschaften; Polish: Naukowe towarzystwo imenia Szevczenka
Chronik. German edition
17–44, 49–59, 1904–14‖
Matematichno prirodopisno-līkars'ka sektsii͡a
Zbīrnik
9–29, 1903–30
NAUKOVI zapiski Naukovo-doslidchikh-matematichnikh katedr Ukraini. *See* ZAPISKI Kharkivs'kogo matematichnogo tovaristva ta Ukrains'kogo institutu matematichnikh nauk
NEW York mathematical society. *See* AMERICAN mathematical society
NIEUW archief voor wiskunde. (Wiskundig genootschap te Amsterdam) Amsterdam. Supersedes Archief
1–20, 1875–93; s2 v1+, 1894+
NIEUW tijdschrift voor wiskunde. Groningen
1+, 1913/14+
NIPPON sugako-buturigakkwai, Tokyo. 1884–1918 as Tokyo sugaku buturigakkwai
Proceedings. 1–9 as the society's Kiji; s2 v1–2 no6 Hokoku; s2 v2 no7–v3 Kiiji-gaiyo. English title: Tokyo mathematico-physical society, later Physico-mathematical society of Japan. s2 v1–3 carry no series number
1–9, 1884–1901; s2 v1–9, 1901–18; s3 v1–22, 1919–40
NIZHNIĬ-Novgorod. *See* GORKI
NORSK matematisk forening, Oslo
Skrifter
s1 no1–18, 1921–28
See also NORSK matematisk tidsskrift
NORSK matematisk tidsskrift. (Norsk matematisk forening) Oslo
1–22, 1919–40
NORSKE videnskaps-akademi, Oslo. 1857–1924 as Videnskabs-selskabet i Kristiania
Årbok. Supersedes its Oversigt over videnskabs-selskabets møder
1925+
Fordhandlinger. Superseded by Avhandlinger of two classes
$1885^{15,18,21}$, $1886^{1,16-17}$, 1888^{9}, 1889^{7}, 1892^{17}, 1898–99, 1901–24‖
Index: 1858–1924
Matematisk-naturvidenskapelig klasse
Avhandlinger
1925+
Skrifter
1898+
Oversigt over videnskabs-selskabets møder. 1868–86 issued with and indexed in its Forhandlinger. Superseded by its Årbok
1898–1924‖

NOTRE Dame mathematical lectures. (University of Notre Dame) Notre Dame, Ind.
  1+, 1942+
  See also REPORTS of a mathematical colloquium
NOUVELLE correspondance mathématique. Brussels
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ON DEGREE OF APPROXIMATION ON A JORDAN CURVE TO A FUNCTION ANALYTIC INTERIOR TO THE CURVE BY FUNCTIONS NOT NECESSARILY ANALYTIC INTERIOR TO THE CURVE

J. L. WALSH

It is our object here to consider the subject mentioned in the title by proving the following theorem.

THEOREM. *Let C be a rectifiable Jordan curve in the plane of the complex variable z, and let the function $f(z)$ be analytic interior to C, continuous in the corresponding closed region. Let each of the functions $f_n(z)$, $n=1, 2, \cdots$, be analytic exterior to C and continuous in the corresponding closed region, except perhaps for poles of total order not greater than n exterior to C. We write*

$$f_n(z) \equiv g_n(z) + h_n(z), \tag{1}$$

for z on and exterior to C, where $g_n(z)$ is a rational function of z with all its poles exterior to C and $h_n(z)$ is analytic in the extended plane exterior to C, continuous in the corresponding closed region, with $h_n(\infty)=0$. If the totality of poles of all the $f_n(z)$ have no limit point on C, and if we have

$$\limsup_{n\to\infty} [\max |f(z) - f_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R < 1, \tag{2}$$

then we have also

$$\limsup_{n\to\infty} [\max |f(z) - g_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R, \tag{3}$$

$$\limsup_{n\to\infty} [\max |h_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R. \tag{4}$$

The significance of this theorem is in part as follows. In numerous situations a function $f(z)$ is approximated on a Jordan curve C by a function of type $f_n(z)$; for instance $f(z)$ may be approximated on the unit circle C: $|z|=1$ by a trigonometric polynomial in the arc length, of order n, which is of type $f_n(z)$. If this approximating function $f_n(z)$ is split into the two components $g_n(z)$ and $h_n(z)$, what does $h_n(z)$ contribute to the degree of approximation? This question is answered by the theorem, asserting that asymptotically $h_n(z)$ contributes nothing to the degree of approximation, insofar as approximation is measured by the first member of (2); it is no more favorable to approximate

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to $f(z)$ on C by functions $f_n(z)$ than by their respective components $g_n(z)$. As a matter of comparison, we note that if C is the unit circle, each of the functions $h_n(z)$ is orthogonal[1] to $f(z)$ on C, and consequently *in the sense of least squares* no $h_n(z)$ can contribute to approximation to $f(z)$ on C.

The present problem, approximation by functions not analytic in a given region to a function analytic in that region, is in a sense a companion piece to a problem studied on various occasions[2] by the present writer, namely approximation by functions analytic in a region to a function not analytic in that region.

The theorem is of course essentially more general than approximation on the unit circle. The theorem is not invariant under arbitrary conformal transformation of either the interior or exterior of C; such a transformation does not leave unchanged the components of $f_n(z)$ indicated in (1).

In connection with this theorem, it is to be noted[3] that any function $f(z)$ which satisfies the given hypothesis and is analytic also on C can be uniformly approximated on C so that (2) is satisfied for suitably chosen R, by rational functions $f_n(z) \equiv g_n(z)$ of respective degrees n with poles in n prescribed points exterior to C, the prescribed poles having no limit point on C.

We proceed to the proof of the theorem. Let Γ_1 and Γ_2 be two analytic Jordan curves chosen with Γ_2 interior to Γ_1, and C interior to Γ_2, and so chosen that all the poles of every $f_n(z)$ lie exterior to Γ_1. For z between Γ_1 and Γ_2 we have

$$f_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f_n(t)dt}{t - z} + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f_n(t)dt}{t - z},$$

the integrals being extended in the positive sense with respect to the annular region bounded by Γ_1 and Γ_2. The function

$$g_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f_n(t)dt}{t - z}, \qquad z \text{ interior to } \Gamma_1, \tag{5}$$

is defined and analytic throughout the interior of Γ_1, and the function

$$h_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f_n(t)dt}{t - z}, \qquad z \text{ exterior to } \Gamma_2, \tag{6}$$

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[1] See for instance Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1935, §6.11.

[2] Loc. cit. §§6.11 et seq., §§11.3 et seq.

[3] This result is due to Shen; loc. cit. §9.7.

is defined and analytic in the extended plane exterior to Γ_2, with $h_n(\infty)=0$. The second members of (5) and (6) for fixed z are independent of continuous deformation of Γ_1 and Γ_2 provided that Γ_1 and Γ_2 in the deformation do not cross z or singularities of $f_n(z)$.

The equation

$$f_n(z) \equiv g_n(z) + h_n(z), \tag{7}$$

valid for z between Γ_1 and Γ_2, is also valid wherever the functions involved can be defined by analytic extension. It follows from (7) that in the extended plane exterior to Γ_2 the singularities of $g_n(z)$ are precisely the singularities of $f_n(z)$, namely poles of total order not greater than n; moreover $g_n(z)$ is analytic interior to Γ_1, hence is a rational function of order not greater than n. The function $h_n(z)$, defined by (6) and suitably extended analytically by (7), is analytic in the extended plane exterior to C, continuous in the corresponding closed region.

The separation of $f_n(z)$ into components as defined by (1) is necessarily unique, for if we have in the closed exterior of C

$$g_n^{(1)}(z) + h_n^{(1)}(z) \equiv g_n^{(2)}(z) + h_n^{(2)}(z),$$

where $g_n^{(1)}(z)$ and $g_n^{(2)}(z)$ satisfy the prescribed conditions for $g_n(z)$ and $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$ satisfy the prescribed conditions for $h_n(z)$, then the function $g_n^{(1)}(z)-g_n^{(2)}(z)$ is rational with its poles (if any) exterior to C, but is also equal to $h_n^{(1)}(z)-h_n^{(2)}(z)$ exterior to C, hence is analytic in the entire extended plane, with the relation $g_n^{(1)}(\infty)-g_n^{(2)}(\infty)=h_n^{(2)}(\infty)-h_n^{(2)}(\infty)=0$; thus we have $g_n^{(1)}(z)\equiv g_n^{(2)}(z)$ and $h_n^{(1)}(z)\equiv h_n^{(2)}(z)$.

We denote generically by C_ρ the image in the z-plane of the circle $|w|=\rho>1$ under the conformal map of the exterior of C onto the region $|w|>1$, with the points at infinity in the two planes corresponding to each other. Let R_1 be chosen so that the poles of all the $f_n(z)$ exterior to C lie exterior also to C_{R_1}.

Choose M and R_0, $1<R_0<R$, so that for $n=1, 2, \cdots$ we have from (2)

$$|f(z) - f_n(z)| \leqq M/R_0^n, \qquad z \text{ on } C;$$

we have also

$$|f(z) - f_{n+1}(z)| \leqq M/R_0^{n+1}, \qquad z \text{ on } C,$$

whence

$$|f_n(z) - f_{n+1}(z)| \leqq 2M/R_0^n, \qquad z \text{ on } C,$$

where $f_n(z)-f_{n+1}(z)$ has at most a total of $2n+1$ poles exterior to C; all such poles lie exterior to C_{R_1}. By consideration of the function $f_n(z)-f_{n+1}(z)$ under the conformal map of the exterior of C in the z-plane onto the exterior of the unit circle in the w-plane, we have[4]

$$(8)\qquad |f_n(z)-f_{n+1}(z)| \leqq \frac{2M}{R_0^n}\left(\frac{R_1\rho-1}{R_1-\rho}\right)^{2n+1} \quad \text{for } 1<|w|=\rho<R_1.$$

Choose ρ so near unity that we have

$$(9)\qquad \frac{1}{R_0}\left(\frac{R_1\rho-1}{R_1-\rho}\right)^2<1.$$

It then follows from (8) that the sequence $f_n(z)$ converges uniformly on C_ρ. By (2) the sequence $f_n(z)$ converges to $f(z)$ uniformly on C, hence converges uniformly in the closed annulus bounded by C and C_ρ, to a limit which in the annulus we now denote by $f(z)$. It is a consequence of the rectifiability of C that the function $f(z)$ as just enlarged in definition is analytic also on C itself, hence analytic throughout the interior of C_ρ. By virtue of (8) and (9) we can write

$$(10)\qquad |f(z)-f_n(z)| \leqq \frac{M_1}{R_0^n}\left(\frac{R_1\rho-1}{R_1-\rho}\right)^{2n}, \qquad z \text{ on } C_\rho,$$

where M_1 depending on ρ is suitably chosen.

If we now set

$$(11)\qquad f(z)=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(t)dt}{t-z}, \qquad z \text{ interior to } C_\rho,$$

and also choose Γ_1 in (5) as C_ρ, inequality (3) follows from (5) and (11) by virtue of (10); it will be noted that (10) implies

$$(12)\qquad \limsup_{n\to\infty}\left[\max |f(z)-g_n(z)|, z \text{ on } C\right]^{1/n} \leqq \frac{1}{R_0}\left(\frac{R_1\rho-1}{R_1-\rho}\right)^2,$$

and in the second member of (12) we can allow R_0 to approach R and ρ to approach unity.

Inequality (4) is a consequence of inequalities (2) and (3), by virtue of (1). The theorem is established.

COROLLARY. *The theorem remains valid if C is not assumed rectifiable, but if $f(z)$ is assumed analytic in the closed interior of C.*

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[4] Loc. cit. p. 231, Lemma I. The requirement in this Lemma I that $P(z)$ be rational may be replaced by the requirement that $P(z)$ be continuous on $|z|=r$, analytic in $r<|z|<\rho r$, meromorphic with not more than n poles in $|z|\geqq\rho r$.

The proof of the corollary follows directly the proof of the theorem; in (9) we choose ρ so near unity that $f(z)$ is analytic on and within C_ρ; the identity in the annulus bounded by C and C_ρ of the given function $f(z)$ and the limit of the sequence $f_n(z)$ follows from the fact that these two functions analytic in the annulus approach the same set of boundary values when z exterior to C approaches C.

Theorem and corollary can be extended to include point sets C much more general than Jordan curves, even the case where C consists of several disjoint continua; these extensions are left to the reader.

The theorem becomes false if the requirement is omitted that C be rectifiable and if no additional requirement is placed on $f(z)$. For under such conditions C may be chosen (Osgood) to have positive area; there exists (Pompeiu) a function $f(z)$ analytic in the extended plane except on C, continuous for all values of z, zero at infinity yet not identically zero. We can choose $f_n(z) \equiv f(z)$, $n=1, 2, \cdots$, from which it follows that (2) is satisfied for arbitrary R. Yet we have $g_n(z) \equiv 0$, $h_n(z) \equiv f_n(z) = f(z)$, so that the first member of both (3) and (4) can be written

$$\limsup_{n\to\infty} [\max |f(z)|, z \text{ on } C]^{1/n},$$

which has the value unity. Inequalities (3) and (4) of the conclusion of the theorem are not satisfied.

HARVARD UNIVERSITY

ON THE NUMBER OF 1-1 DIRECTLY CONFORMAL MAPS WHICH A MULTIPLY-CONNECTED PLANE REGION OF FINITE CONNECTIVITY p (>2) ADMITS ONTO ITSELF

MAURICE HEINS

1. Introduction.

It is well known[1] that a plane multiply-connected region G of finite connectivity greater than two admits only a finite number of 1-1 directly conformal maps *onto* itself (such maps will be termed henceforth *conformal automorphisms* of G); in fact, if G is of connectivity $p(>2)$, then the number of conformal automorphisms of G can in no case exceed $p(p-1)(p-2)$. The object of the present note is to determine the *best* upper bound, $N(p)$, for the number of conformal automorphisms of G as a *function of the connectivity* p. The basic theorems are:

THEOREM A. *The group of conformal automorphisms of a plane region of finite connectivity* $p(>2)$ *is isomorphic to one of the finite groups of linear fractional transformations of the extended plane onto itself.*

THEOREM B. *If* $p(>2)$ *is different from* 4, 6, 8, 12, 20, *then* $N(p)=2p$. *For the exceptional values of* p, *one has*

$$N(4) = 12, \qquad N(6) = N(8) = 24, \qquad N(12) = N(20) = 60.$$

The proofs of these theorems are based upon the following results:[2]

I. *An arbitrary plane region* G *of finite connectivity* p *admits a* 1-1 *directly conformal map onto a canonical plane region* G^* *whose boundary consists of points and complete circles (either possibly absent), in all* p *in number, and mutually disjoint.*

If $\mathcal{G}$ and $\mathcal{G}^*$ denote the groups of conformal automorphisms of G and G^* respectively, then $\mathcal{G}$ is isomorphic to $\mathcal{G}^*$. Hence for the purposes of the present problem it suffices to consider the canonical regions and their associated groups of conformal automorphisms.

II. *A conformal automorphism of a canonical region* G^* *admits an extension in definition throughout the extended complex plane as a linear fractional transformation.*

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[1] Cf. G. Julia, *Leçons sur la représentation conforme des aires multiplement connexes*, Paris, 1934. In particular, see pp. 68–69.

[2] Cf. Hurwitz-Courant, *Funktionentheorie*, Berlin, 1929. See pp. 512–520.

Hence the group $\mathcal{G}^*$ is in essence a *finite group* of linear fractional transformations of the extended plane onto itself. In the next section it will be shown that there exists a region Γ which lies in the extended complex plane, is bounded by p distinct points, and in addition

(i) contains G^* as a subregion,

(ii) remains *invariant* under the automorphisms of $\mathcal{G}^*$.

It will then follow that it suffices to consider the problem for regions bounded by p distinct points. The proof of Theorem B is thus reduced to the determination of the connectivities of regions Γ which are bounded by a finite set of points and which remain invariant under the members of a given finite group of linear fractional transformations of the extended complex plane onto itself.

2. Reduction of problem to the case where the region is bounded by p distinct points. We start then with a canonical region G^* whose boundary consists of p disjoint components which are either points or circles and the associated group of conformal automorphisms $\mathcal{G}^*$. Boundary components which consist of points will be unaltered. If there are circles present among the boundary components, say $\beta_1, \beta_2, \cdots, \beta_m$ $(1 \leqq m \leqq p)$, one proceeds as follows. Suppose β_k $(1 \leqq k \leqq m)$ is carried into itself by some transformation $S \in \mathcal{G}^*$ other than the identity. Note that S is an *elliptic* linear fractional transformation and hence possesses a unique fixed point ζ_k in the region G_k of the extended plane bounded by β_k which is exterior to G^*. Further any transformation of $\mathcal{G}^*$ which carries β_k into itself possesses ζ_k as a fixed point since the subgroup of $\mathcal{G}^*$ whose members preserve β_k is *cyclic*. For any such transformation (not the identity) ζ_k is the unique fixed point in g_k. In this case we replace β_k by the point ζ_k. If β_k is not carried into itself by any transformation of $\mathcal{G}^*$ other than the identity, then β_k and its images with respect to the transformations of $\mathcal{G}^*$ constitute a set of n disjoint circles, where n is the order of $\mathcal{G}^*$. These circles are permuted among themselves by the transformations of $\mathcal{G}^*$. To replace these circles by points, we select any one of them—say β_{k_0}—and fix a point η_{k_0} on β_{k_0} replacing thereby β_{k_0} by η_{k_0}. The image of β_{k_0} with respect to a transformation of $\mathcal{G}^*$ is to be replaced by the image of η_{k_0} with respect to the same transformation of $\mathcal{G}^*$. In this manner G^* is replaced by a region $\Gamma \supset G^*$ of connectivity p whose boundary consists of p distinct points. It is readily verified that Γ remains invariant with respect to the transformations of $\mathcal{G}^*$.

Hence, to determine

$$N(p) \equiv \max_{\{G\}} [\text{order } \mathcal{G}(G)],$$

where G is of connectivity p and $\mathcal{G}(G)$ is the group of conformal automorphisms of G, it suffices to consider regions G bounded by p distinct points.

3. **An observation.** It is to be observed that $N(p)$ is *bounded below by* $2p$. This follows from the fact that the region in the extended z-plane whose boundary consists of the points

$$z = e^{2\pi i k/p} \qquad (k = 0, 1, 2, \cdots, p-1)$$

is carried into itself by the dihedral group of order $2p$ generated by

$$S: z \mid 1/z, \qquad T: z \mid e^{2\pi i/p} z.$$

This fact will be significant for determining $N(p)$.

4. **Determination of $N(p)$.** Given a positive integer p, a finite group $\mathcal{G}$ of linear fractional transformations of the extended complex plane onto itself will be termed *admissible relative to p*, if there exists a region Γ which is bounded by p distinct points and remains invariant under the transformations of $\mathcal{G}$. Given $\mathcal{G}$, the integers p for which $\mathcal{G}$ is *admissible* are listed in the following table:[3]

TABLE 1

If $\mathcal{G}$ is isomorphic to	then p for which $\mathcal{G}$ is admissible are given by	
Cyclic group of order n	$n\left[\frac{a}{n}+\frac{b}{1}\right]$	where $a=0, 1, 2$ $b=0, 1, 2, \cdots; a+b>0$
Dihedral group of order $2n$	$2n\left[\frac{a}{n}+\frac{b}{2}+\frac{c}{1}\right]$	where $a=0, 1$ $b=0, 1$ $c=0, 1, 2, \cdots; a+b+c>0$
Tetrahedral group	$12\left[\frac{a}{3}+\frac{b}{2}+\frac{c}{1}\right]$	where $a=0, 1, 2$ $b=0, 1$ $c=0, 1, 2, \cdots; a+b+c>0$
Octahedral group	$24\left[\frac{a}{4}+\frac{b}{3}+\frac{c}{2}+\frac{d}{1}\right]$	where $a=0, 1$ $b=0, 1$ $c=0, 1$ $d=0, 1, 2, \cdots; a+b+c+d>0$
Icosahedral group	$60\left[\frac{a}{5}+\frac{b}{3}+\frac{c}{2}+\frac{d}{1}\right]$	where $a=0, 1$ $b=0, 1$ $c=0, 1$ $d=0, 1, 2, \cdots; a+b+c+d>0$

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[3] This table is readily verified on reference to the classical results of the theory of finite groups of linear fractional transformations.

Recall that all finite groups $\mathcal{G}$ of linear fractional transformations of the extended complex plane onto itself are considered in Table 1. Since $N(p) \geqq 2p$, it suffices to consider groups $\mathcal{G}$ admissible relative to $p(>2)$ *whose orders are at least* $2p$. These are readily determined from Table 1 and are given below together with their orders in Table 2.

TABLE 2

p	Groups $\mathcal{G}$ admissible relative to p and of order $\geqq 2p$	Order of $\mathcal{G}$
$\neq 4, 6, 8, 12, 20, 30$	Dihedral	$2p$
4	Tetrahedral	12
	Dihedral	8
6	Tetrahedral	12
	Octahedral	24
	Dihedral	12
8	Octahedral	24
	Dihedral	16
12	Octahedral	24
	Icosahedral	60
	Dihedral	24
20	Icosahedral	60
	Dihedral	40
30	Icosahedral	60
	Dihedral	60

Theorem B follows at once from Table 2.

Remark. It would be interesting to deduce Theorems A and B without using the canonical regions G^*.

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SOME PROPERTIES OF ABSOLUTELY MONOTONIC FUNCTIONS

P. C. ROSENBLOOM

In this note we collect several fragmentary results which were obtained as by-products of another investigation. They are rather loosely connected with each other, but still may be of some interest.

We recall that a function $f(x_1, \cdots, x_k)$ is said to be absolutely monotonic in a set D if f and all its partial derivatives exist and are non-negative in D. If D is of the form $0 \leqq x_i < a_i$, $i=1, \cdots, k$, then a necessary and sufficient condition that f be absolutely monotonic in D is that it can be expanded in a power series in $x_1, \cdots, x_k$ with non-negative coefficients converging in D. (The well known theorem of Bernstein [1][1] for the case $k=1$ can be extended in a trivial manner.)

THEOREM 1. *If $f(x)$ is absolutely monotonic in $0 \leqq x < a$, and if $0 \leqq x_1, x_2, \cdots, x_n < a$, and if $L(x)$ is the Lagrange interpolation polynomial of $f(x)$ at the points $x_1, \cdots, x_n$, then*

$$g(x) = \frac{f(x) - L(x)}{\omega(x)}, \qquad \omega(x) = (x - x_1) \cdots (x - x_n),$$

is an absolutely monotonic function of $x, x_1, \cdots, x_n$ in the range $0 \leqq x, x_1, \cdots, x_n < a$.

PROOF. The function $g(x)$ can be expressed as a divided difference of $f(x)$ (see for example, Milne-Thompson [2]):

$$g(x) = [x x_1 \cdots x_n],$$

where

$$[x x_1] = \frac{f(x) - f(x_1)}{x - x_1},$$

and

$$[x x_1 \cdots x_k] = \frac{[x x_1 \cdots x_{k-1}] - [x_k x_1 \cdots x_{k-1}]}{x - x_k}, \qquad k = 2, \cdots, n.$$

It is sufficient, then, to show that if $f(x)$ is absolutely monotonic in $0 \leqq x < a$ then

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$\frac{f(x)-f(x_1)}{x-x_1}$$

is absolutely monotonic in the square $0 \leqq x,\ x_1 < a$. But if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \geqq 0$, then

$$\frac{f(x)-f(x_1)}{x-x_1} = \sum_{n=1}^{\infty} a_n (x^{n-1} + x_1 x^{n-2} + \cdots + x_1^{n-1})$$

is a power series with non-negative coefficients converging for $0 \leqq x$, $x_1 < a$, and is therefore absolutely monotonic there.

COROLLARY. *We have* $f(x) \geqq L(x)$ *for* x *in the intervals* $[x_n, a]$, $[x_{n-2}, x_{n-1}]$, $\cdots$, *while* $f(x) \leqq L(x)$ *in the intervals* $[x_{n-1}, x_n]$, $[x_{n-3}, x_{n-2}]$, $\cdots$; *if the equality sign holds at an interior point of any of these intervals, it holds identically.*

It will be convenient to introduce the notations

$$V(k_1, \cdots, k_n) = \begin{vmatrix} x_1^{k_1} & \cdots & x_1^{k_n} \\ x_n^{k_1} & \cdots & x_n^{k_n} \end{vmatrix},$$

and

$$A(k_1, \cdots, k_n) = \frac{V(k_1, \cdots, k_n)}{V(0, 1, \cdots, n-1)}.$$

LEMMA. *If* $0 \leqq k_1 < k_2 < \cdots < k_n$, *where* $k_1, \cdots, k_n$ *are integers, then* $A(k_1, \cdots, k_n)$ *is a symmetric polynomial in* $x_1, \cdots, x_n$, *with non-negative coefficients.*

PROOF. It is obvious that A is a symmetric function. For $n=1$, $A(k_1) = x_1^{k_1}$, and the lemma is true. Suppose now that $n > 1$. Then setting $m_i = k_{i+1} - k_1$, we have

$$\begin{aligned} A(k_1, \cdots, k_n) &= x_1^{k_1} \cdots x_n^{k_1} A(0, m_1, \cdots, m_{n-1}) \\ &= \frac{x_1^{k_1} \cdots x_n^{k_1} \begin{vmatrix} 0, & x_1^{m_1} - x_n^{m_1}, & \cdots, & x_1^{m_{n-1}} - x_n^{m_{n-1}} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & x_{n-1}^{m_1} - x_n^{m_1}, & \cdots, & x_{n-1}^{m_{n-1}} - x_n^{m_{n-1}} \\ 1, & x_n^{m_1}, & \cdots, & x_n^{m_{n-1}} \end{vmatrix}}{(x_n - x_1) \cdots (x_n - x_{n-1}) V(0, 1, \cdots, n-2)} \\ &= \frac{x_1^{k_1} \cdots x_n^{k_1} \begin{vmatrix} x_1^{m_1-1} + x_n x_1^{m_1-2} + \cdots + x_n^{m_1-1}, & \cdots \\ \cdot & \cdot \end{vmatrix}}{V(0, 1, \cdots, n-2)}. \end{aligned}$$

Now we multiply the first column in the numerator by $x_n^{m_2-m_1}$ and subtract from the second; then we multiply the first column by $x_n^{m_3-m_1}$ and the second by $x_n^{m_3-m_2}$ and subtract from the third and so on. We thus obtain

$$A(k_1, \cdots, k_n)=\frac{x_1^{k_1} \cdots x_n^{k_1}}{V(0, 1, \cdots, n-2)}$$

$$\cdot\begin{vmatrix} x_1^{m_1-1}+x_nx_1^{m_1-2}+\cdots+x_n^{m_1-1}, & x_1^{m_2-1}+\cdots+x_n^{m_2-m_1-1}x_1^{m_1}, & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

$$=x_1^{k_1}\cdots x_n^{k_1}\sum x_n^{\beta}A(\alpha_1, \cdots, \alpha_{n-1}),$$

where $A(\alpha_1, \cdots, \alpha_{n-1})$ is formed with the indicated exponents and the variables $x_1, \cdots, x_{n-1}$, and the summation runs over all systems of α's satisfying

$$0 \leqq \alpha_1 \leqq m_1 - 1, m_1 \leqq \alpha_2 \leqq m_2 - 1, \cdots, m_{n-2} \leqq \alpha_{n-1} \leqq m_{n-1} - 1,$$

and

$$\beta = (m_1 - 1 - \alpha_1) + \cdots + (m_{n-1} - 1 - \alpha_{n-1}).$$

Hence if the lemma is true for $n-1$, it is true for n. The lemma follows, then, by induction.

THEOREM 2. *If $f(x)$ and $g(x)$ are absolutely monotonic in the interval $0 \leqq x < a$, then*

$$\frac{1}{V(0, 1, 2)}\begin{vmatrix} 1 & f(ux_1) & g(vx_2x_3) \\ 1 & f(ux_2) & g(vx_3x_1) \\ 1 & f(ux_3) & g(vx_1x_2) \end{vmatrix}$$

is an absolutely monotonic function of its five arguments for $0 \leqq x_1, x_2, x_3 < a$, $0 \leqq u \leqq 1/a$, $0 \leqq v \leqq 1/a^2$.

PROOF. Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$, $g(x)=\sum_{n=0}^{\infty}b_nx^n$, $a_n \geqq 0$, $b_n \geqq 0$, $n=0, 1, \cdots$. Then

$$\frac{1}{V(0, 1, 2)}\begin{vmatrix} 1 & u^mx_1^m & g(vx_2x_3) \\ 1 & u^mx_2^m & g(vx_3x_1) \\ 1 & u^mx_3^m & g(vx_1x_2) \end{vmatrix}$$

$$= \sum_{n=0}^{\infty}\frac{b_n}{V(0, 1, 2)}\begin{vmatrix} 1 & u^mx_1^m & v^nx_2^nx_3^n \\ 1 & u^mx_2^m & v^nx_3^nx_1^n \\ 1 & u^mx_3^m & v^nx_1^nx_2^n \end{vmatrix}.$$

Now

$$\begin{vmatrix} 1 & u^m x_1^m & v^n x_2^n x_3^n \\ \multicolumn{3}{c}{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot} \end{vmatrix} = \frac{1}{x_1^n x_2^n x_3^n} \begin{vmatrix} x_1^n & u^n x_1^{m+n} & v_n x_1^n x_2^n x_3^n \\ \multicolumn{3}{c}{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot} \end{vmatrix}$$

$$= u^m v^n V(n, m+n, 0).$$

Hence

$$\frac{1}{V(0,1,2)} \begin{vmatrix} 1 & f(ux_1) & g(vx_2x_3) \\ 1 & f(ux_2) & g(vx_3x_1) \\ 1 & f(ux_3) & g(vx_1x_2) \end{vmatrix} = \sum_{m,n=0}^{\infty} a_m b_n u^m v^n A(0, n, m+n).$$

If the terms of the last series are rearranged we obtain a power series in the five variables with non-negative coefficients converging in the range specified above.

THEOREM 3. *If* $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n > 0$, *and* a_{n+1}/a_n *is monotonically non increasing, and if* $f(x)$ *and* $g(x)$ *are absolutely monotonic in* $0 \leqq x < a$, *then*

$$\frac{1}{V(0,1,2)} \begin{vmatrix} 1 & f(x_1) & f(x_1)g(x_1) \\ 1 & f(x_2) & f(x_2)g(x_2) \\ 1 & f(x_3) & f(x_3)g(x_3) \end{vmatrix}$$

is an absolutely monotonic function of all three variables for $0 \leqq x_1, x_2, x_3 < a$.

PROOF. Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$, $b_n \geqq 0$, $0 \leqq x < a$. Let $0 \leqq x_1, x_2, x_3 \leqq r < a$. Then

$$\frac{1}{V(0,1,2)} \begin{vmatrix} 1 & f(x_1) & f(x_1)g(x_1) \\ 1 & f(x_2) & f(x_2)g(x_2) \\ 1 & f(x_3) & f(x_3)g(x_3) \end{vmatrix}$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{a_n a_m}{V(0,1,2)} \begin{vmatrix} 1 & x_1^m & x_1^n g(x_1) \\ 1 & x_2^m & x_2^n g(x_2) \\ 1 & x_3^m & x_3^n g(x_3) \end{vmatrix}$$

$$= \sum_{m=1}^{\infty} \sum_{n+q=2}^{\infty} a_n a_m b_q A(0, m, n+q).$$

Now by elementary estimates

$$\begin{aligned} |A(0, m, n+q)| &\leqq 2^{-1} m(n+q)(m+n+q-2) r^{m+n+q-3} \\ &\leqq m^2 (n+q)(n+q-1) r^{m+n+q-3} \end{aligned}$$

if $m \geqq 1$, $n+q \geqq 2$ and $|a_n| \leqq CR^{-n}$, $|b_n| \leqq CR^{-n}$, where $r < R < a$, and C is a suitably chosen constant. Then this series is dominated by

$$C^2r^{-3}\sum_{m=1}^{\infty}\sum_{n+q=2}^{\infty} m^2(n+q)(n+q-1)(r/R)^{m+n+q}$$
$$= C^2r^{-3}\left(\sum_{m=1}^{\infty} m^2(r/R)^m\right)\left(\sum_{k=2}^{\infty} k(k+1)(k-1)(r/R)^k\right),$$

and is therefore absolutely convergent. Hence we can rearrange it as follows:

$$\sum_{m=0}^{\infty} a_m\left\{-\sum_{\alpha=0}^{m-1} A(0,\alpha,m)\sum_{q=0}^{\alpha} a_{\alpha-q}b_q + \sum_{\beta=m+1}^{\infty} A(0,m,\beta)\sum_{q=0}^{\beta} a_{\beta-q}b_q\right\}$$
$$= \sum_{0<\alpha<\beta} A(0,\alpha,\beta)\left\{\sum_{q=0}^{\infty} b_q a_\alpha a_\beta\left(\frac{a_{\beta-q}}{a_\beta}-\frac{a_{\alpha-q}}{a_\alpha}\right)\right\},$$

and the latter can be rearranged as a power series in x_1, x_2, and x_3. But

$$\frac{a_{\alpha-q}}{a_\alpha} = \frac{a_{\alpha-q}}{a_{\alpha-q+1}}\cdot\frac{a_{\alpha-q+1}}{a_{\alpha-q+2}}\cdot\;\cdots\;\cdot\frac{a_{\alpha-1}}{a_\alpha} \leqq \frac{a_{\beta-q}}{a_{\beta-q+1}}\cdot\;\cdots\;\cdot\frac{a_{\beta-1}}{a_\beta} = \frac{a_{\beta-q}}{a_\beta}.$$

Therefore the resulting power series has only non-negative coefficients.

We feel that the last two theorems are at most mere curiosities as they stand, since we haven't any idea as to what significance the above determinants may have. It is to be hoped that proofs can be found which will be more illuminating than the above purely computational ones.

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TOEPLITZ METHODS WHICH SUM A GIVEN SEQUENCE

P. ERDÖS AND P. C. ROSENBLOOM

The following note arose out of discussions of a paper by Agnew, but is, however, self-contained.

THEOREM. *Let* $\{x_n\}$ *be a bounded divergent sequence. Suppose that* $\{y_n\}$ *is summable by every regular Toeplitz method which sums* $\{x_n\}$. *Then* $\{y_n\}$ *is of the form* $\{cx_n+a_n\}$ *where* $\{a_n\}$ *is convergent.*

PROOF. For typographical convenience we shall often write $x(n)$ for x_n, and so on. Let $\{x(n_k)\}$, $k=1, 2, \cdots$, be any convergent subsequence of $\{x_n\}$. Then $\{x_n\}$ is summable by the matrix $(a(n, k))$ where $a(n, k)=1$ for $n=n_k$ and $a(n, k)=0$ for $n\neq n_k$. Hence $\{y(n_k)\}$ is also convergent.

Let $\{n_k'\}$ and $\{n_k''\}$ be sequences of integers such that $n_k'\neq n_k''$ for all k and

$$\lim_{k\to\infty} x(n_k') = A, \qquad \lim_{k\to\infty} x(n_k'') = B, \qquad A \neq B.$$

These sequences $\{n_k'\}$ and $\{n_k''\}$ will be held fixed throughout the rest of the argument. Then the sequences $\{y(n_k')\}$ and $\{y(n_k'')\}$ are also convergent, say to α and β respectively. Let $\{x(n_k)\}$ be an arbitrary convergent subsequence of $\{x_n\}$ with the limit C. Let λ and μ be determined by the equations

$$\lambda + \mu = 1, \qquad \lambda A + \mu B = C.$$

Then the matrix $(b(n, k))$ with

$$b(n, k) = \begin{cases} \lambda, & n = n_k', \quad k \text{ even}, \\ \mu, & n = n_k'', \quad k \text{ even}, \\ 1, & n = n_k, \quad k \text{ odd}, \\ 0, & \text{for all other values of } n \text{ and } k, \end{cases}$$

sums $\{x_n\}$ to the limit C. Hence it also sums $\{y_n\}$, that is

$$\lim_{k\to\infty} y(n_k) = \lim_{k\to\infty} (\lambda y(n_k') + \mu y(n_k'')) = \lambda\alpha + \mu\beta.$$

Note that the numbers λ and μ depend only on C and not on the particular subsequence $\{x(n_k)\}$ converging to C, and hence $\lim_{k\to\infty} y(n_k)$

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also depends only on C, and is, in fact, a linear function of C.

We now determine constants m and a from the equations

$$\alpha = mA + a, \qquad \beta = mB + a.$$

Let $\{n_k\}$ be an arbitrary sequence of positive integers, and let $\{n_k'''\}$ be a subsequence of $\{n_k\}$ such that $\{x(n_k''')\}$ converges, say to C. We determine λ and μ as before Then

$$\begin{aligned}\lim_{k\to\infty} (y(n_k'') - mx(n_k'')) &= \lambda\alpha + \mu\beta - mC \\ &= \lambda(mA + a) + \mu(mB + a) - mC = a.\end{aligned}$$

Thus every subsequence of $\{y_n - mx_n\}$ contains a subsubsequence converging to a. Hence $\lim_{n\to\infty} (y_n - mx_n) = a$, which proves our theorem.

COROLLARY. *If* $\{x_n\}$ *and* $\{y_n\}$ *are bounded divergent sequences, and* $\{y_n\}$ *is summable by every regular Toeplitz method which sums* $\{x_n\}$, *then* $\{x_n\}$ *is summable by every regular Toeplitz method which sums* $\{y_n\}$.

By a theorem of Agnew,[1] there is no single Toeplitz method which has the sequences of the form $\{cx_n + a_n\}$ as its convergence field. The above theorem shows, however, that this set of sequences is the common part of the convergence fields of Toeplitz methods which sum $\{x_n\}$.

Added November 11, 1945 We have just had the opportunity of seeing the paper of A. Brudno, *Summation of bounded sequences by matrices*, Rec. Math. (Mat. Sbornik) N.S. vol. 16 (1945), pp. 191–247. From the English summary it seems that our result is contained in his Theorem 11, p. 236. His Theorem 11 can clearly be proved by our method. It is difficult to compare the simplicity of the proofs.

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[1] R. P. Agnew, *Convergence fields of methods of summability*, Ann. of Math. vol. 46 (1945) pp. 93–101.

A SUFFICIENCY THEOREM FOR DIFFERENTIAL SYSTEMS

AUGUSTO BOBONIS

1. Introduction. This paper is concerned with a boundary value problem involving differential equations and boundary conditions of the form

$$\begin{aligned} y_i' &= [A_{ij}(x) + \lambda B_{ij}(x)]y_j, \\ (1.1)\quad s_i[y, \lambda] &\equiv (\overset{0}{M}_{ij} + \lambda \overset{1}{M}_{ij})y_j(a) + (\overset{0}{N}_{ij} + \lambda \overset{1}{N}_{ij})y_j(b) = 0 \\ &\qquad (a \leqq x \leqq b;\ i, j = 1, 2, \cdots, n), \end{aligned}$$

where the matrix of constants $\|M_{ij}^0+\lambda M_{ij}^1, N_{ij}^0+\lambda N_{ij}^1\|$ has rank n for all values of the characteristic parameter λ. In his dissertation the author [3][1] extended to such systems the concept of definite self-adjointness introduced by Bliss [2] for problems with boundary conditions independent of the parameter. Earlier, Bliss [1] had formulated a definition of definite self-adjoint systems in such a manner that systems of this type had infinitely many characteristic values. This property is in general no longer true for systems that are definitely self-adjoint in the modified sense of Bliss [2], and the analogous definition of Bobonis [3] is such that definitely self-adjoint systems (1.1) need not possess an infinitude of characteristic values. As shown in [3], however, for definitely self-adjoint systems (1.1) the characteristic values are all real and have indices equal to their multiplicities; moreover, such systems admit expansion theorems analogous to those obtained by Bliss [2].

It is the purpose of the present paper to consider a definitely self-adjoint system (1.1) which satisfies the additional condition that the matrix $\|B_{ij}(x)\|$ is of constant rank on the interval $a \leqq x \leqq b$. Such a system is shown to be equivalent to a boundary value problem associated with the second variation of a calculus of variations problem of the type considered by Reid [4], and the extremizing properties of the characteristic values of the equivalent problem lead to necessary and sufficient conditions for the given problem to have an infinitude of characteristic values. The methods of proof herein used are analogous to those employed by Reid [5] in establishing the corresponding results for definitely self-adjoint systems whose boundary conditions are independent of λ.

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[1] The numbers in brackets refer to the bibliography at the end of this paper.

2. Statement of the problem. In the present paper matrix notation will be used whenever possible. The subscripts i, j, k, α, will have the range $1, 2, \cdots, n$. Capital italic letters denote n-rowed square matrices, the element in the ith row and jth column being denoted by the same letter with the subscript ij. The vector (y_i) is denoted by the lower case letter y; Ay and yA represent the vectors $(A_{ij}y_j)$ and (y_jA_{ji}), respectively, where the repeated subscript j indicates a summation with respect to the subscript over the range $1, 2, \cdots, n$. The scalar product y_jz_j of the vectors y and z will be written yz. The transpose of the matrix A will be denoted by $\bar{A}$. Whenever the elements of the matrix A are differentiable functions, the matrix of derivatives is denoted by A'. For brevity, we shall also write $\mathcal{L}[y]$ and $\mathcal{M}[z]$ for the adjoint differential operators whose components $\mathcal{L}_i[y]$ and $\mathcal{M}_i[z]$ are given by

$$\mathcal{L}_i[y] = y_i' - A_{ij}(x)y_j, \qquad \mathcal{M}_i[z] = z_i' + z_jA_{ji}(x).$$

The elements of the matrices $\|A_{ij}(x)\|$, $\|B_{ij}(x)\|$ are real-valued and continuous functions in $a \leqq x \leqq b$ and the matrices $M(\lambda) = \|M_{ij}(\lambda)\| = \|M_{ij}^0 + \lambda M_{ij}^1\|$, $N(\lambda) = \|N_{ij}(\lambda)\| = \|N_{ij}^0 + \lambda N_{ij}^1\|$ are such that the elements of M^0, M^1, N^0, N^1 are real-valued and the $n \times 2n$ matrix $\|M_{ij}^0 + \lambda M_{ij}^1, N_{ij}^0 + \lambda N_{ij}^1\|$ has rank n for all values of the parameter λ. It is assumed that not all of the elements of M^1, N^1 are zero.

The system of equations with boundary conditions to be considered here is

$$\begin{aligned} y' &= [A(x) + \lambda B(x)]y; \\ s[y, \lambda] &\equiv s^0[y] + \lambda s^1[y] \\ &\equiv (M^0 + \lambda M^1)y(a) + (N^0 + \lambda N^1)y(b) = 0. \end{aligned} \tag{2.1}$$

Throughout the present paper we assume that the boundary conditions of (2.1) satisfy the following hypothesis:

For all values of λ *the* $n \times 2n$ *matrix* $\|M(\lambda)\ N(\lambda)\|$ *has rank* n. *Moreover, there exist matrices* $P(\lambda) = P^0 + \lambda P^1$, $Q(\lambda) = Q^0 + \lambda Q^1$, *together with matrices* M^*, N^*, P^*, Q^* *independent of* λ *such that the* $2n \times 2n$ *matrices*

$$\left\| \begin{matrix} -P^* & -P(\lambda) \\ Q^* & Q(\lambda) \end{matrix} \right\|, \qquad \left\| \begin{matrix} M(\lambda) & N(\lambda) \\ M^* & N^* \end{matrix} \right\| \tag{2.2}$$

are reciprocals for all values of λ.

In particular, the differential system adjoint to (2.1) is given by

$$\begin{aligned} z' &= -z[A(x) + \lambda B(x)]; \\ t[z, \lambda] &\equiv t^0[z] + \lambda t^1[z] \equiv z(a)[P^0 + \lambda P^1] + z(b)[Q^0 + \lambda Q^1] = 0. \end{aligned} \tag{2.3}$$

The condition that the matrices (2.2) are reciprocals leads to the following useful relations

$$
\begin{aligned}
&-P^*M^0 - P^0M^* = I, && Q^*M^0 + Q^0M^* = 0,\\
&-P^*M^1 - P^1M^* = 0, && Q^*M^1 + Q^1M^* = 0,\\
&-P^*N^0 - P^0N^* = 0, && Q^*N^1 + Q^1N^* = 0,\\
&-P^*N^1 - P^1N^* = 0, && Q^*N^0 + Q^0N^* = I.
\end{aligned} \tag{2.4}
$$

$$M(\lambda)P(\lambda) - N(\lambda)Q(\lambda) = 0, \tag{2.5}$$

$$-M^*P^0 + N^*Q^0 = I, \tag{2.6}$$

$$-M^0P^* + N^0Q^* = I, \tag{2.7}$$

$$-M^*P^* + N^*Q^* = 0, \tag{2.8}$$

$$
\begin{aligned}
&-M^*P^1 + N^*Q^1 = 0,\\
&-M^1P^* + N^1Q^* = 0.
\end{aligned} \tag{2.9}
$$

In the above $I=\|I_{ij}\|$ and $0=\|0_{ij}\|$ denote the $n\times n$ identity and 0 matrices respectively.

Setting

$$s^*[y] = M^*y(a) + N^*y(b), \qquad t^*[z] = z(a)P^* + z(b)Q^*,$$

we have, in view of equations (2.4), that

$$t^*[z]s[y,\lambda] + t[z,\lambda]s^*[y] = z(x)y(x)\Big|_a^b.$$

Consequently

$$t^*[z]s^0[y] + t^0[z]s^*[y] = z(x)y(x)\Big|_a^b, \tag{2.10}$$

$$t^*[z]s^1[y] + t^1[z]s^*[y] = 0, \tag{2.11}$$

for arbitrary sets $y(a)$, $y(b)$, $z(a)$, $z(b)$.

It is to be emphasized that the boundary conditions of our problem are unchanged if the matrices $M(\lambda)$ and $N(\lambda)$ are replaced by $\Gamma(\lambda)M(\lambda)$, $\Gamma(\lambda)N(\lambda)$ where the matrix $\Gamma(\lambda)$ is nonsingular for all values of λ and such that the product matrices $\Gamma(\lambda)M(\lambda)$, $\Gamma(\lambda)N(\lambda)$ remain linear in λ.

The hypotheses under which the boundary value problem is to be developed are the following:

(H_1) The system (2.1) is self-adjoint under the nonsingular transformation $z=T(x)y$, where $T(x)$ is a nonsingular matrix with real-valued elements of class C'.

(H_2) $S(x)=\bar{T}(x)B(x)$ is symmetric on $a\leqq x\leqq b$.

(H_3) The matrix

$$\left\| \begin{matrix} \bar{T}(a)P^*M^1 & \bar{T}(a)P^*N^1 \\ \bar{T}(b)Q^*M^1 & \bar{T}(b)Q^*N^1 \end{matrix} \right\| \tag{2.12}$$

belonging to the quadratic form $Q[y(a), y(b)]=t^*[Ty]s^1[y]$ is symmetric.

(H_4) The quadratic expression

$$K[y] \equiv Q[y(a), y(b)] + \int_a^b ySydx \tag{2.13}$$

is non-negative for arbitrary vectors $y(x)$ whose components are continuous on $a\leqq x\leqq b$.

(H_5) For an arbitrary value of λ the only solution of the system (2.1) satisfying $K[y]=0$ is $y(x)\equiv 0$ on $a\leqq x\leqq b$.

(H_6) $B(x)$ is of constant rank $n-m$, $0\leqq m<n$, on $a\leqq x\leqq b$.

The characteristic values of problems satisfying hypotheses (H_1)–(H_5) are all real and the zeros of a permanently convergent power series (see Bobonis [3]), so that there are at most a countable number of such values. Moreover since hypotheses (H_1)–(H_6) remain invariant under a linear change of parameter, it may be assumed without loss of generality that $\lambda=0$ is not a characteristic value of the problem considered.

Necessary and sufficient conditions for the system (2.1) to be self-adjoint under the transformation $z=Ty$ are

$$TA+\bar{A}T+T'=0, \qquad TB+\bar{B}T=0, \tag{2.14}$$

$$M(\lambda)T^{-1}(a)\bar{M}(\lambda) = N(\lambda)T^{-1}(b)\bar{N}(\lambda). \tag{2.15}$$

For the proof of the above conditions see Bobonis [3]. We have further that equations (2.5), (2.15), together with the hypothesis (H_3), justify the relations (Bobonis [3])

$$M(\lambda)T^{-1}(a) = C\bar{P}(\lambda), \qquad N(\lambda)T^{-1}(b) = C\bar{Q}(\lambda), \tag{2.16}$$

where C is a nonsingular constant matrix independent of λ.

We also have as a consequence of the previous hypotheses the following useful lemmas (Bobonis [3]).

LEMMA 2.1. *Hypotheses* (H_2) *and* (H_4) *imply that the quadratic form* $yS(x)y$ *is positive semidefinite on* $a\leqq x\leqq b$.

LEMMA 2.2. *Hypotheses* (H_3) *and* (H_4) *imply that the quadratic form* $Q[y(a), y(b)]$ *is positive semidefinite.*

LEMMA 2.3. *If for continuous* $y(x)$ *we have that* $\mathcal{K}[y]=0$, *then hypotheses* (H_2), (H_3), *and* (H_4) *imply* $By\equiv 0$ *on* $a\leqq x\leqq b$, $s^1[y]=0$.

The previous lemmas enable us to state that if hypotheses (H_2), (H_3), (H_4) are satisfied, hypothesis (H_5) is equivalent to:

(H_5') The only solution of $y'=Ay$, $s^0[y]=0$ satisfying $ySy\equiv 0$ on $a\leqq x\leqq b$, $s^1[y]=0$, is $y(x)\equiv 0$.

We can also see with the help of relations (2.10) and (2.11) that if $y(x)$ and $y^*(x)$ are solutions of (2.1) for distinct values of λ, then

$$t^*[Ty]s^1[y^*]+\int_a^b ySy^*dx=0. \tag{2.17}$$

As a consequence of hypothesis (H_6) it follows that there exist m sets of continuous functions $\Pi_{i\epsilon}(x)$ $(\epsilon=1, 2, \cdots, m)$, such that on $a\leqq x\leqq b$,

$$S_{ij}(x)\Pi_{j\epsilon}(x)\equiv 0. \tag{2.18}$$

The sets $\Pi_{i\epsilon}$ can be chosen orthonormal in the sense that

$$\Pi_{i\epsilon}(x)\Pi_{i\sigma}(x)\equiv I_{\epsilon\sigma} \qquad (\epsilon, \sigma=1, 2, \cdots, m).$$

In view of hypotheses (H_2) and (H_6) it follows that there exists an $n\times n$ matrix $R(x)$ such that the $(n+m)\times(n+m)$ matrices

$$\begin{Vmatrix} S_{ij}(x) & \Pi_{i\sigma}(x) \\ \Pi_{j\epsilon}(x) & 0_{\epsilon\sigma} \end{Vmatrix}, \qquad \begin{Vmatrix} R_{ij}(x) & \Pi_{i\sigma}(x) \\ \Pi_{j\epsilon}(x) & 0_{\epsilon\sigma} \end{Vmatrix}$$

are symmetric reciprocals on $a\leqq x\leqq b$.

Moreover, in view of Lemma 2.1 and relation (2.18), the quadratic form $R_{ij}(x)u_iu_j$ is positive for every non-null set (u_i) satisfying

$$\Pi_{j\epsilon}(x)u_j=0 \qquad (\epsilon=1, 2, \cdots, m).$$

The matrix

$$\begin{Vmatrix} P^*M^1T^{-1}(a) & P^*N^1T^{-1}(b) \\ Q^*M^1T^{-1}(a) & Q^*N^1T^{-1}(b) \end{Vmatrix} \tag{2.19}$$

is symmetric and positive semidefinite since it is obtained by multiplying the positive semidefinite matrix (2.12) on the left by the nonsingular matrix

$$\begin{Vmatrix} \overline{T}^{-1}(a) & 0 \\ 0 & \overline{T}^{-1}(b) \end{Vmatrix},$$

and by the transpose of the latter matrix on the right.

Multiplication of (2.19) on the left by the nonsingular matrix

$$\begin{Vmatrix} -M^0 & N^0 \\ -M^* & N^* \end{Vmatrix},$$

and by the transpose of the latter matrix on the right, with the use of equations (2.7), (2.8), (2.9) and (2.16), show that the matrix $(-M^0P^1+N^0Q^1)\overline{C}$ is symmetric and positive semidefinite. Consequently, the matrix $E=C^{-1}(-M^0P^1+N^0Q^1)$ is also symmetric and positive semidefinite. Equations (2.5), (2.6), together with the second, fourth, sixth and seventh equations of (2.4), imply $P^1=P^*(-M^0P^1+N^0Q^1)=P^*CE$, $Q^1=Q^*(-M^0P^1+N^0Q^1)=Q^*CE$. As the $n\times 2n$ matrix $\|\overline{P}^*\overline{Q}^*\|$ is of rank n, it follows that $E_{ij}g_j=0$ is satisfied by a set (g_i) if and only if $P_{ij}{}^1g_j=0$, $Q_{ij}{}^1g_j=0$. In particular the rank of E is equal to the rank of the $n\times 2n$ matrix, $\|\overline{P}^1\,\overline{Q}^1\|$, which, in turn, is equal to the rank of $\|M^1\,N^1\|$ in view of (2.16). Hence the rank of E is equal to $n-r$, $0\leqq r<n$, and there exist r sets of constants $g_{i\nu}$ $(\nu=1, 2, \cdots, r)$ such that $E_{ij}g_{j\nu}=0$. The constants $g_{i\nu}$ can also be chosen normed and orthogonal in the sense that $g_{i\gamma}g_{i\nu}=I_{\gamma\nu}$. It clearly follows that there exists a constant matrix $\mathcal{K}$ such that the symmetric $(n+r)\times(n+r)$ matrices

$$\begin{Vmatrix} E_{ij} & g_{i\gamma} \\ g_{j\nu} & 0_{\nu\gamma} \end{Vmatrix}, \qquad \begin{Vmatrix} \mathcal{K}_{ij} & g_{i\gamma} \\ g_{j\nu} & 0_{\nu\gamma} \end{Vmatrix}$$

are reciprocals. Moreover, the quadratic form $\mathcal{K}_{ij}u_iu_j$ is positive for all non-null sets (u_i) satisfying $g_{j\nu}u_j=0$ $(\nu=1, 2, \cdots, r)$.

3. A minimum problem and its accessory boundary value problem.

An arc $z(x)$ is admissible if the functions $z_i(x)$ are of class D' on $a\leqq x\leqq b$ and satisfy the differential equations

$$\Pi_{j\epsilon}(x)\mathcal{M}_j[z] = 0 \qquad (\epsilon = 1, 2, \cdots, m). \tag{3.1}$$

The class H_1^* is defined as the totality of admissible functions satisfying the following additional conditions

$$g_{i\gamma}\overset{0}{t_i}[z] = 0 \qquad (\gamma = 1, 2, \cdots, r), \tag{3.2}$$

$$G[z(a), z(b)] + \int_a^b zKzdx = 1, \tag{3.3}$$

where $G[z(a), z(b)]$ is a quadratic form in the arguments $z(a)$, $z(b)$ having (2.19) as matrix of coefficients, and $K=-B\overline{T}^{-1}$. The second equation (2.14) and the symmetry of S show that K is symmetric.

Suppose H_1^* is nonvacuous, and consider the problem of minimizing the expression

$$I[z] = t^0[z]\mathcal{K}t^0[z] + \int_a^b \mathfrak{M}[z]R\mathfrak{M}[z]dx \tag{3.4}$$

in this class. In view of (3.2) the quadratic form $t^0[z]\mathcal{K}t^0[z]$ is positive unless $t^0[z]=0$; similarly, (3.1) implies that the integral of (3.4) is positive unless $\mathfrak{M}[z]=0$ on $a\leqq x\leqq b$. As $\lambda=0$ is not a characteristic value of (2.1), it is also not a characteristic value of the adjoint system and consequently $I[z]>0$ for all arcs of H_1^*. For a minimizing arc $z(x)$ define

$$R_{ij}\mathfrak{M}_j[z] + \Pi_{i\epsilon}\mu_\epsilon = \zeta_i. \tag{3.5}$$

From the first necessary conditions of the above defined calculus of variations problem we know that there exist multipliers $\mu_\epsilon(x)$, Λ, and d_γ such that in addition to (3.1), (3.2), and (3.3) we have

$$\zeta' - A\zeta + \Lambda Kz = 0, \tag{3.6}$$

$$\begin{aligned} \overset{0}{P}_{ij}(\mathcal{K}_{jk}\overset{0}{t}_k[z] + g_{j\gamma}d_\gamma) - \Lambda \overset{*}{P}_{ij}\overset{1}{s}_j[T^{-1}z] - \zeta_i(a) &= 0,\\ \overset{0}{Q}_{ij}(\mathcal{K}_{jk}\overset{0}{t}_k[z] + g_{j\gamma}d_\gamma) - \Lambda \overset{*}{Q}_{ij}\overset{1}{s}_j[T^{-1}z] + \zeta_i(b) &= 0. \end{aligned} \tag{3.7}$$

Solving (3.1) and (3.5) simultaneously we have that

$$\mathfrak{M}[z] = S\zeta, \qquad \mu_\epsilon = \Pi_{j\epsilon}\zeta_j. \tag{3.8}$$

In view of (2.5), (2.6), (2.7), and (2.8) it follows that (3.7) is equivalent to

$$s^0[\zeta] = \Lambda s^1[T^{-1}z], \tag{3.9}$$

$$\mathcal{K}_{ij}\overset{0}{t}_j[z] + g_{i\gamma}d_\gamma = -\overset{*}{s}_i[\zeta]. \tag{3.10}$$

Solution of (3.10) simultaneously with (3.2) and use of equation (2.6) together with the second, fourth, sixth, and seventh of equations (2.4) yield

$$t^0[z] - C^{-1}s^1[\zeta] = 0, \qquad d_\gamma = -g_{i\gamma}\overset{*}{s}_i[\zeta]. \tag{3.11}$$

Therefore the system (3.1), (3.2), (3.5), (3.6), and (3.7) is equivalent to the system

$$\begin{aligned} \mathfrak{M}[z] &= S\zeta, & t^0[z] - C^{-1}s^1[\zeta] &= 0,\\ \mathcal{L}[\zeta] &= -\Lambda Kz, & s^0[\zeta] - \Lambda s^1[T^{-1}z] &= 0. \end{aligned} \tag{3.12}$$

Concerning the above system we shall prove the following result.

THEOREM 3.1. *The system* (3.12) *is normal.*

Suppose the system is not normal. Then functions $z_i(x) \equiv 0$, $\zeta_i(x) \neq 0$ exist satisfying (3.12) for constant Λ. The functions $\zeta_i = \Pi_{ie}\mu_e$ would then satisfy the system

$$\mathcal{L}[\zeta] = 0, \qquad s^0[\zeta] = 0, \qquad s^1[\zeta] = 0.$$

Since $B_{ij}\Pi_{je} \equiv 0$ we have that $B_{ij}\zeta_j \equiv 0$. This implies by hypothesis (H_5') that $\zeta_i(x) \equiv 0$ which is contrary to the hypothesis that $\zeta_i(x) \neq 0$. Therefore, the system is normal.

The positiveness and reality of the characteristic values of the system (3.12) have been proved by Reid [4]. We also know that the characteristic values of such a system are at most denumerably infinite in number, since they are the zeros of a permanently convergent power series.

4. Sufficient conditions for the existence of infinitely many characteristic values. To prove our sufficiency theorem use will be made of two theorems proved by Reid [4]. They will be inserted here for reference.

THEOREM 4.1. *Suppose that the class* H_1^* *is not empty and* $\Lambda = \Lambda_1$ *is the greatest lower bound of* $I[z]$ *in this class; then* $\Lambda_1 > 0$ *and* $\Lambda = \Lambda_1$ *is the least characteristic value of* (3.12).

THEOREM 4.2. *Suppose* $\Lambda_1 < \Lambda_2 < \cdots < \Lambda_{t-1}$ *are consecutive characteristic values of* (3.12) *and corresponding to* $\Lambda = \Lambda_p$ $(p = 1, 2, \cdots, t-1)$; *there are* r_p *linearly independent solutions* z_{iq_p}, ζ_{iq_p} $(q_p = 1, 2, \cdots, r_p)$. *Define the class* H_t^* *as the subclass of arcs belonging to* H_1^* *satisfying the conditons*

$$G[z(a), z_{q_p}(b)] + \int_a^b z_i K_{ij} z_{jq_p} dx = 0.$$

Then, if H_t^* *is not empty and* Λ_t *is the greatest lower bound of* $I[z]$ *in this class,* $\Lambda_t > \Lambda_{t-1}$ *and* $\Lambda = \Lambda_t$ *is a characteristic value of* (3.12).

Consider once more system (3.12). Let z, ζ be a solution of this system for some value of Λ. If we use(2.14) and (2.16) it then follows that $z = (-1/\Lambda^{1/2})T\eta$ defines a vector η such that η, ζ is a solution of

$$\begin{aligned} &\mathcal{L}[\eta] = \Lambda^{1/2}B\zeta, \qquad s^0[\eta] + \Lambda^{1/2}s^1[\zeta] = 0, \\ &\mathcal{L}[\zeta] = \Lambda^{1/2}B\eta, \qquad s^0[\zeta] + \Lambda^{1/2}s^1[\eta] = 0. \end{aligned} \tag{4.1}$$

Now, if η^*, ζ^* is a solution of (4.1) the functions $\eta = \eta^* + \zeta^*$, $\zeta = \eta^* + \zeta^*$ and $\eta = \eta^* - \zeta^*$, $\zeta = -\eta^* + \zeta^*$ are also solutions of this system. Hence if Λ is a characteristic value of (3.12) of index r, it follows that there exist r linearly independent solutions η, ζ of (4.1) such that for each of these solutions we have either $\eta=\zeta$ or $\eta=-\zeta$. Suppose that η, $\zeta=\eta$ is a solution of (4.1). Then $y=\eta$ is a solution of (2.1) for $\lambda=\Lambda^{1/2}$. Likewise, if η, $\zeta=-\eta$ is a solution of (4.1), $y=\eta$ is a solution of (2.1) for $\lambda=\Lambda^{1/2}$. Therefore, the sum of the indices of $\Lambda^{1/2}$ and $-\Lambda^{1/2}$ as characteristic values of (2.1) is not smaller than the index of Λ as a characteristic value of (3.12). On the other hand, if $y(x)$ is a characteristic solution of (2.1) for some $\lambda\neq 0$, $z=(-1/\lambda)Ty$, $\zeta=y$ is a solution of (3.12) for $\Lambda=\lambda^2$. Relation (2.17) implies that the set of characteristic solutions corresponding to values $\Lambda^{1/2}$ and $-\Lambda^{1/2}$ are linearly independent and, therefore, we have that the index of Λ as a characteristic value of (3.12) is not less than the sum of the indices of $\Lambda^{1/2}$ and $-\Lambda^{1/2}$ as characteristic values of (2.1). Hence, we have the following result.

THEOREM 4.3. *If* Λ *is a characteristic value of* (3.12), *then either* $\Lambda^{1/2}$ *or* $-\Lambda^{1/2}$ *is a characteristic value of* (2.1); *conversely, if* λ *is a characteristic value of* (2.1), *then* $\Lambda=\lambda^2$ *is a characteristic value of* (3.12) *with index which equals the sum of the indices of* λ *and* $-\lambda$ *as characteristic values of* (2.1).

The three preceding theorems imply the analogue of Theorem 4.1 of Reid [5].

THEOREM 4.4. *A system* (2.1) *satisfying hypotheses* (H_1)–(H_6) *has an infinity of characteristic values if and only if there are infinitely many arcs* $z=w_p(x)$ $(p=1, 2, \cdots)$ *satisfying* (3.1) *and* (3.2) *and such that for each* r *and arbitrary constants* $f_t\neq 0_t$ $(t=1, 2, \cdots, r)$ *the arcs* $w=w_t(x)f_t$ *satisfy the condition*

$$G[w(a), w(b)] + \int_a^b wKwdx > 0.$$

Finally, it is to be noted that if z is admissible then there exists a vector $h(x)$ with components piecewise continuous on $a\leqq x\leqq b$ and such that $\mathfrak{M}[z]=-h(x)B(x)$. Furthermore, if the end values of z satisfy (3.2) it follows that there exists a constant vector $k\equiv(k_\iota)$ such that $t^0[z]=kE$. From the form of the matrix E it then follows that $h(a)=kC^{-1}M^0$, $h(b)=-kC^{-1}N^0$ satisfy $t^0[z]+t^1[h]=0$. That is, an admissible z satisfies (3.1) if and only if

$$\mathfrak{M}[z] = -h(x)B(x), \qquad t^0[z]+t^1[h]=0, \tag{4.2}$$

where $h(x)$ is piecewise continuous on $a\leqq x\leqq b$, it being understood, in particular, that the components of $h(x)$ may be discontinuous at $x=a$ and $x=b$.

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ON THE SUMMATION OF MULTIPLE FOURIER SERIES. III[1]

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Let $f(x)=f(x_1, \cdots, x_k)$ be a function of the Lebesgue class L, which is periodic in each of the k-variables, having the period 2π. Let

$$a_{\nu_1\cdots\nu_k} = \frac{1}{(2\pi)^k}$$

$$\cdot \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} f(x) \exp \left\{- i(\nu_1 x_1 + \cdots + \nu_k x_k)\right\} dx_1 \cdots dx_k,$$

where $\{\nu_k\}$ are all integers. Then the series $\sum a_{\nu_1\cdots\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k)$ is called the multiple Fourier series of the function $f(x)$, and we write

$$f(x) \sim \sum a_{\nu_1\cdots\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k).$$

Let the numbers $(\nu_1^2 + \cdots + \nu_k^2)$, when arranged in increasing order of magnitude, be denoted by $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, and let

$$C_n(x) = \sum a_{\nu_1\cdots\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k),$$

where the sum is taken over all $\nu_1^2 + \cdots + \nu_k^2 = \lambda_n$,

$$\phi(x, t) = \sum C_n(x) \exp(-\lambda_n t),$$

$$S_R(x) = \sum_{\lambda_n \leqq R^2} C_n(x), \qquad \lambda_n \leqq R^2 < \lambda_{n+1}.$$

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[1] Papers I and II with the same title are to appear in Proc. London Math. Soc.

Also, let $R_k(\lambda_s)$ and $r_k(\lambda_s)$ represent respectively the number of solutions of $\nu_1^2+\cdots+\nu_k^2\leqq\lambda_s$ and of $\nu_1^2+\cdots+\nu_k^2=\lambda_s$.

The object of this note is to study the convergence of multiple Fourier series, when summed up spherically by Bochner's method, that is, of the series $\sum C_n(x)$. We prove the following results.

THEOREM I. *If*

$$\sum (\nu_1^2+\cdots+\nu_k^2)^{k/2}\left|a_{\nu_1\cdots\nu_k}\right|^2<\infty,$$

then the series $\sum_{n=0}^{\infty} C_n$ *converges at every point of continuity of* $f(x)$.

THEOREM II. *If*

$$\sum (\nu_1^2+\cdots+\nu_k^2)^{k/2+\epsilon}\left|a_{\nu_1\cdots\nu_k}\right|^2<\infty, \qquad \epsilon>0,$$

then the series $\sum_{n=0}^{\infty} C_n$ *converges absolutely.*

The following result of Bochner[2] is used in the proof of the above theorems.

LEMMA. *At a point of continuity of* $f(x)$, $\phi(x, t)$ *tends to a limit as* t *tends to zero.*

PROOF OF THEOREM I. We shall first prove that

$$\lim_{R\to\infty} S_R(x) = \lim_{t\to+0} \phi(x, t), \tag{1}$$

whenever the limit on the right exists. Next, by the application of the above lemma, we deduce that at a point where $f(x)$ is continuous, $\sum C_n(x)$ is convergent.

Now

$$\begin{aligned} S_R(x) - \phi(x, t) &= \sum_{s=0}^{n} C_s[1 - \exp(-\lambda_s t)] - \sum_{s=n+1}^{\infty} C_s \exp(-\lambda_s t) \\ &\equiv \sum J_1 - J_2, \end{aligned} \tag{2}$$

say. We have,

$$\begin{aligned} J_1 &= \sum_{s=0}^{n} C_s[1-\exp(-\lambda_s t)] \\ &= \sum_{s=0}^{n} [1-\exp(-\lambda_s t)]\sum a_{\nu_1\cdots\nu_k} \exp[i(\nu_1x_1+\cdots+\nu_kx_k)] \\ &= \sum a_{\nu_1\cdots\nu_k} \exp i(\nu_1x_1+\cdots+\nu_kx_k)[1-\exp(-\nu_1^2-\cdots-\nu_k^2)t], \end{aligned}$$

---

[2] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207.

where the third sum runs over $\lambda_s = \nu_1^2 + \cdots + \nu_k^2$ and the last sum runs over $\nu_1^2 + \cdots + \nu_k^2 \leqq \lambda_n$, so that

$$
\begin{aligned}
|J_1| &\leqq \sum |a_{\nu_1\cdots\nu_k}\{1 - \exp(-\nu_1^2 - \cdots - \nu_k^2)t\}| \\
&\leqq [\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} |a_{\nu_1\cdots\nu_k}|^2 \\
&\quad \times \sum \{1 - \exp(-\nu_1^2 - \cdots - \nu_k^2)t\}^2 (\nu_1^2 + \cdots + \nu_k^2)^{-k/2}]^{1/2} \\
&\leqq O(1)\cdot t\left[\sum_{s=0}^{n} r_k(\lambda_s)\lambda_s^{2-k/2}\right]^{1/2},
\end{aligned} \tag{3}
$$

where the first sum runs over $\nu_1^2 + \cdots + \nu_k^2 \leqq \lambda_n$.

Now,

$$
\begin{aligned}
\sum_{s=0}^{n} r_k(\lambda_s)\lambda_s^{2-k/2} &= \sum_{s=0}^{n-1} R_k(\lambda_s)\{\lambda_s^{2-k/2} - \lambda_{s+1}^{2-k/2}\} + R_k(\lambda_n)\lambda_n^{2-k/2} \\
&= O\left(\int_0^{\lambda_n} x\,dx\right) + O(\lambda_n^2) \\
&= O(\lambda_n^2).
\end{aligned} \tag{4}
$$

Hence, from (3), we obtain,

$$
|J_1| = O(t\lambda_n). \tag{5}
$$

Again,

$$
\begin{aligned}
|J_2| &= \left|\sum_{s=n+1}^{\infty} C_s \exp(-\lambda_s t)\right| \\
&\leqq \lambda_n^{-k/4} \sum_{s=n+1}^{\infty} \lambda_s^{k/4} |C_s \exp(-\lambda_s t)| \\
&\leqq \lambda_n^{-k/4} [\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} |a_{\nu_1\cdots\nu_k}|^2 \\
&\qquad\qquad \times \sum \exp\{-2(\nu_1^2 + \cdots + \nu_k^2)t\}]^{1/2} \\
&\leqq \epsilon_n^{1/2} (t\lambda_n)^{-k/4}
\end{aligned} \tag{6}
$$

(in the last two sums $\nu_1^2 + \cdots + \nu_k^2$ runs from λ_{n+1} to ∞), where

$$
\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} |a_{\nu_1\cdots\nu_k}|^2 = \epsilon_n
$$

($\nu_1^2 + \cdots + \nu_k^2$ runs from λ_{n+1} to ∞), and $\epsilon_n \to 0$ as $n \to \infty$, since $\sum e^{-2\nu^2 t} = O(t^{-1/2})$ as $t \to 0$. Thus, we have, from (5) and (6),

$$S_R(x) - \phi(x, t) = O(t\lambda_n) + o\left[\epsilon_n^{1/2}(t\lambda_n)^{-k/4}\right]. \tag{7}$$

If t is so chosen that $t\lambda_n = \delta_n = \epsilon_n{}^{1/k}$, then,

$$S_R(x) - \phi(x, t) = O(\delta_n) + O(\epsilon_n^{1/2} \cdot \delta_n^{-k/4}) = o(1), \qquad \text{as } n \to \infty.$$

PROOF OF THEOREM II.

$$\begin{aligned} \sum \left| C_s(x) \right| &\leqq \sum \left| a_{\nu_1 \cdots \nu_k} \right| \\ &\leqq \left\{ \sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2+\epsilon} \left| a_{\nu_1 \cdots \nu_k} \right|^2 \right\}^{1/2} \\ &\quad \times \left\{ \sum (\nu_1^2 + \cdots + \nu_k^2)^{-k/2-\epsilon} \right\}^{1/2} \\ &= O(1) \sum \left(r_k(\lambda_s) \lambda_s^{-k/2-\epsilon} \right)^{1/2} \\ &= O\left(\left(\int^\infty R_k(x) x^{-k/2-1-\epsilon} dx \right)^{1/2} \right) \\ &= O\left(\left(\int^\infty x^{-1-\epsilon} dx \right)^{1/2} \right) < \infty. \end{aligned}$$

On using Hölder's inequality instead of Schwarz's in (3) and (6), we can easily generalize Theorem I as follows:

If

$$\sum (\nu_1^2 + \cdots + \nu_k^2)^{k(p-1)/2} \left| a_{\nu_1 \cdots \nu_k} \right|^p < \infty,$$

where $1 < p \leqq 2$, *then* $\sum C_n$ *converges at every point of continuity of* $f(x)$.

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GENERALIZATIONS OF TWO THEOREMS OF JANISZEWSKI. II

R. H. BING

The purpose of this note is to strengthen Theorems 5 and 6 of [1][1] and to make corrections regarding assumptions of compactness in that paper. The following theorems hold in the plane.

THEOREM 1. *If neither of the domains D_1, D_2 separates the point A from the point B, the boundary of D_1 is compact and the common part of D_2 and each component of D_1 is connected or does not exist, then D_1+D_2 does not separate A from B.*

PROOF. Assume that D_1+D_2 separates A from B. Considering there to be a point P at infinity, we find that D_1+D_2+P contains a simple closed curve J separating A from B. Let d_2 be a component of D_2 intersecting J. We find [1, Theorem 4] that $J-J\cdot d_2$ contains a continuum M cutting A from B in the complement of d_2 and such that any open arc of J containing M separates A from B in the complement of d_2. Let d_1 be a component of D_1 covering a point of M on the boundary of d_2. Now d_1 covers M or else it would intersect two components of D_2. But by Theorem 5 of [1], d_1+d_2 does not separate A from B.

Instead of assuming that the boundary of D_1 is compact, we could assume that the part of D_1 in the complement of D_2 is compact.

THEOREM 2. *If neither of the domains D_1, D_2 cuts the point A from the point B, the boundary of D_1 is compact and the common part of D_2 and each component of D_1 is connected or does not exist, then D_1+D_2 does not cut A from B.*

PROOF. Let C_i $(i=1, 2)$ be the component of the complement of D_i containing $A+B$, let D_i' be the complement of C_i and let D_2'' be the sum of all components of D_2' that are not covered by D_1'. Neither D_1' nor D_2'' separates the plane. The boundary of D_1' is a subset of the boundary of D_1 and is therefore compact. If d' is a component of D_1', we shall show that $d'\cdot D_2''$ is connected or does not exist. It will follow from Theorem 1 that $D_1'+D_2''$ does not separate the plane. Hence, its complement is a continuum containing $A+B$ and its subset D_1+D_2 does not cut A from B.

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[1] Number in brackets refers to the reference cited at the end of the paper.

Assume that $d'\cdot D_2''$ contains two components c_1 and c_2. There exist an arc in d' from a point of c_1 to a point of c_2 and a simple closed curve J in D_1 separating this arc from the boundary of d'. Let d be the component of D_1 containing J and let P_iQ_i be an arc in J irreducible from a point P_i of $J\cdot c_i$ to the boundary of c_i. Since Q_i is a point of C_2, P_iQ_i must contain a point of D_2. Then both c_1 and c_2 contain points of $d\cdot D_2$. But it is contrary to a hypothesis of this theorem that $d\cdot D_2$ not be connected. Hence, $d'\cdot D_2''$ is connected or does not exist.

Example. Theorem 2 would not be true if instead of assuming that the boundary of D_1 is compact, we assume that the part of D_1 in the complement of D_2 is compact. Let $D_1\cdot D_2$ be the set of all points having positive ordinates less than 1 other than those on the lines joining $(1, 1/n)$ to $(n, 1/n)$, $(n, 1/n)$ to $(1, 1)$, $(-1, 1/n)$ to $(-n, 1/n)$ and $(-n, 1/n)$ to $(-1, 1)$ for $n=2, 3, \cdots$; let D_i $(i=1, 2)$ be the sum of $D_1\cdot D_2$ and the interior of a unit circle with center at $([-1]^i, 1)$. Neither D_1 nor D_2 cuts $(0, 0)$ from $(0, 1)$ but their sum does.

THEOREM 3. *Suppose that neither of the sets H, K cuts the point A from the point B, that the boundary of H is compact, that the junction of H and K is equal to $H\cdot K$ and that H is the sum of a collection of mutually exclusive sets no one of which contains either a limit point of the sum of the others or two components of $H\cdot K$. Then $H+K$ does not cut A from B.*

PROOF. We note that H is contained by a domain D, no component of which contains two components of $H\cdot K$. Let C_H and C_K be two continua in the complements of H and K respectively such that each contains $A+B$. Let D_0 be a subdomain of D with a compact boundary such that D_0 contains $H\cdot K$ but no point of C_H+C_K and each component of D_0 contains a point of $H\cdot K$. There exist domains D_1 and D_2 such that D_1 is a subset of D having a compact boundary and containing $H-H\cdot D_0$ but no point of C_H, D_2 contains $K-K\cdot D_0$ but no point of C_K and $D_1\cdot D_2$ is a subset of D_0. Considering D_0+D_1 and D_0+D_2 as the domains of Theorem 2, we find that $D_0+D_1+D_2$ does not cut A from B. Hence, its subset $H+K$ does not.

THEOREM 4. *If H is a compact closed set cutting the point A from the point B in the complement of the connected set K, then H contains a subset H' irreducible with respect to being a closed set cutting A from B in the complement of K. If K is compact, H' is a continuum that is not separated by any subset of the closure of K.*

The proof is as given in Theorem 7 of [1]. If K is not compact, H' need not be a continuum as is shown by the following example.

Let H be the sum of the points $(1, 1)$ and $(-1, 1)$; let K be the common part of domains D_1 and D_2 described in the example in Theorem 2; let A and B be the points $(0, 0)$ and $(0, 1)$.

Corrections to [1]. The example given in Theorem 2 shows that 6, 7, 10 should have been omitted from the third footnote of [1]. As pointed out in Theorem 4, it is necessary to suppose that K is compact in Theorem 7 of [1]. Accordingly, D must be assumed compact in the fourth footnote of [1].

REFERENCE

1. R. H. Bing, *Generalizations of two theorems of Janiszewski*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 954–960.

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TOPOLOGICAL GROUPS IN WHICH MULTIPLICATION OF ONE SIDE IS DIFFERENTIABLE

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The present note is concerned with the problem of determining the formally weakest conditions on a group which can be proved to result in the group being a Lie group. We show essentially that it is sufficient for this purpose to require that the group be a (real) manifold of class C^1 in which right multiplication is of class C^1 (our result is slightly stronger—Theorem 2 is the precise statement). The weakest previously existing condition is contained in work of P. A. Smith [2],[1] and required, in addition to the preceding condition, that left multiplication satisfy a Lipschitz condition.[2]

We should point out that our result is obtained by combining previous work on Lie groups with a theorem which we prove independently of previous work. Specifically, we first prove that if right multiplication is of class C^1 on a group G which is a manifold of class C^1, then it follows that left multiplication is of class C^1 (Theorem 1). We then deduce that G is a Lie group either from the result of Smith quoted above, or from the result of G. Birkhoff [1] that if both left and right multiplication are of class C^1, then G is a Lie group. We do not introduce any kind of canonical parameters, and, so far as we know, neither the work of Smith nor that of Birkhoff can be simplified by the use of our results.

Our method is novel in that it utilizes the existence of Haar measure and operates in the large. (Smith and Birkhoff were both concerned exclusively with the theory in the small, a standpoint incompatible with the use of Haar measure.) In this way it is shown that a wide collection of functions of x of class C^1, x being a variable group element, are also of class C^1 as functions of x^{-1}, and most of the proof is occupied with showing that this collection is sufficiently wide to allow the conclusion to be drawn that x^{-1} is of class C^1 as a function of x. It follows that left multiplication is of class C^1. Only minor use is made of the theory of differential equations.

We are indebted to D. Montgomery for highly suggestive conversations about the foundations of the theory of Lie groups.

The following theorem is valid for functions and manifolds of class

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Smith had suggested that the Lipschitz condition might be inessential.

C^k ($k \geqq 1$) with an almost unchanged proof, but we state it for the case $k=1$ only.

THEOREM 1. *Let G be a topological group which is a manifold of class C^1, and such that xa as a function of x on G is of class C^1, for every fixed a in G. Then ax is likewise of class C^1.*

Suppose that f and g are (real-valued) functions of class C^1 on G which vanish except on a compact set. Then it is clear that the function h defined by:

$$h(x) = \int_G f(xy)g(y)dy \tag{1}$$

is also of class C^1, dy being left-invariant Haar measure, and it is easy to see that

$$h(x^{-1}) = \int_G g(xy)f(y)dy, \tag{2}$$

showing that $h(x^{-1})$ is of class C^1 as a function of x, also. It is first shown (Lemma 1), using properties of Haar measure, that the collection of functions having the form (1) has the property that for any two given points of the group there exists a function in the collection which assumes different values at the points. It then follows (Lemma 2), by a method applicable to a general manifold, that x^{-1} is of class C^1 as a function of x. Noting that:

$$ax = (x^{-1}a^{-1})^{-1},$$

the conclusion follows from the fact that functional composition of functions of class C^1 results in a function of class C^1.

Since xa is a function of x of class C^1 we lose no generality by assuming now that if $(x|y)^i$ denotes the ith coordinate of the point x with respect to the local system at y ($i=1, \cdots, r$), then $(x|y)^i = (xy^{-1}|e)^i$; here e is the identity in G, x and y are elements of G, and r is the dimension of G. We also assume that $(e|e)^i = 0$.

LEMMA 1. *Let Γ be the collection of functions on G of the form $\int_G f(xy)g(y)dy$, where f and g are real-valued functions on G of class C^1, which vanish except on compact sets. Then for every two points of G there exists an element of Γ which assumes different values at the points.*[3]

The proof is indirect: suppose that $\gamma(a) = \gamma(b)$ for all $\gamma \in \Gamma$, where

---

[3] We are indebted to C. Chevalley and S. Eilenberg for suggesting a modification in the proof of the lemma which resulted in a considerably more elementary proof.

$a \neq b$. Clearly, if $h(x)$ is a function of class C^1 on G, then the same is true of the function $h(xa)$, for every fixed a in G. It follows that

$$\int_G f(ay)g(yx)dy = \int_G f(by)g(yx)dy, \tag{3}$$

where f and g are general functions of class C^1 vanishing except on compact sets. Putting h_p and $\bar{h}$ for the functions defined respectively by $(h_p)(x) = (h)(p^{-1}x)$ and $(\bar{h})(x) = \rho(x)(h)(x^{-1})$, where $dx^{-1} = \rho(x)dx$, (3) can be written in the following form:

$$\bar{f} * g_a = \bar{f} * g_b. \tag{4}$$

Here $r * s$, for general complex-valued functions r and s, denotes the function defined by:

$$(r * s)(x) = \int_G r(y)s(y^{-1}x)dy,$$

provided the integral exists. We now show that (4) is valid if f and g are integrable functions vanishing outside of any right translations of a certain neighborhood R of the identity.

Specifically, let P be the neighborhood of the identity defined by the inequalities $|(x|e)^i| < a$ $(i = 1, \cdots, r)$, where a is chosen sufficiently small so that the closure of P is compact; let Q be the same with a replaced by $a/2$, and let R be a neighborhood of the identity such that $R^2 \subset Q$. If h is a continuous function which vanishes outside of Qd, then, putting $(x|d)^i = x$, and $h*(x_1, \cdots, x_r) = h(x)$ in the present sentence, we define h_e by

$$h_e(x) = (2e)^{-n} \int_{x_r - e}^{x_r + e} \cdots \int_{x_1 - e}^{x_1 + e} h*(t_1, \cdots, t_r)dt_1, \cdots, dt_r.$$

It is clear that if $0 < e < 1/2a$, h_e is defined, of class C^1, and converges uniformly to h on Pd as $e \to 0$. If we set $\|r\|$ for the L_1 norm of r (that is, the integral of the absolute value of r with respect to Haar measure), it is plain that $\|h_e - h\| \to 0$ as $e \to 0$. Since the operators $h \to h_p$ and $h \to \bar{h}$ are linear and isometric on L_1, it follows that (3) is valid if f and g are continuous and vanish outside right translations of Q. Now if h is integrable and vanishes outside Rd, and if $u_S(x) = (m(S))^{-1}f_S(x)$, where m is Haar measure, S is a neighborhood of the identity, and f_S is the characteristic function of S, it is easy to verify that $u_S * h$ vanishes outside SRd. On the other hand, for any positive number e there exists such an S contained in R and such that $\|u_S * h - h\| < e$. It follows that (3) is valid if f and g are integrable and vanish outside a right translation of R.

Now let T be a neighborhood of the identity such that $T=T^{-1}$, $T\subset R$, $a^{-1}Ta\subset R$, and $a^{-1}b$ is not contained in T^3. Let f and g be the characteristic functions of T and $a^{-1}T$ respectively. It is easy to verify that:

$$(\bar{f}*g_a)(x) = m(a^{-1}T \cap a^{-1}Tx^{-1}),$$
$$(\bar{f}*g_b)(x) = m(b^{-1}T \cap a^{-1}Tx^{-1}).$$

Clearly $\bar{f}*g_a$ and $\bar{f}*g_b$ are continuous functions, and it is easy to see that the first function is not zero at the identity and that the second function vanishes in T. Hence $\bar{f}*g_a\neq\bar{f}*g_b$, a contradiction.[4]

LEMMA 2. *Let M be a manifold of class C^1 and dimension r. Let Φ be a collection of functions of class C^1 on M, with the property that for any two given points of M there exists a function in Φ which assumes different values at the points. Then there exists a point of M and r elements of Φ whose Jacobian with respect to local coordinates at p, evaluated at p, is not zero.*

Again our proof is indirect: assume that the Jacobian of any r elements of Φ with respect to local coordinates at x, evaluated at x, vanishes for every $x\in M$. Let s be the maximum rank of the matrices of all such Jacobians, let $\bar{x}$ be a point at which the rank is attained, and let $f_1, \cdots, f_s$ be elements of Φ involved in a nonvanishing minor of order s in a Jacobian matrix at $\bar{x}$. For the remainder of the lemma, put $x_i=(x|\bar{x})^i$, and then re-index the coordinates so that $x_1, \cdots, x_s$ are the coordinates involved in the nonvanishing minor just mentioned. Then bordering this minor by adding any $f\in\Phi$ and any local coordinate around $\bar{x}$ not yet involved in the minor, say x_{s+1}, the resulting minor of order $s+1$ vanishes in a neighborhood of $\bar{x}$, clearly. Expanding the minor as a sum of products of row elements involving f with the corresponding cofactors, we obtain an equation of the form

$$\sum_{j=1}^{r} \frac{\partial f}{\partial x_j} A_j(x_1, \cdots, x_r) = 0,$$

where the A_j are continuous functions and independent of f, with $A_j=0$ for $j>s+1$, and $A_{s+1}(0, \cdots, 0)\neq 0$. Now let $\{x_i(t)\}$ be a solution in a neighborhood of $t=0$ of the system of equations:

$$\frac{dx_i}{dt} = A_i(x_1, \cdots, x_r), \qquad x_i(0) = 0 \qquad (i = 1, \cdots, r).$$

---

[4] It is interesting to note that the set of functions of class C^1 which vanish outside compact sets is L_1-dense in L_1. This fact could be proved, and the proof of the lemma shortened, by using theorems due to H. Whitney.

Plainly, in no neighborhood of $t=0$ do all $x_i(t)$ vanish identically. On the other hand, it is clear from the partial differential equation in f that $f(x(t))=f(\bar{x})$, where $x(t)$ is the point whose coordinates are $(x_1(t), \cdots, x_r(t))$, for all t in some neighborhood of $t=0$. This equation clearly contradicts the hypothesis that Φ contains a function assuming different values at any two given points.

To conclude the proof of Theorem 1, let M be an arbitrary open set in G, set $\Gamma=\Phi$, and let p be a point of M at which there exist r elements $\{\phi_i\}$ of Γ which have a nonvanishing Jacobian; by Lemmas 1 and 2, p and $\{\phi_i\}$ exist, evidently. By the theory of implicit functions, $\{(x|p)^i\}$ is a function of $\{\phi_i(x)\}$ of class C^1 in a neighborhood of p, say

$$(5) \qquad (x \mid a)^i = F_i(\phi_1(x), \cdots, \phi_r(x)) \qquad (i = 1, \cdots, r).$$

Putting $x=y^{-1}$, $a=b^{-1}$, (5) becomes:

$$(6) \qquad (y^{-1} \mid b^{-1})^i = F_i(\phi_i(y^{-1}), \cdots, \phi_r(y^{-1})) \qquad (i = 1, \cdots, r).$$

Recalling that $\phi_i(y^{-1})$ is of class C^1 as a function of y, it is clear from (6) that y^{-1} is of class C^1 as a function of y, for y in a neighborhood of b. Since M is an arbitrary open set, the set S of points at which there exists a set of r elements of Φ with a nonvanishing Jacobian is dense in G; hence the same is true of the set S^{-1}, and therefore G is covered by neighborhoods in which y^{-1} is of class C^1 as a function of y.

Theorem 1 (which is a result in the large) is now combined with an existing result in the theory of local Lie groups, to obtain a condition that a group be a Lie group. At the same time we show that in order for right multiplication to be of class C^1 on a group which is a manifold of class C^1, it is sufficient for the first-order derivatives of right multiplication to exist, and be continuous at one point.

THEOREM 2. *Let G be a connected topological group. Let there exist a neighborhood N of the identity in G which is a manifold of class C^1 and dimension r, and a neighborhood of the identity P such that $P=P^{-1}$, $P^2 \subset N$, and the first-order derivatives of $\{(xa|ya)^i; i=1, \cdots, r\}$ with respect to $\{(x|y)^i; i=1, \cdots, r\}$ exist for every fixed a and y in P, at $x=y$, and for one value of y and all values of a in P are continuous; here $\{(x|y)^i; i=1, \cdots, r\}$ are the local coordinates at y of x. Then G is a Lie group.*

Let $f(s)$ be a real-valued function whose derivative (by derivative we mean first-order derivative in the following) exists for s in an interval which includes the range of values of the real-valued function $g(t)$, as t ranges over an interval containing the point t_0 in its interior; assume also that the derivative of $g(t)$ exists for t in such an interval, and that the derivative of $f(s)$ is continuous at $s=g(t_0)$. Then it is easy to conclude from the mean value theorem of the differential calculus that the derivative of $f(g(t))$ exists at $t=t_0$ and equals $f'(g(t_0))g'(t_0)$. Now the identity transformation on G can clearly be obtained by first performing the transformation $x \to xa^{-1}$, and then following by the transformation $x \to xa$. If we put x_0 for the point at which xa has continuous derivatives as a function of x for all a in P, then the derivatives of the first transformation exist at $x=x_0a=v_a$, if $v_a \in P$; and the derivatives of the second transformation exist and are continuous at $x=x_0$. It follows that:

$$I = \left[\frac{\partial(xa^{-1})}{\partial(x)}\right]_{x=v_a} \cdot \left[\frac{\partial(xa)}{\partial(x)}\right]_{x=x_0},$$

where I is the identity matrix, and $[\partial(xc)/\partial(x)]_{x=y}$ stands for the Jacobian matrix of the function xc, with respect to local coordinates at y and yc, evaluated at $x=y$. Since $[\partial(xa)/\partial(x)]_{x=u}$ is continuous at $u=x_0$, it follows that $[\partial(xa^{-1})/\partial x]_{x=u}$ is continuous at $u=v_a$, if $v_a \in P$. That is, xa^{-1} as a function of x has continuous derivatives at $x=v_a$, if $v_a \in P$. Hence $xa^{-1}b$ as a function of x has continuous derivatives at $x=v_a$, if a, b, and v_a are all in P. Plainly, x_0P is an open set containing e; therefore there exists a neighborhood Q of e such that $Q=Q^{-1}$, $Q \subset x_0P$. Let R be a neighborhood of e such that $R=R^{-1}$, and $R^2 \subset Q$. It is plain that as a ranges over $x_0^{-1}R$ (which is a subset of P), v_a ranges over R, and it is easy to verify that if $a \in x_0^{-1}R$, then $a^{-1}P \supset R$. It follows that xa is of class C^1 as a function of x, for x and a in R.

Now let S be a cubical neighborhood of e such that $S^{-1}S \subset R$. We now regard G as a manifold with local coordinates $[x|a]^i$ defined to be $(xa^{-1}|e)^i$ $(i=1, \cdots, r)$ in the neighborhood Sa of a. At the identity this system of coordinates is the same as the original system, in S, and we now show that G is a manifold of class C^1. If the neighborhoods Sa_1 and Sa_2 overlap, it is easy to verify that $a_2a_1^{-1} \in R$. Since the coordinates of x relative to the Sa_j system are $\{(xa_j^{-1}|e)^i;$ $i=1, \cdots, r\}$, with $xa_j^{-1} \in S$, and multiplication on the right by the element $a_2a_1^{-1}$ of G is of class C^1 on N, it follows that the $[x|a_1]^i$ $(i=1, \cdots, r)$ are of class C^1 as functions of the $[x|a_2]^i$ $(i=1, \cdots, r)$. Clearly, xa is of class C^1 as a function of x, at all points of G and for all fixed a.

Then the hypothesis of Theorem 1 is satisfied. If we combine the conclusion of the theorem either with Smith's Theorem 12.9 [2] or Birkhoff's Corollary 15.7 [1], it follows that G is a Lie group.

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A NOTE ON POINTWISE NONWANDERING TRANSFORMATIONS

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Let X be a T_1-space and let f be a continuous transformation of X into X In the terminology of G. D. Birkhoff [1, p. 191],[1] a point x of X is said to be *nonwandering* under f provided that to each neighborhood U of x there correspond infinitely many positive integers n for which $U \cap f^n(U) \neq \emptyset$; also, the transformation f is said to be *pointwise nonwandering* provided that each point of X is nonwandering under f.

THEOREM 1. *If f is pointwise nonwandering, then so also is f^k for every positive integer k.*

PROOF. (We make use of a technique employed by Erdös and Stone [2, pp. 126–127].) Suppose k is a positive integer, $x_0 \in X$, and U_0 is a neighborhood (= open neighborhood) of x_0. Let n_1 be a positive integer for which $U_0 \cap f^{n_1}(U_0) \neq \emptyset$. Choose $x_1 \in U_0$ so that $f^{n_1}(x_1) \in U_0$ and a neighborhood U_1 of x_1 so that $U_1 \subset U_0$ and $f^{n_1}(U_1) \subset U_0$. Let n_2 be a positive integer for which $U_1 \cap f^{n_2}(U_1) \neq \emptyset$. Choose $x_2 \in U_1$ so that $f^{n_2}(x_2) \in U_1$ and a neighborhood U_2 of x_2 so that $U_2 \subset U_1$ and $f^{n_2}(U_2) \subset U_1$. Continuing this process, we obtain a sequence $\{n_i\}$ of positive integers and a sequence $\{U_i\}$ of neighborhoods so that $U_i \subset U_{i-1}$ and $f^{n_i}(U_i) \subset U_{i-1}$ $(i=1, 2, \cdots)$. Let r_i denote the integer for which $1 \leqq r_i \leqq k$ and $n_i \equiv r_i \bmod k$. Infinitely many of the r_i are equal to some integer, say r. We may suppose $r_i = r$, $U_i \subset U_{i-1}$ and $f^{n_i}(U_i) \subset U_{i-1}$ $(i=1, 2, \cdots)$. Choose an arbitrary positive integer p. Define $n = \sum_{i=1}^{pk} n_i$. Now $n \equiv 0 \bmod k$. Choose $x \in U_{pk}$. Clearly, $x \in U_0$ and $f^n(x) \in U_0$. Hence, $U_0 \cap f^n(U_0) \neq \emptyset$. Since $n \geqq p$, the proof is completed.

LEMMA 1. *If $f(X)=X$ is a homeomorphism, if A and B are closed connected sets for which $A \cup B = X$, $A \cap B = x \in X$ and $A \cap f(A) \neq \emptyset \neq B \cap f(B)$, and if x is nonwandering, then x is fixed.*

PROOF. Assume x is not fixed. We may suppose that $f(x) \in B$. Now $x \notin f^{-1}(A)$ for in the contrary case $f(x) \in A \cap B = x$. The set $f(A)$ is connected and intersects both A and B. Hence, $x \in f(A)$. There exists a neighborhood U of x such that $U \cap f^{-1}(A) = \emptyset$ and such that

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$U\subset f(A)$ for if the latter condition could not be satisfied, $x\in f(B)$ and $f(x)=f(A\cap B)=f(A)\cap f(B)\ni x$. Now $A\subset f(A)$ since otherwise $A\cap f(B)\neq\emptyset\neq B\cap f(B)$ whence $x\in f(B)$ and as before $f(x)=x$. Thus $A\subset f^n(A)$ for each integer $n\geqq 1$. We conclude that for each integer $n\geqq 1, f^{n+1}(U)\cap f^n(A)=\emptyset$ and, since also $U\subset f(A)\subset f^n(A)$, $U\cap f^{n+1}(U)=\emptyset$. This contradicts the hypothesis that x is nonwandering.

THEOREM 2. *If X is connected and if $f(X)=X$ is a pointwise nonwandering homeomorphism, then each cut point of X is periodic.*

PROOF. Let x be a cut point of X. There exist closed connected sets A and B such that $A\cup B=X$, $A\cap B=x$ and $A\neq x\neq B$. The proof is split into two exhaustive cases. Case I: Either $A\cap f^i(A)\neq\emptyset$ $(i=1, 2, \cdots)$, or $B\cap f^i(B)\neq\emptyset$ $(i=1, 2, \cdots)$. Case II: There exist positive integers m and n such that $A\cap f^m(A)=\emptyset$ and $B\cap f^n(B)=\emptyset$.

Suppose Case I occurs. We may assume that $A\cap f^i(A)\neq\emptyset$ $(i=1, 2, \cdots)$. Since $B-x$ is open, there exists a positive integer k such that $B\cap f^k(B)\neq\emptyset$. By Theorem 1, x is nonwandering under f^k and by Lemma 1, x is therefore fixed under f^k.

Suppose Case II occurs. Now $B\supset f^m(A)$ and $A\supset f^n(B)$. Hence $f^n(B)\supset f^{m+n}(A)$ and $f^m(A)\supset f^{m+n}(B)$. It follows that $A\supset f^{m+n}(A)$ and $B\supset f^{m+n}(B)$. Thus, $f^{m+n}(x)=f^{m+n}(A\cap B)=f^{m+n}(A)\cap f^{m+n}(B)\subset A\cap B=x$. (Actually Case II can never occur.)

COROLLARY 1. *If X is a compact connected semi-locally connected metric space and if $f(X)=X$ is a pointwise nonwandering homeomorphism, then every cyclic element of X which is not an end point is periodic.*

A theorem [3, p. 242] of Schweigert's shows that Corollary 1 permits a weakening of hypothesis from pointwise recurrence (= "pointwise almost periodicity" in the sense of Ayres) to pointwise nonwandering in certain theorems of Ayres and Whyburn on the behavior of cyclic elements under a homeomorphism. The reader is referred to Schweigert's paper [3] for complete references to the work of Ayres and Whyburn.

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THE ASANO POSTULATES FOR THE INTEGRAL DOMAINS OF A LINEAR ALGEBRA

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1. **Introduction.** The multiplicative ideal theory for a noncommutative ring A as developed by Asano[1] postulates the existence in A of a maximal bounded order R which satisfies the maximal chain condition for two-sided R-ideals contained in R and the minimal chain condition for one-sided R-ideals in R containing any fixed two-sided R-ideal. Let A be a separable algebra over the field P, and let P be the quotient field of the domain of integrity g. It has been shown [2, pp. 123–126] that if g has a Noether ideal theory, then a maximal domain of g-integers exists in A and satisfies the conditions of the Asano theory. It is the purpose of this paper to prove that the condition of separability can be removed from A and that it need only be postulated that A shall have an identity.

2. **Subgroups of direct sums.** Let G be a commutative group with operator domain Ω. Let G be the direct sum of the Ω-subgroups $G_1, G_2, \cdots, G_n$. We shall write $G=G_1+G_2+\cdots+G_n$. The direct summand G_i gives rise to a projection α_i which is an endomorphism of G on G_i: if $g=g_1+g_2+\cdots+g_n$, $g_j\in G_j$, then $\alpha_i g=g_i$. The sum $\alpha_1+\alpha_2+\cdots+\alpha_n$ is the identity operator I. Furthermore the sum of any subset of the projections $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a projection. We shall label in particular the operators $\delta_i=\sum_{j=1}^{i}\alpha_j$. Then $\delta_1=\alpha_1$, and $\delta_n=I$. In general $\delta_{i+1}=\delta_i+\alpha_{i+1}$. If $\omega\in\Omega$, then $\omega\alpha_i=\alpha_i\omega$, and as a result $\omega\delta_i=\delta_i\omega$; that is, α_i and δ_i are Ω-operators. It follows that $\alpha_i H$ and $\delta_i H$ are Ω-subgroups whenever H is an Ω-subgroup.

LEMMA 1. *Let the commutative group* $G=G_1+G_2+\cdots+G_n$ *contain the* Ω*-subgroups* H *and* K. *If* $H\supseteq K$, *then* $\alpha_i H\supseteq\alpha_i K$, $\delta_i H\supseteq\delta_i K$, *and* $\delta_i H\cap G_i\supseteq\delta_i K\cap G_i$.

Since $H\supseteq K$, the image $\alpha_i K$ of K under the homomorphism of H on $\alpha_i H$ must be contained in $\alpha_i H$. By the same argument $\delta_i H\supseteq\delta_i K$, and therefore $\delta_i H\cap G_i\supseteq\delta_i K\cap G_i$.

LEMMA 2. *Let the commutative group* $G=G_1+G_2+\cdots+G_n$ *contain the* Ω*-subgroups* H *and* K. *If* $H\supseteq K$ *and if* $\alpha_i H=\alpha_i K$, $\delta_i H\cap G_i=\delta_i K\cap G_i$, *then* $H=K$.

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[1] Cf. Asano [1], Jacobson [2]. We use here the formulation of these postulates given by Jacobson. Numbers in brackets refer to the references at the end of the paper.

Since $\delta_1=\alpha_1$ and $\alpha_1H=\alpha_1K$, it follows that $\delta_1H=\delta_1K$.

We shall assume that $\delta_iH=\delta_iK$ and prove that under this assumption $\delta_{i+1}H=\delta_{i+1}K$. Since $\delta_{i+1}K=(\delta_i+\alpha_{i+1})K\subseteq\delta_iK+\alpha_{i+1}K$ and $\delta_{i+1}K\subseteq\delta_{i+1}H$, it is obvious that $\delta_{i+1}K\subseteq(\delta_iK+\alpha_{i+1}K)\cap\delta_{i+1}H$. On the other hand let $\delta_ik_1+\alpha_{i+1}k_2$ be an element of $\delta_iK+\alpha_{i+1}K$ contained in $\delta_{i+1}H$. Consider that $(\delta_i+\alpha_{i+1})k_1$ is an element of $\delta_{i+1}K$ and therefore an element of $\delta_{i+1}H$. Then $\delta_{i+1}H$ contains $\delta_ik_1+\alpha_{i+1}k_2-(\delta_i+\alpha_{i+1})k_1=\alpha_{i+1}(k_2-k_1)$ which lies in $\delta_{i+1}H\cap G_{i+1}=\delta_{i+1}K\cap G_{i+1}\subseteq\delta_{i+1}K$. It follows immediately that $\delta_ik_1+\alpha_{i+1}k_2=(\delta_i+\alpha_{i+1})k_1+\alpha_{i+1}(k_2-k_1)$ lies in $\delta_{i+1}K$, and $\delta_{i+1}H\cap(\delta_iK+\alpha_{i+1}K)=\delta_{i+1}K$. However, since $\delta_iK=\delta_iH$ and $\alpha_{i+1}K=\alpha_{i+1}H$, then $\delta_iK+\alpha_{i+1}K=\delta_iH+\alpha_{i+1}H$, and $\delta_{i+1}K=\delta_{i+1}H\cap(\delta_iH+\alpha_{i+1}H)=\delta_{i+1}H$.

The lemma follows by finite induction; for $\delta_nH=H$, $\delta_nK=K$.

LEMMA 3. *Let the commutative group $G=G_1+G_2+\cdots+G_n$ contain the Ω-subgroup H. Let γ be an automorphism of G contained in the centrum of Ω. Then $H\supseteq\gamma H$, $\alpha_iH\supseteq\alpha_i(\gamma H)=\gamma(\alpha_iH)$, and $\delta_iH\cap G_i\supseteq\delta_i(\gamma H)\cap G_i=\gamma(\delta_iH\cap G_i)$.*

The automorphism γ lies in the centrum of Ω and therefore γH will be an Ω-subgroup of H. It follows by Lemma 1 that $\alpha_iH\supseteq\alpha_i(\gamma H)$, $\delta_iH\supseteq\delta_i(\gamma H)$, and $\delta_iH\cap G_i\supseteq\delta_i(\gamma H)\cap G_i$. Since γ lies in Ω and α_i and δ_i are Ω-operators, $\alpha_i(\gamma H)=\gamma(\alpha_iH)$ and $\delta_i(\gamma H)=\gamma(\delta_iH)$.

It remains to prove that $\delta_i(\gamma H)\cap G_i=\gamma(\delta_iH\cap G_i)$. Consider that $\gamma G_i=\gamma\alpha_iG=\alpha_i\gamma G=\alpha_iG=G_i$. Then $\delta_i(\gamma H)\cap G_i=\delta_i(\gamma H)\cap\gamma G_i=\gamma(\delta_iH)\cap\gamma G_i$. Let $\gamma\delta_ih=\gamma g_i$; γ is an automorphism, and $\delta_ih=g_i$. It follows that $\gamma(\delta_iH)\cap\gamma G_i\supseteq\gamma(\delta_iH\cap G_i)$. But certainly $\gamma(\delta_iH\cap G_i)\subseteq\gamma(\delta_iH)\cap\gamma G_i$ for any operator γ.

THEOREM 1. *Let G be a commutative Ω-group, and let Ω contain an automorphism γ in its centrum. Let G be the direct sum of the Ω-subgroups $G_1, G_2, \cdots, G_n$, and let G contain the Ω-subgroup H. If for every Ω-subgroup A_i of G_i the Ω-group $A_i/\gamma A_i$ satisfies the minimal (maximal) chain condition for Ω-subgroups of $A_i/\gamma A_i$, then the Ω-group $H/\gamma H$ satisfies the minimal (maximal) chain condition for Ω-subgroups of $H/\gamma H$.*

A chain of Ω-subgroups

$$\text{(A)}\qquad H\supset H_1\supset H_2\supset\cdots\supset\gamma H$$

implies, by Lemmas 1 and 3, the existence of the $2n$ chains

$$\alpha_iH\supseteq\alpha_iH_1\supseteq\alpha_iH_2\supseteq\cdots\supseteq\gamma(\alpha_iH),\qquad i=1,2,\cdots,n,$$

$$\text{(B)}\quad \begin{aligned}&\delta_iH\cap G_i\supseteq\delta_iH_1\cap G_i\supseteq\delta_iH_2\cap G_i\\&\qquad\supseteq\cdots\supseteq\gamma(\delta_iH\cap G_i),\qquad i=1,2,\cdots,n.\end{aligned}$$

Lemma 2 implies that if the chain (A) is infinite, at least one of the chains (B) must be nontrivially infinite. If the minimal chain condition fails in $H/\gamma H$, it must fail in one of the groups $\alpha_i H/\gamma(\alpha_i H)$ or $\delta_i H\cap G_i/\gamma(\delta_i H\cap G_i)$ where $\alpha_i H$ and $\delta_i H\cap G_i$ are Ω-subgroups of G_i.

The statement of the theorem for maximal chains follows by the same argument.

3. **Chains of g-modules.** Let g be a domain of integrity with Noether ideal theory. This implies that in g every ideal is the product of powers of prime ideals and that a prime ideal is divisorless. If P is the quotient field of g, fractional ideals are defined in P. The set of all ideals in P forms a group under multiplication. In particular if $\mathfrak{a}$ is an ideal, $\mathfrak{a}^{-1}$ will exist such that $\mathfrak{a}\mathfrak{a}^{-1}=g$, and if $\mathfrak{a}\mathfrak{c}=\mathfrak{b}\mathfrak{c}$, then $\mathfrak{a}=\mathfrak{b}$.

A g-module in P is a set of elements of P which forms a group under addition and is closed under multiplication by elements of g. The g-module $\mathfrak{a}$ is an ideal if $\alpha\mathfrak{a}\subseteq g$ for some element $\alpha\neq 0$ of g. The product of an ideal contained in g and a g-module $\mathfrak{a}$ is contained in $\mathfrak{a}$. If $\mathfrak{a}\supset\mathfrak{b}$, the group $\mathfrak{a}/\mathfrak{b}$ is a g-module (not contained in P).

LEMMA 4. *If g has a Noether ideal theory, and if $\mathfrak{a}$ is a g-module in the quotient field P of g, the g-module $\mathfrak{a}/\alpha\mathfrak{a}$ has a composition series for any element $\alpha\neq 0$ of g.*

Let $\mathfrak{a}$ be a g-module contained in P, and let α be an element not equal to 0 of g. If the principal ideal (α) has the factorization $\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_s^{r_s}$ in g, we shall prove that the chain of g-modules

$$\mathfrak{a}\supseteq\mathfrak{p}_1\mathfrak{a}\supseteq\mathfrak{p}_1^2\mathfrak{a}\supseteq\cdots\supseteq\mathfrak{p}_1^{r_1}\mathfrak{a}\supseteq\mathfrak{p}_1^{r_1}\mathfrak{p}_2\mathfrak{a}\supseteq\cdots\supseteq\alpha\mathfrak{p}_s^{-1}\mathfrak{a}\supseteq\alpha\mathfrak{a}$$

allows no nontrivial refinement. The series

$$\begin{aligned}\mathfrak{a}/\alpha\mathfrak{a}\supseteq\mathfrak{p}_1\mathfrak{a}/\alpha\mathfrak{a}\supseteq\mathfrak{p}_1^2\mathfrak{a}/\alpha\mathfrak{a}\supseteq\cdots\supseteq\mathfrak{p}_1^{r_1}\mathfrak{a}/\alpha\mathfrak{a}&\\ \supseteq\mathfrak{p}_1^{r_1}\mathfrak{p}_2\mathfrak{a}/\alpha\mathfrak{a}\supseteq\cdots\supseteq\alpha\mathfrak{p}_s^{-1}\mathfrak{a}/\alpha\mathfrak{a}\supseteq(0)&\end{aligned}$$

will include a composition series for $\mathfrak{a}/\alpha\mathfrak{a}$.

Let $\mathfrak{p}$ be a prime ideal in g, and let $\mathfrak{b}$ be a g-module contained in P. Assume that between $\mathfrak{b}$ and $\mathfrak{p}\mathfrak{b}$ there lies a g-module $\mathfrak{c}$ equal to neither: $\mathfrak{b}\supset\mathfrak{c}\supset\mathfrak{p}\mathfrak{b}$. Then there is an element β of $\mathfrak{b}$ not contained in $\mathfrak{c}$ and an element γ of $\mathfrak{c}$ not contained in $\mathfrak{p}\mathfrak{b}$. We form the chain of ideals of P: $(\beta,\gamma)\supset(\mathfrak{p}\beta,\gamma)\supset\mathfrak{p}(\beta,\gamma)$. Since $\mathfrak{p}\beta\subseteq\mathfrak{p}\mathfrak{b}\subset\mathfrak{c}$ and $\gamma\in\mathfrak{c}$, $(\mathfrak{p}\beta,\gamma)\subseteq\mathfrak{c}$. But β is not an element of $\mathfrak{c}$, and therefore $(\mathfrak{p}\beta,\gamma)$ and (β,γ) are distinct. Since $\mathfrak{p}(\beta,\gamma)\subseteq\mathfrak{p}\mathfrak{b}$ and γ is not an element of $\mathfrak{p}\mathfrak{b}$, $\mathfrak{p}(\beta,\gamma)$ and $(\mathfrak{p}\beta,\gamma)$ are distinct. It would follow that $g\supset(\mathfrak{p}\beta,\gamma)(\beta,\gamma)^{-1}\supset\mathfrak{p}$ is a chain of distinct ideals in g. However, the prime ideal $\mathfrak{p}$ is divisorless. It follows that $\mathfrak{a} \supset \mathfrak{p}\mathfrak{a}$ allows no nontrivial refinement.

If M is a P-module with linearly independent P-basis $x_1, x_2, \cdots, x_n$ we shall write $M = Px_1 + Px_2 + \cdots + Px_n$.

THEOREM 2. *Let* $M = Px_1 + Px_2 + \cdots + Px_n$ *contain the g-module* N. *Then if* γ *is an element not equal to* 0 *of* g, *the g-module* $N/\gamma N$ *has a composition series.*

The module N is a g-submodule of the direct sum $Px_1 + Px_2 + \cdots + Px_n$. The element γ of g is an automorphism of M, and the operator domain g is commutative. Lemma 4 assures us that for every g-subgroup $\mathfrak{a}x_i$ of Px_i the g-module $\mathfrak{a}x_i/\gamma(\mathfrak{a}x_i) \cong \mathfrak{a}/\gamma\mathfrak{a}$ has a composition series. The conditions of Theorem 1 are satisfied, and the g-module $N/\gamma N$ must have a composition series.

4. **Orders of finite linear algebras.** We shall again assume that g is a domain of integrity with Noether ideal theory and that P is the quotient field of g. We consider a linear algebra A with identity e of order n over the field P.

An order R of A which contains g can be defined to be a subring of A which contains g and a basis for A [2, p. 124]. We shall consider only orders of A which contain g. A left (right) R-ideal of R is a submodule $\mathfrak{M}$ of R such that $R\mathfrak{M} \subseteq \mathfrak{M}$ ($\mathfrak{M}R \subseteq \mathfrak{M}$) and which contains a regular element of A. Then $\mathfrak{M}$ contains an element $\gamma \neq 0$ of g and contains the two-sided ideal γR: every order R is bounded. Since R contains g, R and every R-ideal of R are g-modules.

THEOREM 3. *Let g be a domain of integrity with Noether ideal theory, and let P be the quotient field of g. If A is a linear algebra with identity of finite order over P, every order of A which contains g will satisfy the maximal condition for any chain of left (right) R-ideals contained in R and the minimal condition for any chain of left (right) R-ideals in R containing a fixed left (right) R-ideal.*

We may consider the algebra A to be the P-module $Px_1 + Px_2 + \cdots + Px_n$ where $x_1, x_2, \cdots, x_n$ constitute a linearly independent basis for A over P, and R as a g-submodule of A. An R-ideal $\mathfrak{M}$ of R contains an element $\gamma \neq 0$ of g so that $R \supseteq \mathfrak{M} \supseteq \gamma R$. By Theorem 2 every chain of g-modules between R and γR must be finite. In particular a chain of R-ideals between R and $\mathfrak{M}$ must be finite since an R-ideal is a g-module if R contains g.

Two orders R and R' are said to be equivalent if there exist regular elements a, b, c, d of A such that $aRb \subseteq R'$, $cR'd \subseteq R$. An order is said to be maximal if it is contained in no equivalent order.

The Asano treatment of the ideal theory of a class of equivalent orders depends on three postulates:

I. There exists a maximal bounded order R in the class.

II. The minimal chain condition holds for left R-ideals in R which contain a fixed two-sided R-ideal.

III. The maximal chain condition holds for two-sided R-ideals contained in R.

In Theorem 3 we have shown that postulates II and III are satisfied by any order of A which contains g. If a maximal order exists, it must be bounded since every order is bounded.

An order of A which contains g and contains only integral elements of A is called an integral domain. A maximal integral domain is an integral domain which is contained in no other integral domain.

LEMMA 5. *If the order R contains g and is equivalent to the integral domain S, then R is an integral domain.*

Since R is equivalent to S there exist regular elements a, b such that $aRb \subseteq S$. Since R is an order of A there exists in g an element $\beta \neq 0$ such that βb^{-1} is an element of R. Then $\beta b^{-1}R \subseteq R$. Similarly S, which is an order of A, must contain αa^{-1} for some element $\alpha \neq 0$ of g, and $\alpha S a^{-1} \subseteq S$. Then

$$\alpha[a(\beta b^{-1}R)b]a^{-1} \subseteq \alpha[aRb]a^{-1} \subseteq \alpha S a^{-1} \subseteq S,$$

or

$$(ab^{-1})(\alpha\beta)R(ba^{-1}) \subseteq S.$$

Set $\alpha\beta = \gamma$, $ab^{-1} = c$; then $c(\gamma R)c^{-1} \subseteq S$, and $\gamma R \subseteq c^{-1}Sc$ where c is a regular element of A. It follows that γR consists only of integral elements of A.

Let r be an element of R. Let $g[r]$ indicate the polynomial domain generated by r with coefficients in g; $g[r]$ is a commutative ring contained in R. Further $\gamma g[r]$ is a ring of integers. If we consider that $g[r]$ is a g-module contained in the P-module $A = Px_1 + Px_2 + \cdots + Px_n$ we may apply Theorem 2 to $g[r]$ and obtain that every chain of g-modules between $\gamma g[r]$ and $g[r]$ is finite. If H is the union of g and $\gamma g[r]$, H is a ring of integers, and $\gamma g[r] \subseteq H \subseteq g[r]$. Since $g \subset H$, the chain of H-modules

$$H \subseteq Hr \subseteq (Hr, Hr^2) \subseteq \cdots \subseteq g[r]$$

is a chain of g-modules between H and $g[r]$ and must be finite in length. It follows that r satisfies an equation $r^k = h_1 r^{k-1} + h_2 r^{k-2} + \cdots + h_k r$ with coefficients in H. Then r is g-integral, and R is an integral domain [3, p. 90].

COROLLARY. *A maximal integral domain S is a maximal order in the class of orders equivalent to S.*

We can now establish the existence in A of a maximal order by the following argument: Let all integral domains S_α of A be well-ordered. Construct a chain

$$S \subset S_{\sigma_1} \subset S_{\sigma_2} \subset \cdots$$

of domains containing a fixed domain S by choosing S_{σ_1} to be the first which contains S, S_{σ_2} to be the first which contains S_{σ_1}, and so on. The union R of the S_{σ_i} will be a maximal integral domain and, by the above corollary, R is a maximal order. The class of orders equivalent to R will satisfy the Asano postulates.

THEOREM 4. *Let g be a domain of integrity with Noether ideal theory, and let P be the quotient field of g. Every linear algebra with identity of finite order P contains a nontrivial class of orders which satisfy the Asano postulates and which contain only integral elements of the algebra.*

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PURDUE UNIVERSITY

ON A PROBLEM OF KUROSCH AND JACOBSON

IRVING KAPLANSKY

1. Introduction. Let A be an algebraic algebra over a field K, that is, an algebra over K each of whose elements satisfies a polynomial equation with coefficients in K. In analogy to Burnside's problem for groups, Kurosch[1] has raised the following question: if A is finitely generated, does it necessarily have a finite basis? Jacobson[2] studied the question for the case where the elements of A are of bounded degree, and reduced it to the consideration of certain specific nil algebras defined as follows: $A(r, n) = F(r) - I(r, n)$, where $F(r)$ is the free algebra generated over K by indeterminates $u_1, \cdots, u_r$, and $I(r, n)$ is the (two-sided) ideal generated by all nth powers in $F(r)$. In this note we shall prove the following theorem.

THEOREM 1. *If K has at least n elements, $A(r, n)$ has a finite basis.*

Thus Kurosch's question for algebraic algebras of bounded degree receives an affirmative answer if K is large enough, and in particular if it is infinite.

In §3, by a different method suggested by Kurosch's treatment of $n=3$, we prove that $A(r, 4)$ has a finite basis over $GF(3)$. In §4 we discuss a special case of another question proposed by Jacobson: if the dimension $d(r, n)$ of $A(r, n)$ is finite, what is its precise value? We show that $d(2, 3)$ may be equal to 16 or 17, depending on K.

2. Proof of Theorem 1. Throughout this section we shall assume that the coefficient field K has at least n elements.

The algebra $F(r)$ consists of all (noncommutative) polynomials in the u's with coefficients in K: that is, linear combinations of terms $u_i u_j u_k \cdots$ which we shall call monomials. The degree of a monomial is the number of u's it contains, and a polynomial is homogeneous if its monomials all have the same degree. We now prove the following lemma.

LEMMA 1. *The ideal $I(r, n)$ has a basis of homogeneous polynomials.*

PROOF. Specifically, $I = I(r, n)$ has a basis consisting of all

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[1] *Ringtheoretische Probleme die mit dem Burnsideschen Problem über periodische Gruppen in Zusammenhang stehen*, Bull. Acad. Sci. URSS. Ser. Math. vol. 5 (1941) pp. 233–240.

[2] *Structure theory for algebraic algebras of bounded degree*, Ann. of Math. (2) vol. 46 (1945) pp. 695–707.

$S(x^iy^jz^k \cdots)$ where $x, y, z, \cdots$ are monomials, $i+j+k+\cdots = n$, and S denotes (as it will throughout the paper) the sum of products taken in all the $n!/i!j!k! \cdots$ possible orders. That these polynomials generate at least $I(r, n)$ follows from the fact that for any scalars $\alpha, \beta, \gamma, \cdots, (\alpha x+\beta y+\gamma z+\cdots)^n$ is a linear combination of such polynomials.

We show conversely that every $S(x^iy^jz^k \cdots)$ is in I. We have $T=(\alpha x+\beta y+\gamma z+\cdots)^n \in I$ for any scalars $\alpha, \beta, \gamma, \cdots$. Let $T_i = T_i(\alpha)$ be the sum of all terms in the expansion of T which are of degree i in x. Then $T=\sum_{i=0}^{n} \alpha^i T_i(1)$. Since T_0 and T_n are in I, the sum from 1 to $n-1$ is likewise in I. Applying this with $n-1$ different nonzero values of α, and multiplying by the minors of the resulting Vandermonde determinant, we see that $T_i \in I$. We now set $\alpha=1$ and let $T_{ij}=T_{ij}(\beta)$ be the subset of T_i containing those terms of degree j in y; then $T_i=\sum_{j=0}^{n-1} \beta^j T_{ij}(1)$. We may assume by induction on the number of terms $x, y, z, \cdots$ that $T_{i0} \in I$. Taking $n-i$ different nonzero values for β, we obtain $T_{ij} \in I$. Continuing in this fashion, we prove that $T_{ijk\cdots} = S(x^iy^jz^k \cdots) \in I$.

Any element of $F(r)$ may evidently be written as a unique sum of homogeneous polynomials, which we may call its homogeneous parts. From Lemma 1 we have at once the following lemma.

LEMMA 2. *Any element of $F(r)$ is in $I(r, n)$ if and only if each of its homogeneous parts is in $I(r, n)$.*

In order to prove Theorem 1 by induction, we prove a stronger result.

LEMMA 3. *Suppose that B is an ideal in $A=A(r, n)$ generated by a finite number of ith powers of elements of A, and that $A-B$ is nilpotent. Then A is nilpotent.*

PROOF. Lemma 3 is certainly true for $i=n$. We make a descending induction on i. Suppose B is generated by $y_1^i, \cdots, y_m^i$ and that $(A-B)^k=0$. Let M be the set of all monomials of degree less than $k(m+1)$. Let C be the ideal generated by $y_1^{i+1}, \cdots, y_m^{i+1}$, and by all x^{i+1}, $(y+\alpha x)^{i+1}$, where y ranges over y_i, x ranges over M, and α runs through i different nonzero scalars. Use of the Vandermonde determinant shows that

$$S(y^ix) = y^ix + y^{i-1}xy + \cdots + xy^i$$

is in C. Right-multiplying by y^i, we have $y^ixy^i \in C$.

We shall now show that $A-C$ is nilpotent, whence by induction A is. Let t be any monomial of degree $k(m+1)$. We may write

$t=z_1z_2\cdots z_{m+1}$, with each z a monomial of degree k. Since $(A-B)^k=0$ each z is in B and can be written as a sum of terms ay^ib; moreover by Lemma 2 we may assume that each of these terms ay^ib has degree k. Now in a product of $m+1$ terms, each of the form ay^ib, at least one y must get repeated; hence such a product contains y^ixy^i with x a monomial of degree less than $k(m+1)$. As shown above, $y^ixy^i\in C$, whence $t\in C$, and $A-C$ is nilpotent of index at most $k(m+1)$.

Theorem 1 is an immediate consequence of Lemma 3. We take the case $i=1$, $B=(u_1,\cdots,u_r)=A$. The hypothesis that $A-B$ is nilpotent is fulfilled, and we conclude that A is nilpotent and has a finite basis.

3. **The case** $n=4$. Kurosch gave a direct combinatorial proof that $A(r,3)$ has a finite basis, which, unlike the above proof, is also valid over $GF(2)$. We shall show that his argument can be extended so as to push the case $n=4$ nearer completion.

Theorem 2. *$A(r, 4)$ has a finite basis over $GF(3)$.*

Proof. Define

$$f(x,y,z)=(x+y+z)^4-(x+y)^4-(x+z)^4 \\ -(y+z)^4+x^4+y^4+z^4.$$

Then

$$x^3yx^3=x^2[f(x,y,x^2)+f(-x,y,x^2)-x^2yx^2-x^3yx]=0.$$

Hence the (two-sided) ideal generated by x^3 is nilpotent. More generally, any ideal B generated by a finite number of cubes is nilpotent. To show that $A=A(r,4)$ is nilpotent it therefore suffices to prove $A-B$ nilpotent, that is, we may take any finite number of cubes to be zero without loss of generality.

We may suppose by induction that the subalgebra C generated by $u_1,\cdots,u_{r-1}$ has a finite basis $\{t_i\}$, and let us write u for u_r. Set u^3, t_i^3, and $(u\pm t_i)^3$ all equal to zero. Then $u^2t_i+ut_iu+t_iu^2=0$, whence for any $x\in C$

$$u^2x+uxu+xu^2=0, \tag{1}$$

and right-multiplying by u^2 we have

$$u^2xu^2=0. \tag{2}$$

Using (1) to replace uxu by $-u^2x-xu^2$ and then using (2), we see that any monomial containing four u's vanishes. This proves the theorem.

4. The order of $A(2, 3)$. In the case $n=2$, we have $(x+y)^2=0$ and hence $xy=-yx$ for any x, y. In the case of characteristic 2, $A(r, 2)$ is commutative, and its order $d(r, 2)$ is 2^r-1 as observed by Jacobson.[3] For characteristic not equal to 2, we have $(xy)z=-z(xy)$ but also $x(yz)=-x(zy)=z(xy)$, so that $xyz=0$. It follows that the elements u_i and u_iu_j $(i<j)$ constitute a basis, that is, $d(r, 2)=r+{}_rC_2={}_{r+1}C_2$.

We shall now evaluate $d(r, n)$ in the next simplest case: $r=2$, $n=3$. The result seems to indicate that a complete evaluation of $d(r, n)$ will involve formidable combinatorial difficulties.

THEOREM 3. *If the coefficient field is $GF(2)$ or if it has characteristic 3, then $d(2, 3)=17$; otherwise $d(2, 3)=16$.*

PROOF. From $(x+y)^3=0$ we obtain

$$S(x^2y) + S(xy^2) = 0, \tag{3}$$

and from

$$(x + y + z)^3 = (x + y)^3 + (x + z)^3 + (y + z)^3 + S(xyz) - x^3 - y^3 - z^3, \tag{4}$$

we have

$$S(xyz) = 0. \tag{5}$$

Take $x=u$, $y=v$, $z=u^2$ in (5):

$$uvu^2 + u^2vu = 0.$$

Together with

$$S(u^2v) + S(uv^2) = 0 \tag{6}$$

this gives

(7) $S(u^2v)$ and $S(uv^2)$ are annihilated by u and v on both left and right.

Next, by taking $x=u$, $y=v$, $z=uv$ in (5) and using (7), we have

$$vuvu = u^2v^2, \tag{8}$$

which implies

$$uvuvu = vuvuv = 0. \tag{9}$$

From (7), (8), and (9) it is easy to see that

(10) all monomials of degree $\geqq 7$ vanish.

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[3] Loc. cit. p. 707.

Finally take $x=u$, $y=v$, $z=uvuv$ in (5) and make repeated use of (7) and (9). We find

$$3u^2vuv^2 = 0. \tag{11}$$

We now assert that for $K=GF(2)$, (6)–(11) is an exhaustive set of relations on u, v; for $K\neq GF(2)$, (6) and (7) may be strengthened to

$$S(u^2v) = S(uv^2) = 0, \tag{12}$$

and (8)–(12) is an exhaustive set of relations. To prove this it suffices to assume the relations in question, and show that they ensure the evanescence of every cube. For the latter it suffices to prove that the cube of any monomial vanishes and that (3) and (5) hold for any monomials x, y, z. By (10), there are only a finite number of verifications to make; we omit the details of the computation.

We can now make a list of basis elements. It is convenient to note, by systematic use of (7), that candidates for basis elements of degree not less than 4 need contain $u^2(v^2)$ only at the beginning (end).

Degree
1: u, v
2: u^2, uv, vu, v^2
3: u^2v, uvu, vuv, uv^2 [add vu^2 for $K=GF(2)$]
4: $u^2v^2, uvuv, u^2vu, vuv^2$
5: $u^2vuv, uvuv^2$
6: u^2vuv^2 [characteristic 3 only].

Added in proof: Professor Jacobson has pointed out to me that (by §9 of his paper) an affirmative answer to Burnside's problem for fourth powers would imply that $A(r, 4)$ has a finite basis over $GF(2)$, and thus complete the case $n=4$. Moreover the known affirmative answer for the case of two generators shows that $d(2, 4)$ over $GF(2)$ is finite, and has the explicit (though probably much too large) upper bound of 1024.

UNIVERSITY OF CHICAGO

ABSOLUTE RETRACTS IN GROUP THEORY

REINHOLD BAER

The subgroup R of the group G has been termed a retract[1] of the group G whenever there exists an idempotent endomorphism of G which maps G upon R. This definition is in strict analogy to the topological concept of retract. Thus one may be tempted to define absolute retracts in like similarity to topological usage. However, we shall prove in the course of the present note that the identity is the only group which is a retract of every containing group. Consequently only modifications of the topological concept will be useful, and we shall show that each of the following classes of groups may in a certain sense lay claim to the title of absolute retract: the complete groups, the abelian groups the orders of whose elements are finite and square free, and the free groups.

The following definition of the concept of retract is equivalent to the one given above, but it will be a little bit easier to handle: The group R is a retract of the group G if R is a subgroup of G, and if there exists an endomorphism ϵ of G with the following properties:

$$G^{\epsilon} \leqq R \quad \text{and} \quad r^{\epsilon} = r$$

for r in R.

Complete groups (Carmichael [2, p. 81]) have been defined as groups with the following properties: each of their automorphisms is an inner automorphism and their center consists of the identity only. Thus groups are complete if they are "essentially" identical with their group of automorphisms.

THEOREM 1. *The group G is complete if, and only if, it meets the following requirement:*

(*) *If G is a normal subgroup of the group E, then G is a retract of E.*

PROOF. Assume first the validity of condition (*). Consider the automorphism α of G. Then there exists (Zassenhaus [1, pp. 93, 94]) one and essentially only one group A which is obtained by adjoining to G an element t subject to the relations

$$t^{-1}gt = g^{\alpha}$$

for g in G. Clearly G is a normal subgroup of A and A/G is an infinite

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[1] See Baer [2, chap. II, §3] for elementary properties of retracts. Many applications of the concept of retract may be found there too. Numbers in brackets refer to the Bibliography at the end of the paper.

cyclic group. Hence there exists by (*) an endomorphism β of A with the following properties:

$$A^\beta \leqq G \quad \text{and} \quad g^\beta = g$$

for g in G. In particular $s = t^\beta$ is an element in G, and this element satisfies, for every element g in G,

$$s^{-1}gs = (t^{-1})^\beta g^\beta t^\beta = (t^{-1}gt)^\beta = g^{\alpha\beta} = g^\alpha,$$

since g and g^α belong to G. Thus the automorphism α is induced by the element s in G, proving that every automorphism of G is inner.

Consider next the element z in the center of G. Then there exists (Zassenhaus [1, p. 96, Satz 21]) one and essentially only one group Z which is obtained by adjoining to G elements x and y subject to the following relations:

$$x^{-1}y^{-1}xy = z, \quad xg = gx \quad \text{and} \quad yg = gy$$

for every g in G. Then G is a normal subgroup of Z and Z/G is a free abelian group of rank 2. Hence there exists by (*) an endomorphism ϵ of Z with the following properties:

$$Z^\epsilon \leqq G \quad \text{and} \quad g^\epsilon = g$$

for every g in G. If g is an element in G, then

$$x^\epsilon g = x^\epsilon g^\epsilon = (xg)^\epsilon = (gx)^\epsilon = g^\epsilon x^\epsilon = gx^\epsilon,$$

proving that x^ϵ belongs to the center of G, and likewise we show that y^ϵ belongs to the center of G. Consequently

$$z = z^\epsilon = x^{-\epsilon}y^{-\epsilon}x^\epsilon y^\epsilon = 1.$$

Thus we have shown that 1 is the only element in the center of G, and this completes the proof of the fact that every group with the property (*) is complete.

Assume conversely that the group G is complete and that G is a normal subgroup of the group T. Denote by S the centralizer of G in T (which consists of all those elements in T which commute with every element in G). Since every automorphism of G is an inner automorphism, it follows that $T = GS = SG$; and since the identity is the only element in the center of G, it follows that 1 is the cross cut of G and S. Hence T is the direct product (Carmichael [1, p. 96, no. 10]) of G and S, and this fact clearly implies that G is a retract of T, completing the proof.

Groups containing a given group G as normal subgroups are termed (Zassenhaus [1, p. 89 ff.]) extensions of G. It may be noted that in the first part of the proof of Theorem 1 we made use only of extensions of G by abelian groups. This shows the validity of the following proposition:

COROLLARY 1. *The group G is complete if (and only if) it is a retract of every extension of G by an abelian group.*

REMARK. Examples of complete groups are provided by all the symmetric groups of finite or infinite degree,[2] not less than 7. This shows in particular that every group is a subgroup of a complete group and thus a subgroup of a group, meeting requirement (*) of Theorem 1.

THEOREM 2. *The identity is the only group which is a retract of every containing group.*

PROOF. Suppose that the group G is a retract of every group which contains G as a subgroup. Denote by D the direct product of G and of a group H isomorphic to G, and denote by η a definite isomorphism of G upon H. Denote by W the essentially uniquely determined group (Zassenhaus [1, pp. 93, 94]) which is obtained by adjoining to D an element w subject to the relations:

$$w^2 = 1, \qquad wgw = g^\eta$$

for g in G. Then the element w of order 2 induces in D the automorphism which interchanges the elements g and g^η for g in G. Clearly D is a normal subgroup of W and W/D of order 2. Since, therefore, G is a subgroup of the group W, there exists an endomorphism δ of W with the following properties:

$$W^\delta \leqq G \quad \text{and} \quad g^\delta = g$$

for g in G. We note next that every element in G commutes with every element in H. If x is an element in H, and if g is an element in G, then

$$g\,x^\delta = g^\delta x^\delta = (gx)^\delta = (xg)^\delta = x^\delta g^\delta = x^\delta g,$$

proving that x^δ is in the center of G, since x^δ belongs to G. Hence H^δ is part of the center of G.

If g is any element in G, then g^η is in H and $g^{\eta\delta}$ belongs, therefore, to the center of G. Hence

$$w^\delta g = w^\delta g^\delta = (wg)^\delta = (g^\eta w)^\delta = g^{\eta\delta} w^\delta = w^\delta g^{\eta\delta},$$

since w^δ is in G too. Consequently $g = g^{\eta\delta}$, proving that $G = H^\delta$. Thus

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[2] See Baer [1], Schreier and Ulam [1, 2].

G is commutative, since we showed before that H^δ is part of the center of G.

But G is a retract of each of its extensions, and thus it follows from Theorem 1 that the center of G is 1. Hence $G=1$, as we intended to prove.

REMARK. The situation would be changed considerably if we restricted the class of admissible groups. We mention in this respect the following well known theorem: The abelian group A is a retract of every abelian group if, and only if, $A=A^p$ for every prime number p. On the other hand the preceding theorems would not be affected at all by restricting oneself to finite groups.

THEOREM 3. *Every subgroup of the group G is a retract of G if, and only if, G is an abelian group the orders of whose elements are finite and square free.*

PROOF. If G is an abelian group the orders of whose elements are square free integers, then every subgroup of G is known to be a direct factor of G. But direct factors are retracts, proving the sufficiency of our condition.

Assume conversely that every subgroup of G is a retract of G. Then there exists to every element g in G an endomorphism ϵ of G with the following properties: $g^\epsilon = g$ and G^ϵ is part of the cyclic subgroup generated by g.

Denote by E the kernel of the endomorphism ϵ. Then G/E is a cyclic group. This implies in particular that the commutator subgroup of G is part of E. If $g \neq 1$, then g is certainly not in E. Thus it is impossible that an element, not 1, is in the commutator subgroup of G. Consequently G is abelian.

Consider an element z in G and a positive integer n. Then there exists an endomorphism ν of G with the following properties: $(z^n)^\nu = z^n$ and G^ν is part of the cyclic subgroup generated by z^n.

The element z itself is therefore mapped by ν upon a power of z^n. Hence there exists an integer $m = m(n, z)$ such that $z^\nu = z^{nm}$. Consequently we find that

$$z^n = (z^\nu)^n = z^{nmn} \quad \text{or} \quad z^{n(1-mn)} = 1.$$

Let in particular $n=2$. Then $1-2m(2, z)$ is an odd number r such that $z^{2r}=1$, proving that z is of finite order. Next we take for n any divisor of the order of z. Then n and $1-mn(n, z)$ are relatively prime numbers, and we infer from $z^{n(1-mn)}=1$ that the order of z is square free. This completes the proof.

Dualizing the concept of retract of a group, we say that *the quotient* *group G/N is retractable* whenever there exists an idempotent endomorphism of G whose kernel is exactly N. This is equivalent to saying that G is a splitting extension of N by G/N. We note the following known theorem:[3]

The group F is a free group if, and only if, every quotient group, isomorphic to F, is retractable.

This theorem may be considered a dual of either or both Theorems 1 and 2. In order to find a dual of Theorem 3 one has to determine the class of *totally retractable groups*. Here we term a group totally retractable if each of its quotient groups is retractable. A complete characterization of this class of groups does not seem to be available as yet. Thus the following criteria may be of interest.

THEOREM 4. *The abelian group A is totally retractable if, and only if, the orders of the elements in A are finite and square free.*

PROOF. If the orders of the elements in A are finite and square free, then every subgroup of A is a direct factor of A. Consequently every quotient group is retractable and A is totally retractable.

Assume conversely that A is totally retractable. Denote by F the subgroup A which consists of the elements of finite order in A. If $F \neq A$, then it is possible to construct (in many ways) a subgroup W of A such that $F \leqq W < A$ and such that every element in A/W is of finite order. But every endomorphism with kernel W would map A into F. Such an endomorphism cannot then be idempotent, a contradiction which proves that $F = A$, that is, that every element in A is of finite order.

Denote by P the subgroup of A which consists of all the elements in A whose order is a factor of the given prime number p. Then there exists an idempotent endomorphism ϵ of A whose kernel is P. Since 1 is the cross cut of P and A^{ϵ}, it is impossible that A^{ϵ} contains elements of order p. Thus 1 is the only element of order a power of p in A^{ϵ}. Since A/P and A^{ϵ} are isomorphic groups, it follows that P contains all the elements of order a power of p in A, and this fact implies that the orders of the elements in A are square free, as we intended to prove.

THEOREM 5. *If the group G is totally retractable, then it has the following properties:*

(a) *Every quotient group of G is totally retractable.*

(b) *Every subgroup of the center of G is a direct factor of G.*

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[3] See, for example, Baer [2, chap. II, §1].

(c) *The commutator subgroup G' of G is generated by the commutators of the form $x^{-1}y^{-1}xy$ for x in G and y in G'.*

PROOF. Proposition (a) is an almost immediate inference from the fact that every normal subgroup of the quotient group G/N has the form M/N for M a normal subgroup of G. To prove (b) consider a subgroup S of the center of G. Then S is a normal subgroup of G. Hence there exists an idempotent endomorphism ϵ of G whose kernel is S. Since the elements in S commute with every element in G, it follows that G is the direct product of S and G^{ϵ}. To prove (c) denote by G'' the subgroup generated by the commutators $x^{-1}y^{-1}xy$ for x in G and y in G'. Clearly $G'' \leqq G'$ and G'' is a normal subgroup of G. It follows from (a) that G/G'' is totally retractable. Since G'/G'' is part of the center of G/G'', it follows from (b) that G'/G'' is a direct factor of G/G''. But G'/G'' is the commutator subgroup of G/G'', and a commutator subgroup which is an abelian direct factor is necessarily the identity. Hence $G' = G''$, completing the proof.

Proposition (b) and (c) of Theorem 5 imply in particular the fact that nilpotent, totally retractable groups are always abelian. On the other hand already the non-abelian group of order 6 provides an example of a soluble, non-abelian group which is totally retractable.

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LATTICES, EQUIVALENCE RELATIONS, AND SUBGROUPS

PHILIP M. WHITMAN

1. **Introduction.** A lattice[1] is a set of elements A, B, $\cdots$ together with a relation $\leqq$ defined between some or all pairs of the elements such that $A \leqq A$ for each A; if $A \leqq B$ and $B \leqq A$ then $A = B$; if $A \leqq B$ and $B \leqq C$ then $A \leqq C$; and each pair of elements A and B have a least upper bound $A \cup B$ and a greatest lower bound $A \cap B$.

An equivalence relation[2] R' is a relation $\sim$ defined between some or all pairs of elements ("points") of a set $\mathfrak{S}$ such that $p \sim p$ for every point p; if $p \sim q$ then $q \sim p$; if $p \sim q$ and $q \sim r$ then $p \sim r$. If more than one relation is under discussion, we may specify the one used at the moment by writing "$p \sim q$ in R'."

It is well known that the collection of all equivalence relations on a given set $\mathfrak{S}$ forms a lattice. Ore [4] has characterized lattices which are isomorphic to the lattice $\mathfrak{L}^*$ of all equivalence relations on some set $\mathfrak{S}$. One could go farther and ask what lattices are isomorphic to sublattices of $\mathfrak{L}^*$. Our answer to this is: *any* lattice is isomorphic to a sublattice of the lattice of all equivalence relations on some set; more concisely, any lattice is a lattice of equivalence relations.

Garrett Birkhoff has shown [1] that any lattice of equivalence relations is isomorphic to a lattice of subgroups. Therefore the result stated in the previous paragraph implies: any lattice is isomorphic to a sublattice of the lattice of all subgroups of a suitable group.

2. **Outline.** The first and larger part of the paper relates lattices and equivalence relations. Since the formal construction and proof in this part are somewhat lengthy and complicated, we first outline the main steps and indicate the motivation.

A lattice $\mathfrak{L}$ is given; we wish to show that there is some set $\mathfrak{S}$, and some sublattice $\mathfrak{L}'$ of the lattice $\mathfrak{L}^*$ of all equivalence relations on $\mathfrak{S}$, such that $\mathfrak{L}$ and $\mathfrak{L}'$ are isomorphic.

We shall take $\mathfrak{S}$ as the union of disjoint subsets $\mathfrak{A}$, $\mathfrak{B}$, $\cdots$, one for each element A, B, $\cdots$ of $\mathfrak{L}$. The equivalence relations A', B', $\cdots$ corresponding to A, B, $\cdots$ must be chosen in such a manner that

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[1] Sometimes called a *structure*. For a thorough discussion of lattices, see Birkhoff [2]. Numbers in brackets refer to the references cited at the end of the paper.

[2] For a discussion of the relevant properties of equivalence relations (or *congruence relations* as they are sometimes called), see Birkhoff [1], Dubreil and Dubreil-Jacotin [3], or Ore [4].

$A \leqq B$ implies $A' \leqq B'$ and that $(A \cup B)' = A' \cup B'$ and $(A \cap B)' = A' \cap B'$; this will make $\mathfrak{L}'$ a homomorphic image of $\mathfrak{L}$. For an isomorphism, $A' \leqq B'$ must imply $A \leqq B$. To insure this we propose that the subset $\mathfrak{A}$ shall contain two points, $l(A)$ and $r(A)$, such that $l(A) \sim r(A)$ in B' if and only if $A \leqq B$. The notations $l(A)$ and $r(A)$ may be thought of as standing for "left end of $\mathfrak{A}$" and "right end of $\mathfrak{A}$." Similarly $\mathfrak{B}$ is to contain points $l(B)$ and $r(B)$ which are equivalent in Z' if and only if $B \leqq Z$, and so on. However, for simplicity we shall devote our attention to $\mathfrak{A}$ and the points and subsets thereof except when it becomes necessary to refer to $\mathfrak{B}$, and so on.

One's first thought might be to have $\mathfrak{A}$ consist merely of $l(A)$ and $r(A)$, and define B' insofar as it concerns points of $\mathfrak{A}$ by specifying $l(A) \sim r(A)$ in B' if and only if $A \leqq B$. But suppose $A \leqq C \cup D$ though neither $A \leqq C$ nor $A \leqq D$; by this definition one would have $l(A)$ and $r(A)$ equivalent in $(C \cup D)'$ but not in C' or D' and so not in $C' \cup D'$, contradicting $C' \cup D' = (C \cup D)'$. To overcome this difficulty we may enlarge $\mathfrak{A}$ to include another point $p(C, D)$ and specify that $l(A) \sim p(C, D)$ in C' and in $(C \cup D)'$ while $p(C, D) \sim r(A)$ in D' and in $(C \cup D)'$; we may think of $l(A)$ and $p(C, D)$ as being in this respect images of $l(C)$ and $r(C)$ and think of $p(C, D)$ and $r(A)$ as images of $l(D)$ and $r(D)$, with some images having been consolidated with each other or existing points for simplicity.

Now however we have introduced somewhat the same difficulty all over again (one step farther toward the background) for suppose $l(A) \sim p(C, D)$ in C' and $C \leqq E \cup F$. Then necessarily $C' \leqq (E \cup F)'$ so we need $l(A) \sim p(C, D)$ in $(E \cup F)' = E' \cup F'$ yet we dare not (for instance) have this relation automatically hold in E', for otherwise—since $p(C, D) \sim r(A)$ in D'—we shall get $l(A) \sim r(A)$ in $E' \cup D' = (E \cup D)'$ by transitivity and yet it need not be true that $A \leqq E \cup D$. So we must in a manner similar to that of the previous paragraph introduce a new point between $l(A)$ and $p(C, D)$. Evidently we can never stop this process at any finite point without having the same trouble; we are driven to making an infinite succession of enlargements of $\mathfrak{A}$ and hoping (or rather, proving) that their set union will have the desired properties though no one of them does.

In the above discussion we put our attention on one pair of elements C and D with $A \leqq C \cup D$. But of course if $A \leqq G \cup H$ then we must also introduce a point $p(G, H)$ in a similar manner. But now we have difficulty in trying to prove $(P \cap Q)' = P' \cap Q'$. For it might be that

$$p(C, D) \sim l(A) \sim p(G, H) \quad \text{in } P' \tag{1}$$

while

$$p(C, D) \sim r(A) \sim p(G, H) \quad \text{in} \quad Q'. \tag{2}$$

Thus in $P'\cap Q'$, $p(C, D)\sim p(G, H)$ but these need not be equivalent to either $l(A)$ or $r(A)$, whereas in $(P\cap Q)'$ by definition one or the other of the "paths" (1) or (2) must be followed. This trouble can be overcome by having not one but three images of $l(C)$, $r(C)$, $l(D)$, and so on. That is, instead of the one new point $p(C, D)$ we have $p^i(C, D)$, $i=0, 1, 2, 3, 4, 5, 6$, where $p^0(C, D)=l(A)$ and $p^6(C, D)=r(A)$ and $p^i(C, D)\sim p^{i+1}(C, D)$ in C' if i is even and in D' if i is odd. Also of course we must include additional points if $C\leqq E\cup F$, and so on, as discussed above. With this change, consider again the type of situation which was causing trouble; for instance

$$\begin{aligned} p^3(C, D) \sim p^2(C, D) \sim p^1(C, D) \sim l(A) \\ \sim p^1(G, H) \sim p^2(G, H) \quad \text{in} \quad P' \end{aligned} \tag{3}$$

while

$$\begin{aligned} p^3(C, D) \sim p^4(C, D) \sim p^5(C, D) \sim r(A) \sim p^5(G, H) \\ \sim p^4(G, H) \sim p^3(G, H) \sim p^2(G, H) \quad \text{in} \quad Q'. \end{aligned} \tag{4}$$

We propose to arrange matters so that $p^i(C, D)\sim p^{i+1}(C, D)$ in J' will imply $C\leqq J$ if i is even and $D\leqq J$ if i is odd. Then for instance $p^3(C, D)\sim p^4(C, D)$ in Q' will imply $p^2(C, D)\sim p^1(C, D)$ in Q' while $p^4(C, D)\sim p^5(C, D)$ in Q' will imply $p^3(C, D)\sim p^2(C, D)$ in Q' so that the possibility of the path (4) in Q' will imply that the equivalences (3) also hold in Q' and so also in $P'\cap Q'$ as desired.

We may describe the above construction in the terminology of electricity by saying that $l(A)$ and $r(A)$ are connected by several wires or paths in parallel, one path for each pair of lattice elements C and D with $A\leqq C\cup D$. Each path has six filters connected in series, three of which pass current (permit equivalence) of frequency (relation) J' if and only if $C\leqq J$ in the lattice, while the other three pass current in J' if and only if $D\leqq J'$. The filters themselves are complex arrangements of a similar structure.

This gives a general idea of the procedure necessary; our task is now to make the construction precise and prove the existence of the properties that the construction is designed to provide.

3. Construction[3] of the sets and equivalence relations. A lattice

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[3] If the given lattice is finite, Definitions 1–5 involve only finite processes; if the lattice is infinite, the word "construction" must be understood in a broader sense. But in both cases, Definitions 6–7 involve infinite processes. One hopes that a proof can be found which will involve only finite processes if the given lattice is finite.

$\mathfrak{L}$ is given. We shall suppose that in addition to the lattice relation of inclusion, denoted $\leqq$, there is also a relation of preceding and following which linearly orders all elements of the lattice[4] (if the lattice has a finite number of elements, the existence of such a relation is trivial; otherwise we must invoke the axiom of choice). In Definitions 1–4 reference is made to a pair of elements C and D of the lattice; when writing the pair in these definitions we shall write first the one which "precedes" the other under the relation just discussed.

DEFINITION 1. *If* $A\in\mathfrak{L}$, *denote by* $\mathfrak{A}_1$ *the set of points*[5] $p^i(C, D)$ *for* $i=0, 1, 2, 3, 4, 5, 6$ *and all pairs* $C, D\in\mathfrak{L}$ *such that* $A\leqq C\cup D$, *where*

$$p^0(C, D) = p^0(G, H) = l(A)$$

for all pairs C, D *and* G, H *and likewise*

$$p^6(C, D) = p^6(G, H) = r(A)$$

but otherwise $p^i(C, D)\neq p^j(G, H)$ *if* $i\neq j$ *or if* C, D *and* G, H *are distinct pairs. Similarly, all points of* $\mathfrak{A}_1$ *are distinct from the points of* $\mathfrak{B}_1, \mathfrak{C}_1, \cdots$.

DEFINITION 2. *The subset of* $\mathfrak{A}_1$ *consisting of the two points* $p^i(C, D)$ *and* $p^{i+1}(C, D)$ *is denoted* $(C_1, D_1)^i$; *the subset of* $\mathfrak{A}_1$ *consisting of* $l(A)$ *and* $r(A)$ *is denoted* A_0. $(C_0, D_0)^i$ *is a null set.*

Obviously the subsets in Definition 2 overlap but this causes no trouble.

We now proceed to define by induction sets $\mathfrak{A}_i$. Suppose that for each i $(0\leqq i<n)$ and each $A\in\mathfrak{L}$ there has been defined a set $\mathfrak{A}_i$ in such a manner that (for $1\leqq i<n$) if $A\leqq C\cup D$ then $\mathfrak{A}_i$ has subsets $(C_i, D_i)^j$, $j=0, 1, \cdots, 5$, with $(C_{i-1}, D_{i-1})^j\subset(C_i, D_i)^j$ and with the points of $(C_i, D_i)^j$ in one-to-one correspondence with those of $\mathfrak{C}_{i-1}$

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[4] This relation of "preceding" and "following" is not important to the argument and is merely for convenience in stating Definitions 1–4.

[5] Strictly speaking, this notation should also include a symbol to indicate that the points are in the set corresponding to A, as distinguished from the sets corresponding to $B, C, \cdots$, but to simplify notation we shall fix our attention on the sets related to a fixed A and subsets thereof, and on equivalence relations $A', B', \cdots$, only so far as they concern points of this set, unless otherwise indicated. We arrange that no point of the set corresponding to A shall be equivalent to any point of the sets corresponding to $B, C, \cdots$. It is permitted that $C=D$, $D=A$, and so on. It would be possible to reduce the number of points in such cases, but the present definition will minimize the number of separate cases to be considered in proofs. The symbol $=$ when used between names of points means that the names stand for the same point; when used between names of lattice elements it means the names stand for the same element of the lattice.

if j is even and with those of $\mathfrak{D}_{i-1}$ if j is odd, and with no overlapping except as required by these assumptions and Definition 2. Then for $n>1$, we have the following definition.

DEFINITION 3. *$\mathfrak{A}_n$ consists of $\mathfrak{A}_{n-1}$ with each of the latter's subsets $(C_{n-1}, D_{n-1})^j$ augmented by as many points as are in $\mathfrak{C}_{n-1}$ but not in $\mathfrak{C}_{n-2}$ (if j is even) or in $\mathfrak{D}_{n-1}$ but not in $\mathfrak{D}_{n-2}$ (if j is odd), the augmented subsets being called $(C_n, D_n)^j$.*

It is to be observed that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ satisfy the hypotheses, and that if the hypotheses hold then $\mathfrak{A}_n$ again has the same properties; thus the induction is effective. In view of the one-to-one correspondence specified, we may denote by q^{-1} the point of $\mathfrak{C}_{n-1}$ or $\mathfrak{D}_{n-1}$ corresponding to $q \in (C_n, D_n)^j$. Because of the overlapping of the subsets of $\mathfrak{A}_1$ and hence of $\mathfrak{A}_n$, $[p^2(C, D)]^{-1}$, for instance, may be taken as either $r(D)$ or $l(C)$ depending on whether we are at the moment concerned with $p^2(C, D)$ as a point of $(C_n, D_n)^1$ or as a point of $(C_n, D_n)^2$.

Corresponding to each $B \in \mathfrak{L}$, a relation B_1' is next defined so far as it concerns points of $\mathfrak{A}$.

DEFINITION 4. *$p \sim q$ in B_1' if there is some $A \in \mathfrak{L}$ such that p, $q \in \mathfrak{A}_1$ and one of* (5), (6) *or* (7) *holds.*

(5) *For some pair C, $D \in \mathfrak{L}$ with $A \leqq C \cup D$, $p = p^i(C, D)$ and $q = p^{i+1}(C, D)$ or vice versa, and either*

(5a) *i is even and $C \leqq B$ or*

(5b) *i is odd and $D \leqq B$.*

(6) $p = q$.

(7) *There exist points $q_1, \cdots, q_k$ such that $q_1 = p$, $q_k = q$, and for each i $(i = 1, 2, \cdots, k-1)$ q_i and q_{i+1} satisfy the conditions of* (5).

Suppose that, for $1 \leqq i < n$ and each $B \in \mathfrak{L}$, equivalence relations B_i' have been defined. Then by induction we define B_n':

DEFINITION 5. *$p \sim q$ in B_n' for $n>1$ if $p = q$ or there exist $A \in \mathfrak{L}$ and points $q_1, q_2, \cdots, q_k \in \mathfrak{A}_n$ with $q_1 = p$, $q_k = q$, and for each i $(i = 1, \cdots, k-1)$ there exist j and C, $D \in \mathfrak{L}$ with $A \leqq C \cup D$; q_i, $q_{i+1} \in (C_n, D_n)^j$ and $q_i^{-1} \sim q_{i+1}^{-1}$ in B_{n-1}'.*

It is to be observed that B_1' is an equivalence relation, and that if B_{n-1}' is an equivalence relation so is B_n', so the induction is effective. Having thus defined relations with subscripts 1, 2, $\cdots$ and sets with subscripts 0, 1, 2, $\cdots$ we now define ones without subscript, using $\sum$ to denote set union.

DEFINITION 6. $\mathfrak{A} = \sum_{i=0}^{\infty} \mathfrak{A}_i$; $\mathfrak{S} = \sum_{\text{all } A} \mathfrak{A}$; $(C, D)^j = \sum_{i=1}^{\infty} (C_i, D_i)^j$.

DEFINITION 7. *$p \sim q$ in B' if for some n, $p \sim q$ in B_n'.*

That B' as thus defined is an equivalence relation will appear shortly (Lemma 3).

DEFINITION 8. *A point $s \in \mathfrak{A}_1$ is between points p, $q \in \mathfrak{A}$ if for some C and D, $s \in (C, D)^i$, $p \in (C, D)^j$, and $q \in (C, D)^k$ with $j < i < k$ or $k < i < j$, or if $k = j+1$ and $s = p^k(C, D)$, or if $j = k+1$ and $s = p^j(C, D)$, provided in all cases that p, q, and s are distinct.*

For example, $p^1(C, D)$ is between $p^2(C, D)$ and $p^0(C, D)$, for we may take $i = 1$, $j = 2$, and $k = 0$. Likewise $p^5(C, D)$ is between $p^3(C, D)$ and $r(A)$, for we may take $i = 4$, $j = 3$, and $k = 5$ since $p^5(C, D)$ belongs to $(C, D)^4$ as well as to $(C, D)^5$. In fact, "between" means just what its name implies. One should emphasize the restriction in the definition that s must be not only in $\mathfrak{A}$ but indeed in $\mathfrak{A}_1$; otherwise only a partial ordering would be possible.

4. **Properties of $\mathfrak{L}'$.** It is required that the equivalence relations A', B', $\cdots$ form a lattice $\mathfrak{L}'$ isomorphic to the given lattice $\mathfrak{L}$. To prove that this requirement is satisfied, we must first examine in some detail the consequences of the above definitions, with attention to how it is possible to have $p \sim q$.

LEMMA 1. *If $A \leqq B$ then $l(A) \sim r(A)$ in B_1'.*

PROOF. Take $B = C = D$ in Definitions 1 and 4. Then in B_1',

$$l(A) \sim p^1(C, D) \sim p^2(C, D) \sim p^3(C, D) \sim p^4(C, D) \sim p^5(C, D) \sim r(A)$$

by (5) so $l(A) \sim r(A)$ in B_1' by (7).

LEMMA 2. *If $p \sim q$ in B_m' then $p \sim q$ in B_n' for $m \leqq n$ and $p \sim q$ in B'.*

PROOF. The case $m = n$ is trivial.

If $m = 1$ but $n > 1$, we proceed by induction on n; suppose the lemma is true for $n < \nu$ and we wish to prove it true for $n = \nu$. If $p \sim q$ in B_1' then by Definition 4 there exist $A \in \mathfrak{L}$ and $q_1, \cdots, q_k \in \mathfrak{A}_1$ with $q_1 = p$, $q_k = q$, and $q_i \sim q_{i+1}$ by (5), or else $p = q$. If $p = q$ they are equivalent in B_n' by Definition 5, so suppose the other alternative holds. Now if $q_i \sim q_{i+1}$ by (5a) then $C \leqq B$; by Lemma 1, $l(C) \sim r(C)$ in B_1'. By induction, $l(C) \sim r(C)$ in $B_{\nu-1}'$. By similar reasoning, if the equivalence in B_1' is by (5b) then $l(D) \sim r(D)$ in $B_{\nu-1}'$, so for all i, $q_i^{-1} \sim q_{i+1}^{-1}$ in $B_{\nu-1}'$; hence $p \sim q$ in B_ν' as desired, by Definition 5.

If $m > 1$, we proceed by induction on m; suppose $p \sim q$ in B_m'. By Definition 5 there exist $A \in \mathfrak{L}$ and $q_1, \cdots, q_k \in \mathfrak{A}_m$ with $q_i^{-1} \sim q_{i+1}^{-1}$ in B_{m-1}' (as above, the case $p = q$ is trivial). But then by induction on m, $q_i^{-1} \sim q_{i+1}^{-1}$ in $B_{\nu-1}'$, for all i, so by Definition 5, $p \sim q$ in B_ν' as desired.

That the equivalence also holds in B' follows from Definition 7.

LEMMA 3. *B' is an equivalence relation.*

PROOF. We must show that B' has the properties stated in §1. Suppose first that $p=q$. Then p and q are in $\mathfrak{A}$ and hence in $\mathfrak{A}_n$ for some A and n, by Definition 6; by Definition 5, $p \sim q$ in B_n' so, by Definition 7, $p \sim q$ in B' as desired. Suppose next that $p \sim q$ in B'; then $p \sim q$ in B_n' for some n, by Definition 7; since B_n' is an equivalence relation, $q \sim p$ in B_n' so $q \sim p$ in B', as desired. Suppose lastly that $p \sim q \sim r$ in B'; then $p \sim q$ in B_n' for some n and $q \sim r$ in B_m' for some m; by Lemma 2, $p \sim q \sim r$ in $B_{\max(m,n)}'$ so $p \sim r$ in that relation which is known to be an equivalence relation; hence $p \sim r$ in B' as desired. Thus B' has all the properties required of an equivalence relation.

LEMMA 4. *If $p \sim q$ in B_n' then there exist $A \in \mathfrak{L}$ and $s_1, s_2, \cdots, s_k \in \mathfrak{A}_n$ such that $s_1 = p$, $s_k = q$, $s_i \in \mathfrak{A}_1$ $(i \neq 1, k)$ and that the set of s_i is minimal (that is, none can be omitted without violating the requirements placed on them) and that either $n=1$ and $s_i \sim s_{i+1}$ by* (5) *for all i $(i=1, \cdots, k-1)$ or else $n>1$ and s_i and s_{i+1} for all i $(i=1, \cdots, k-1)$ satisfy the requirements imposed on q_i and q_{i+1} in Definition* 5.

Note. By Definitions 4 and 5 there exist points q_i with these properties except $q_i \in \mathfrak{A}_1$, $i \neq 1, k$ (in those definitions a priori only $q_i \in \mathfrak{A}_n$), and minimality. Minimality can easily be attained since the number of q_i is finite, but it is more difficult to show that we can take $s_i \in \mathfrak{A}_1$, $i \neq 1, k$; that is, $s_i = p^\nu(C, D)$ for some ν, C, and D depending on i. It may also be remarked that by virtue of the requirement that $s_i \sim s_{i+1}$ by (5) or Definition 5—more precisely, by (5) or certain conditions stated in Definition 5—it follows that s_i and s_{i+1} must be in the same subset $(C, D)^\nu$ of $\mathfrak{A}$, for both in (5) and Definition 5 this is required. For $1 < i \leqq k-2$, this fact and the requirement $s_i \in \mathfrak{A}_1$, $i \neq 1, k$, mean that $s_i = p^\nu(C, D)$ and either $s_{i+1} = p^{\nu+1}(C, D)$ or $s_{i+1} = p^{\nu-1}(C, D)$ for some ν.

PROOF. If $n=1$, the conclusion follows immediately from Definition 4; minimality can obviously be attained since only a finite number of points enter into (7).

Now proceed by induction on n; suppose $p \sim q$ in B_n' and the lemma is known to be true for smaller values of the subscript. By Definition 5 there exist points $q_i = s_i$ with all the desired properties except minimality and the requirement $s_i \in \mathfrak{A}_1$ for $i \neq 1, k$. Suppose $q_i \notin \mathfrak{A}_1$ and $i \neq 1, k$;

we shall show that q_i can be omitted. But now in Definition 5, q_i and q_{i+1} must be in the same subset of $\mathfrak{A}$, and $q_i^{-1}\sim q_{i+1}^{-1}$ in B_{n-1}' (again omitting the trivial case $p=q$); likewise q_i and q_{i-1} are in the same subset and $q_{i-1}^{-1}\sim q_i^{-1}$ in B_{n-1}'. By Definitions 1, 2, and 3, q_i can be in two subsets of $\mathfrak{A}$ only if $q_i=p^j(C, D)$ for some j, but then q_i is in $\mathfrak{A}_1$ as desired. Otherwise, q_i is in only one subset, and this one must also contain q_{i-1} and q_{i+1} and $q_{i-1}^{-1}\sim q_i^{-1}\sim q_{i+1}^{-1}$ in B_{n-1}'. Hence $q_{i-1}^{-1}\sim q_{i+1}^{-1}$ in B_{n-1}' so we may simply omit q_i from the set of q's. Thus by omitting some of the points provided by Definition 5 we can obtain a set which satisfies the conclusions of the lemma.

LEMMA 5. (i) *If* $p\in(C, D)^i$ *and* $q\in(C, D)^j$ *and* $p\sim q$ *in* B_n' *and* $|i-j|\leqq 3$, *then the points* $s_2, \cdots, s_{k-1}$ *of Lemma* 4 *may be taken as the set of all points of* $\mathfrak{A}_1$ *between* p *and* q.

(ii) *If* $p^i(C, D)\sim p^{i+1}(C, D)$ *in* B_n' *then* $C\leqq B$ *if* i *is even,* $D\leqq B$ *if* i *is odd.*

(iii) *If* $l(A)\sim r(A)$ *in* B_n' *then* $A\leqq B$.

PROOF. First suppose $n=1$.

Suppose the hypotheses of (i) are satisfied. Since $n=1$, we may by Definitions 1 and 2 replace these hyptheses by the following which are implied by them: $p=p^i(C, D)$, $q=p^j(C, D)$, $p\sim q$ in B_1', and $|i-j|\leqq 4$; by "replace" we mean that one whole set of hypotheses replaces the other set, but i and j need not have the same meaning in both. Indeed, the fact that $(C, D)^j$ contains both $p^j(C, D)$ and $p^{j+1}(C, D)$ is the reason why in the second set we must allow a greater difference between i and j. Since $n=1$, $\mathfrak{A}_n=\mathfrak{A}_1$, so in Lemma 4, $s_h\in\mathfrak{A}_1$ for all h. As in previous lemmas, we may ignore the trivial case $p=q$.

First consider the case $p=p^0(C, D)$. Then by the assumptions, $q=p^j(C, D)$ with $1\leqq j\leqq 4$. The s_h of Lemma 4 are all in $\mathfrak{A}_1$ for $h\neq 1, k$. Moreover, s_h and s_{h+1} must be in the same subset of $\mathfrak{A}$; in particular s_{k-1} is in the same subset as s_k, so s_{k-1} is either $p^{j+1}(C, D)$ or $p^{j-1}(C, D)$. We shall assume the latter but the same type of proof would apply in the former case; likewise we shall assume j odd but a similar argument would hold if j were even. In Lemma 4, $p^{j-1}(C, D)\sim p^j(C, D)$ in B_1' implies by (5) that $C\leqq B$ since if j is odd then $j-1$ is even. If $j-1=0$ then $p=s_{k-1}$ and (i) holds, the set referred to being vacuous; otherwise s_{k-2} exists. Both $s_{k-2}=q$ and $s_{k-2}=s_{k-1}$ are prohibited by minimality, so by Lemma 4 and (5), $s_{k-2}=p^{j-2}(C, D)$ and $D\leqq B$. Thus both $D\leqq B$ and $C\leqq B$. Then by (5), $p^\nu(C, D)\sim p^{\nu+1}(C, D)$ in B_n' for all ν by (5), so that if we take $s_2, \cdots, s_{k-1}$ as the set of points of $\mathfrak{A}_1$ between p and q, they will have all the properties required of them by Lemma 4; obviously they are minimal, since only successive

$p^\nu(C, D)$ can be equivalent by (5). Thus we have disposed of the case $p=p^0(C, D)$.

Now consider the case $p=p^1(C, D)$. If $q=p^0(C, D)$, the previous paragraph applies in view of the symmetry of our definitions as far as p and q are concerned. Otherwise, $q=p^j(C, D)$ with $2\leqq j\leqq 5$. Either $s_{k-1}=p^{j-1}(C, D)$ or $s_{k-1}=p^{j+1}(C, D)$. In the former case the argument of the previous paragraph is still applicable; in the latter case the argument fails if $j+1=6$. But then $D\leqq B$ by Lemma 4 and (5) since $p^5(C, D)\sim p^6(C, D)$; also, either $s_2=p^0(C, D)$ or $s_2=p^2(C, D)$. If $s_2=p^0(C, D)$ then by the same reasoning, $C\leqq B$; since we already have found $D\leqq B$, then as in the previous paragraph, $p^\nu(C, D)\sim p^{\nu+1}(C, D)$ for all ν and (i) holds. If on the other hand $s_2=p^2(C, D)$ then either $s_2=q$ and the desired result (i) holds immediately, or $s_3=p^3(C, D)$ by minimality and so by Lemma 4 and (5), $C\leqq B$ and then just as in the last sentence (i) holds. Thus the case $p=p^1(C, D)$ is disposed of.

If $p=p^i(C, D)$ for $i=5$ or $i=6$, the situation is symmetric to the cases $i=1$ and $i=0$ of the last two paragraphs. If $i=2, 3, 4$, the same argument as with $i=0$ may be used if p and q are interchanged, for the essential part of that argument was that p was between q and one end of $\mathfrak{A}$ and that q was far enough from the other end.

Thus for $n=1$, (i) holds in all cases.

Now suppose the hypothesis of (ii) holds, with still $n=1$. By (i) which has been proved for this value of n, we then have $p^i(C, D)\sim p^{i+1}(C, D)$ by (5); and (5) gives precisely the desired conclusion, proving (ii) for $n=1$.

Now suppose the hypothesis of (iii) holds, with still $n=1$. By Lemma 4, $s_1=p$, $s_2=p^1(C, D)$ for some C and D, and, by (5), $C\leqq B$. By minimality $s_3=p^2(C, D)$ so $D\leqq B$. Thus $A\leqq C\cup D\leqq B$ and (iii) holds.

For $n>1$ we proceed by induction; suppose the lemma has been proved for all smaller values of n, and that the hypotheses of (i) hold. For the moment let us suppose also that neither p nor q is in $\mathfrak{A}_1$, and consider the s_i of Lemma 4. Both s_2 and s_{k-1} are in $\mathfrak{A}_1$. If we consider these two points in the role of p and q, then the same argument as for $n=1$ may be applied, except that when there we appealed to (5), now we must look to Definition 5 as the way to have $s_i\sim s_{i+1}$ in B_n'.

Consider the case $s_2=p^0(C, D)$. Then $s_{k-1}=p^j(C, D)$, where just as with $n=1$, j is not necessarily the j of the statement of the lemma, but $1\leqq j\leqq 4$. Then s_{k-2} must be in the same subset of $\mathfrak{A}$ as s_{k-1}, so s_{k-2} is either $p^{j+1}(C, D)$ or $p^{j-1}(C, D)$; we shall assume that it is the latter and that j is odd. By Lemma 4 and Definition 5, $p^{j-1}(C, D)\sim p^j(C, D)$ in B_n' implies $l(C)\sim r(C)$ in B_{n-1}'. If $j-1=0$ then $s_2=s_{k-2}$ and (i) holds as applied to s_2 and s_{k-1}; otherwise s_{k-3} exists and by minimality it is not p so $s_{k-3}=p^{j-2}(C, D)$, and by the same argument as in the previous sentence, $l(D)\sim r(D)$ in B_{n-1}'. Hence if we take $s_3, \cdots, s_{k-2}$ as the set of points of $\mathfrak{A}_1$ between s_2 and s_{k-1}, they will have all the properties required for Lemma 4, for s_h^{-1} and s_{h+1}^{-1} will be either $l(C)$ and $r(C)$ or $l(D)$ and $r(D)$ or vice versa, and these have been shown equivalent.

Now consider the case $s_2=p^1(C, D)$. If $s_{k-1}=p^0(C, D)$, the previous paragraph applies by symmetry. Otherwise $s_{k-1}=p^j(C, D)$ with $2\leqq j\leqq 5$. Either $s_{k-2}=p^{j-1}(C, D)$ or $s_{k-2}=p^{j+1}(C, D)$. In the former case the argument of the previous paragraph is still applicable; in the latter case the argument fails if $j+1=6$. But then by Lemma 4 and Definition 5 we have $l(D)\sim r(D)$ in B_{n-1}'. Also, either $s_3=p^0(C, D)$ or $s_3=p^2(C, D)$. If $s_3=p^0(C, D)$ then by the same reasoning $l(C)\sim r(C)$ in B_{n-1}'; since we already found $l(D)\sim r(D)$, then as in the previous paragraph, (i) holds as applied to s_2 and s_{k-1}. If on the other hand $s_3=p^2(C, D)$ then either $s_3=s_{k-1}$ and the desired result holds immediately, or $s_4=p^3(C, D)$ by minimality and so from $s_3\sim s_4$ we have $l(C)\sim r(C)$ in B_{n-1}', and just as in the last sentence (i) holds as applied to s_2 and s_{k-1}.

If $s_2=p^i(C, D)$ for $i>1$, the proof may be referred to the cases $i=0$ and $i=1$ above, just as was done for $n=1$. Thus (i) holds in all cases, as applied to s_2 and s_{k-1}; now we must show that (i) holds as applied to p and q.

Since we assumed that neither p nor q was in $\mathfrak{A}_1$, s_2 must be in the same subset of $\mathfrak{A}$ as p by the conditions of Lemma 4, and s_{k-1} in the same subset as q. Then by Definition 8, $s_3, \cdots, s_{k-2}$ (which by the above may be taken between s_2 and s_{k-1}) will be between p and q. Suppose however that s_2 is not between p and q though s_3 is. In view of the conditions imposed on the s_i in Lemma 4, s_2 and s_3 must then both be in the same subset of $\mathfrak{A}$ as p; $p^{-1}\sim s_2^{-1}\sim s_3^{-1}$ so $p^{-1}\sim s_3^{-1}$ and s_2 may be omitted. Likewise if s_{k-1} is not between p and q it may be omitted. Thus (i) holds if neither p nor q is in $\mathfrak{A}_1$; the argument can readily be modified to take care of the possibility that one or both is in $\mathfrak{A}_1$.

Now suppose the hypothesis of (ii) holds. By (i), $p^i(C, D)\sim p^{i+1}(C, D)$ in B_n' implies that their antecedents are equivalent in B_{n-1}'; that is, $l(C)\sim r(C)$ or $l(D)\sim r(D)$ in B_{n-1}' as i is even or odd. But by (iii), which is assumed already proved for B_{n-1}', this implies $C\leqq B$ or $D\leqq B$ as i is even or odd, so (ii) holds.

Now suppose the hypothesis of (iii) holds. In Lemma 4, $s_1=p=l(A)$; $s_2=p^1(C, D)$ for some C and D since $s_2\in\mathfrak{A}_1$ and s_2 must be in the same subset as s_1. But by (i), $l(A) \sim p^1(C, D)$ in B_n' implies $C \leqq B$. Also $s_3 = p^2(C, D)$ by minimality so similarly $D \leqq B$. Then $A \leqq C \cup D \leqq B$ so (iii) holds.

LEMMA 6. *Suppose* $p \sim q$ *in* B_n', $p \in (C, D)^i \subset \mathfrak{A}$, *and* $q \in (E, F)^j \subset \mathfrak{A}$, *where* C, D *and* E, F *are distinct pairs.*

(i) *If* $i > 1$ *then* $p \sim r(A)$ *in* B_n'.

(ii) *If* $i < 4$ *then* $p \sim l(A)$ *in* B_n'.

(iii) *If* $j > 1$ *then* $q \sim r(A)$ *in* B_n'.

(iv) *If* $j < 4$ *then* $q \sim l(A)$ *in* B_n'.

PROOF. Suppose the hypothesis of (i) holds: $i > 1$. By Lemma 4 there exist points s_h with the properties specified there; in particular, $s_h \in \mathfrak{A}_1$ for $h \neq 1, k$. We shall suppose that p is not in $\mathfrak{A}_1$; the necessary modifications if $p \in \mathfrak{A}_1$ will be obvious. By Definition 2, either $s_2 = p^i(C, D)$ or $s_2 = p^{i+1}(C, D)$; we shall first suppose the latter. If $s_2 = r(A)$, that is, $i = 5$, then the conclusion of (i) holds immediately; otherwise by minimality $s_3 = p^{i+2}(C, D)$. If $s_3 = r(A)$ then (i) holds; otherwise $s_4 = p^{i+3}(C, D)$. But $p^{i+1}(C, D) \sim p^{i+2}(C, D) \sim p^{i+3}(C, D)$ in B_n' imply $C \leqq B$ and $D \leqq B$ by Lemma 5 (ii), so $p^{\nu}(C, D) \sim p^{\nu+1}(C, D)$ for all ν, and $s_2 \sim r(A)$ in B_n'. Since $p \sim s_2$ we have $p \sim r(A)$ as desired.

If $s_2 = p^i(C, D)$ the same type of argument applies; the hypothesis $i > 1$ insures that we shall get both $C \leqq B$ and $D \leqq B$.

Thus (i) holds. The proofs of (ii), (iii), and (iv) are similar.

5. **Proof of isomorphism.** Having developed some tools in the previous section, we now proceed to prove that the partially ordered system of equivalence relations of Definition 7 is a lattice (a sublattice of the lattice of all equivalence relations on $\mathfrak{S}$) and is isomorphic to the given lattice $\mathfrak{L}$. Using the natural ordering among equivalence relations, we write $B' \leqq G'$ if and only if $p \sim q$ in B' implies $p \sim q$ in G'.

LEMMA 7. *If* $B \leqq G$ *then* $B' \leqq G'$.

PROOF. From Definition 4 it is obvious that $B \leqq G$ implies $B_1' \leqq G_1'$. By Definition 5 and induction it follows that then $B_n' \leqq G_n'$, so by Definition 7, $B' \leqq G'$.

Thus Lemma 7 holds; this proves that there is an order-homomorphism between the lattices. To prove it a lattice-homomorphism we need the next two lemmas.

Any two equivalence relations, G' and H', have[6] a greatest lower bound, $G' \cap H'$, as equivalence relations—namely, $p \sim q$ in $G' \cap H'$ if and only if $p \sim q$ in both G' and H'—though it is not apparent a priori that this new equivalence relation is one of those which we have constructed to correspond to the elements of the given lattice. Likewise G' and H' have a least upper bound $G' \cup H'$—$p \sim q$ in $G' \cup H'$ if and only if there exist $r_1, r_2, \cdots, r_k$ with $r_1 = p$, $r_k = q$, and for each i, $r_i \sim r_{i+1}$ in G' or in H'—which need not a priori correspond to an element of the given lattice, but we proceed to show that it is in fact identical with the relation constructed to correspond to $G \cup H$.

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[6] See for instance Birkhoff [1], or the other references cited in footnote 2.

LEMMA 8. $(G \cup H)' = G' \cup H'$.

PROOF. $G \leqq G \cup H$ so, by Lemma 7, $G' \leqq (G \cup H)'$; similarly $H' \leqq (G \cup H)'$; hence $G' \cup H' \leqq (G \cup H)'$ since $G' \cup H'$ is the least upper bound of G' and H' as equivalence relations. As for the converse, we propose to prove by induction on n that

$$(G \cup H)_n' \leqq G_{n+1}' \cup H_{n+1}'.$$

Suppose $n = 1$. If $p \sim q$ in $(G \cup H)_1'$ then by Lemma 4 there exist the usual points s_j with $s_j \sim s_{j+1}$ by (5). Suppose it is the alternative (5a) which holds; then for an even i, $s_j = p^i(C, D)$, $s_{j+1} = p^{i+1}(C, D)$ or vice versa, and $C \leqq G \cup H$. Now consider the set $\mathfrak{C}_1$; it has subsets $(G, H)^\nu$ since $C \leqq G \cup H$, and $p^\nu(G, H) \sim p^{\nu+1}(G, H)$ in G_1' if ν is even and in H_1' if ν is odd, by Definition 4. By Definition 5 the images of these points in $\mathfrak{A}_2$ are equivalent in G_2' or H_2' as the case may be; if the prefix I denotes their images, then

$$\begin{aligned} s_j = p^i(C, D) &= Ip^0(G, H) \sim Ip^1(G, H) \sim \cdots \sim Ip^5(G, H) \sim Ip^6(G, H) \\ &= p^{i+1}(C, D) = s_{j+1} \end{aligned}$$

in $G_2' \cup H_2'$. Thus for each j, $s_j \sim s_{j+1}$ in $G_2' \cup H_2'$ so $p \sim q$ in $G_2' \cup H_2'$ as desired.

If $n > 1$, we proceed by induction. If $p \sim q$ in $(G \cup H)_n'$ then by Lemma 4 there exist the usual s_j with $s_j^{-1} \sim s_{j+1}^{-1}$ in $(G \cup H)_{n-1}'$ by Definition 5. By induction, $s_j^{-1} \sim s_{j+1}^{-1}$ in $G_n' \cup H_n'$. By the nature of the least upper bound, there exist $r_1, \cdots, r_k$ in $\mathfrak{C}$ or in $\mathfrak{D}$ with $r_1 = s_j^{-1}$, $r_k = s_{j+1}^{-1}$, $r_i \sim r_{i+1}$ in G_n' or in H_n'. By Definition 5, their images in $\mathfrak{A}$ are equivalent: $Ir_i \sim Ir_{i+1}$ in G_{n+1}' or in H_{n+1}'. Hence

$$s_j = Ir_1 \sim Ir_k = s_{j+1} \quad \text{in} \quad G_{n+1}' \cup H_{n+1}',$$

for each j. Hence $p \sim q$ in $G_{n+1}' \cup H_{n+1}'$, as desired.

Thus for all n, $(G \cup H)_n' \leqq G_{n+1}' \cup H_{n+1}'$. But $G_{n+1}' \leqq G'$ and $H_{n+1}' \leqq H'$ so

$$(G \cup H)_n' \leqq G_{n+1}' \cup H_{n+1}' \leqq G' \cup H'.$$

If $p \sim q$ in $(G \cup H)'$ then $p \sim q$ in $(G \cup H)_n'$ for some n by Definition 7, and then by the above inclusion relations, $p\sim q$ in $G'\cup H'$. This proves Lemma 8.

LEMMA 9. $(G\cap H)'=G'\cap H'$.

PROOF. By Lemma 7, $(G\cap H)'\leqq G'$ and $(G\cap H)'\leqq H'$, so $(G\cap H)'\leqq G'\cap H'$. For the converse, suppose $p\sim q$ in $G'\cap H'$. Then $p\sim q$ in G' and in H'. By Definition 7, $p\sim q$ in G_n' and in H_m' for some n and m; suppose for instance $n\geqq m$; then $p\sim q$ in G_n' and in H_n'. It will therefore prove the lemma if we can show that $p\sim q$ in G_n' and in H_n' imply $p\sim q$ in $(G\cap H)_n'$, which we proceed to do by induction.

Suppose $n=1$.

Case 1. $p=p^i(C, D)$, $q=p^j(C, D)$, $|i-j|\leqq 3$. Now $p\sim q$ in G_1' and in H_1'. By Lemma 5 (i), the s_h of Lemma 4 for both G_1' and H_1' may be taken as the set of points of $\mathfrak{A}_1$ between p and q, and $s_h\sim s_{h+1}$ by (5a) or (5b), depending on whether the index is even or odd. Hence $C\leqq G$ and $C\leqq H$, or $D\leqq G$ and $D\leqq H$—depending on whether it is (5a) or (5b) that applies—so $C\leqq G\cap H$ or $D\leqq G\cap H$. Hence by (5), $s_h\sim s_{h+1}$ in $(G\cap H)_1'$. This holds for each h, so $p\sim q$ in $(G\cap H)_1'$ as desired.

Case 2. $p=p^i(C, D)$, $q=p^j(C, D)$, $|i-j|>3$. We may assume $i<j$. By Lemma 4 there exist the usual s_h with $s_h\sim s_{h+1}$ in G_n' by (5). Then $i=0$ or $s_2=p^{i-1}(C, D)$ or $s_2=p^{i+1}(C, D)$.

Case 2a. $s_2=p^{i+1}(C, D)$. Suppose for instance i is even; then by (5), $C\leqq G$. Also $s_3=s_1$ and $s_3=s_2$ are excluded by minimality, and $s_2=q$ and $s_3=q$ are excluded by $|i-j|>3$, so $s_3=p^{i+2}(C, D)$ and $D\leqq G$ by (5). Hence using (5) in the opposite direction, $p^\nu(C, D)\sim p^{\nu+1}(C, D)$ for all ν; and p, $l(A)$, $r(A)$, and q are all equivalent in G_n'. Before proceeding, we bring the other subcases of Case 2 to a corresponding situation.

Case 2b. $s_2=p^{i-1}(C, D)$. If $i=1$, we have immediately $p^0(C, D)\sim p^1(C, D)$ and $C\leqq G$. Otherwise $i=2$, since $i>2$ is excluded by $|i-j|>3$. But by minimality,

$$s_3 = p^{i-2}(C, D) = l(A)$$

and $C\leqq G$ and $D\leqq G$ from $s_2\sim s_3$ and $s_1\sim s_2$.

Case 2c. $i=0$. Here $p=l(A)$.

Thus in all subcases of Case 2, if $p^\nu(C, D)$ and $p^{\nu+1}(C, D)$ are both between p and $l(A)$ then $C\leqq G$ or $D\leqq G$ as ν is even or odd. Similarly $C\leqq H$ or $D\leqq H$, so $C\leqq G\cap H$ or $D\leqq G\cap H$, and $p^\nu(C, D)\sim p^{\nu+1}(C, D)$ in $(G\cap H)_1'$, so $p\sim l(A)$ in $(G\cap H)_1'$, as well as in G_1' and H_1'. Similarly $q\sim r(A)$ in G_1', in H_1', and in $(G\cap H)_1'$. Then

$$l(A) \sim p \sim q \sim r(A) \quad \text{in} \quad G_1' \quad \text{and in} \quad H_1';$$

by Lemma 5 (iii), $A \leqq G$ and $A \leqq H$. Hence $A \leqq G \cap H$, so by Lemma 1, $l(A) \sim r(A)$ in $(G \cap H)_1'$. Thus

$$p \sim l(A) \sim r(A) \sim q \quad \text{in} \quad (G \cap H)_1'$$

as desired.

Case 3. $p \in (C, D)^i$, $q \in (E, F)^j$ with C, D and E, F distinct pairs. This breaks up into four overlapping subcases.

Case 3a. $i > 1$, $j > 1$. By Lemma 6, $p \sim r(A) \sim q$ in G_1' and in H_1'. By Cases 1 and 2, $p \sim r(A) \sim q$ in $(G \cap H)_1'$ so $p \sim q$ in $(G \cap H)_1'$ as desired.

Case 3b. $i > 1$, $j < 4$. By Lemma 6, $p \sim r(A)$ and $q \sim l(A)$ in G_1' and in H_1'; hence $r(A) \sim l(A)$ in G_1' and in H_1'. By Lemma 5 (iii), $A \leqq G$ and $A \leqq H$ so $A \leqq G \cap H$ and by Lemma 1, $l(A) \sim r(A)$ in $(G \cap H)_1'$. By Cases 1 and 2, $p \sim r(A)$ and $q \sim l(A)$ in $(G \cap H)_1'$, so by transitivity $p \sim q$ in $(G \cap H)_1'$ as desired.

Cases 3c *and* 3d. $i < 4$, $j > 1$ or $j < 4$. Similar to Cases 3a and 3b.

This concludes the proof for $n = 1$. If $n > 1$, we proceed by induction.

Case 1. $p \in (C, D)^i$, $q \in (C, D)^j$, $|i - j| \leqq 3$. By Lemma 5 (i), the s_h of Lemma 4 for both G_n' and H_n' may be taken as the points of $\mathfrak{A}_1$ between p and q. But $s_h^{-1} \sim s_{h+1}^{-1}$ in G_{n-1}' and in H_{n-1}' and (by induction) in $(G \cap H)_{n-1}'$, for all h. By Definition 5, $p \sim q$ in $(G \cap H)_n'$.

Case 2. $p \in (C, D)^i$, $q \in (C, D)^j$, $|i - j| > 3$. We may suppose $i < j$. Let us also suppose that neither p nor q is in $\mathfrak{A}_1$; the modifications otherwise necessary will be apparent. By Lemma 4 the usual s_h exist, with $s_h^{-1} \sim s_{h+1}^{-1}$ in G_{n-1}', and $s_2 \in (C, D)^i$ but $s_3 \notin (C, D)^i$ by minimality. Also $s_h \in \mathfrak{A}_1$ for $h \neq 1, k$. Then $s_3 \in (C, D)^{i+1}$ or $s_3 \in (C, D)^{i-1}$ or $i = 0$.

Case 2a. $s_3 \in (C, D)^{i+1}$. Then $s_2 = p^i(C, D)$ and $s_3 = p^{i+1}(C, D)$ is impossible, for this would put s_1, s_2, and s_3 in the same $(C, D)^i$ contrary to minimality. Hence, $s_2 = p^{i+1}(C, D)$ and $s_3 = p^{i+2}(C, D)$. If, say, i is even, then $s_2^{-1} \sim s_3^{-1}$ implies $l(D) \sim r(D)$ in G_{n-1}'. Likewise $s_4 = p^{i+3}(C, D)$ and $l(C) \sim r(C)$ in G_{n-1}'. Also $p^{-1} \sim s_2^{-1}$ and by transitivity $p^{-1} \sim l(C)$ in G_{n-1}'; if i were odd this last would be $p^{-1} \sim l(D)$ in G_{n-1}'.

Case 2b. $s_3 \in (C, D)^{i-1}$. In a similar manner, if $p^\nu(C, D)$ and $p^{\nu+1}(C, D)$ are between p and $l(A)$ then $l(C) \sim r(C)$ or $l(D) \sim r(D)$ in G_{n-1}', as ν is even or odd, and $p^{-1} \sim l(C)$ or $p^{-1} \sim l(D)$ in G_{n-1}'.

Case 2c. $i = 0$. In a similar manner, the same conclusion holds as in Case 2b.

Thus in all subcases of Case 2, if $p^\nu(C, D)$ and $p^{\nu+1}(C, D)$ are between p and $l(A)$ then $l(C) \sim r(C)$ or $l(D) \sim r(D)$ in G_{n-1}' as ν is even or odd, and $p^{-1} \sim l(C)$ or $p^{-1} \sim l(D)$ in G_{n-1}' as i is even or odd. Similarly these equivalences hold in H_{n-1}' and so by induction in $(G \cap H)_{n-1}'$; hence $p \sim l(A)$ in G_n', in H_n', and in $(G \cap H)_n'$. Similarly $q \sim r(A)$ in G_n', in H_n', and in $(G \cap H)_n'$. Then

$$l(A) \sim p \sim q \sim r(A) \quad \text{in} \quad G_n' \quad \text{and in} \quad H_n' ;$$

by Lemma 5 (iii), $A \leqq G$ and $A \leqq H$ so $A \leqq G \cap H$ and by Lemmas 1 and 2, $l(A) \sim r(A)$ in $(G \cap H)_n'$. Thus

$$p \sim l(A) \sim r(A) \sim q \quad \text{in} \quad (G \cap H)_n'$$

as desired.

Case 3. $p \in (C, D)^i$, $q \in (E, F)^i$. The same argument as for $n = 1$ may be used.

Thus Lemma 9 is proved. This shows that we have a lattice-homomorphism. To prove this an isomorphism we need the following lemma.

LEMMA 10. *If* $B' \leqq G'$ *then* $B \leqq G$.

PROOF. By Lemma 1, $l(B) \sim r(B)$ in B', so by hypothesis $l(B) \sim r(B)$ in G'. By Definition 7, $l(B) \sim r(B)$ in G_n' for some n. By Lemma 5 (iii), $B \leqq G$ as desired.

From Lemmas 7–10, the equivalence relations which were constructed form a lattice isomorphic to the given lattice. Hence we have the main result:

THEOREM 1. *Any lattice is isomorphic to a sublattice of the lattice of all equivalence relations on some set.*

COROLLARY 1. *If a lattice identity holds in the lattice of all equivalence relations on the set* $\mathfrak{S}$, *for every* $\mathfrak{S}$, *then this identity holds in every lattice.*

6. **Connection with groups.** Birkhoff has shown [1] that any sublattice of the lattice of all equivalence relations on a set is isomorphic to a sublattice of the lattice of all subgroups of some group.[7] In view of Theorem 1, we immediately have the following theorem.

THEOREM 2. *Any lattice is isomorphic to a sublattice of the lattice of all subgroups of some group.*

COROLLARY 2. *If a lattice identity holds in the lattice of all subgroups of the group* $\mathfrak{G}$, *for every* $\mathfrak{G}$, *then this identity holds in every lattice.*

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[7] The converse is better known; Birkhoff [1] has a proof of it as well as references.

Since Birkhoff gives only an indication of the proof of his result, it is perhaps worth giving a proof here. We are given a lattice of equivalence relations on a set $\mathfrak{S}$. If B is one of these relations, let B^* be the group of all permutations $P: p \to f(p)$ of the points p of $\mathfrak{S}$ such that

(8) for each $p, f(p) \sim p$ in B, and

(9) P leaves unchanged all but a finite number of the points of $\mathfrak{S}$.

Evidently B^* is a subgroup of the group of all permutations of $\mathfrak{S}$.

Then $B^* \cap C^* = (B \cap C)^*$, for any permutation which satisfies (9) and has $f(p) \sim p$ in B and in C also has $f(p) \sim p$ in $B \cap C$ and conversely.

Also $B^* \cup C^* = (B \cup C)^*$. For certainly $B^* \leqq (B \cup C)^*$ and $C^* \leqq (B \cup C)^*$; on the other hand suppose $P \in (B \cup C)^*$; that is, P is a permutation which involves only a finite number of points and in which $f(p) \sim p$ in $B \cup C$. By definition of join of equivalence relations, there exist for each p affected a finite number of points $w_1, \cdots, w_k$ with $w_1 = p$, $w_k = f(p)$, and $w_i \sim w_{i+1}$ in B or in C. Consider, for a given p, the permutation which is the following product:

$$(w_1w_2)(w_2w_3)(w_3w_4) \cdots (w_{k-1}w_k)(w_{k-2}w_{k-1})(w_{k-3}w_{k-2}) \cdots (w_1w_2).$$

We observe that this product equals (w_1w_k); that is, it simply transposes p and $f(p)$. On the other hand each factor (w_iw_{i+1}) is a member of B^* or C^* according as $w_i \sim w_{i+1}$ in B or in C. Thus any transposition (w_1w_k) of points equivalent under $B \cup C$ is a member of $B^* \cup C^*$. But by (9), any member of $(B \cup C)^*$ affects only a finite number of points of $\mathfrak{S}$, and hence is the product of a finite number of transpositions (pq); it is evident that these transpositions can be so chosen that $p \sim q$ in $B \cup C$; by the above argument each of these transpositions (and so also their product) is in $B^* \cup C^*$, so $(B \cup C)^* \leqq B^* \cup C^*$. Then by the first part of this paragraph, $(B \cup C)^* = B^* \cup C^*$. Thus the given lattice of equivalence relations has an isomorphic lattice of subgroups as asserted.

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ON THE COMMUTATIVITY OF CERTAIN RINGS

ALEXANDRA FORSYTHE AND NEAL H. McCOY

1. Introduction. In his fundamental paper [5][1] on Boolean rings, Stone showed that a Boolean ring, that is, a ring R with the property that $a^2=a$ for every element a of R, is necessarily commutative. Recently, Kaplansky [3] announced that this result can be extended and that, under certain rather strong conditions on the positive integer n, a ring R is commutative if $a^n=a$ for every element a of R. Furthermore, Jacobson [2] has now shown that this is true without restriction on n. In fact, he has established the following more general result:

THEOREM 1 (JACOBSON). *If for each element a of a ring R there exists an integer $n(a)>1$, depending on a, such that $a^{n(a)}=a$, then R is commutative.*

A simple calculation [2, p. 702] shows that every element of a ring satisfying the hypothesis of this theorem has finite additive order. Thus if a *division ring* satisfies the hypothesis, its prime field is necessarily finite. Theorem 8 of [2] then furnishes a short and elegant proof of Theorem 1 for the case of division rings. The proof of the theorem can then be completed by obtaining a "reduction to division rings," that is, a proof that the theorem is true for all rings if it is true for all division rings. Jacobson accomplishes this by use of some rather deep results on algebraic algebras. The principal purpose of the present note is to present a simple reduction to division rings which was obtained independently by the present authors. This, coupled with Jacobson's proof of the result for division rings, furnishes a short and simple proof of Theorem 1.

We shall give, in §3, an entirely elementary proof of this theorem for the special case in which R is of prime characteristic p and $a^p=a$ for every element a of R. Such a ring is a *p-ring* [4], which is perhaps the simplest and most natural generalization of a Boolean ring.

2. The reduction to division rings. We shall, in fact, treat a somewhat more general case which may be of some interest in itself.

We recall that, according to Birkhoff [1], a ring R is *subdirectly irreducible* if the intersection of all nonzero two-sided ideals in R is a nonzero ideal. The following fundamental theorem is essential for our purpose:

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[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 2 (BIRKHOFF). *Any ring is isomorphic to a subdirect sum of subdirectly irreducible rings.*

We shall begin by proving the following lemma:

LEMMA 1. *In a ring S without nonzero nilpotent elements, every idempotent is in the center.*

Let e be an idempotent in the ring S, and x an arbitrary element of S. Then

$$(exe - ex)^2 = exexe - exex - exexe + exex = 0.$$

Thus $exe = ex$, and a similar calculation shows that also $exe = xe$. Hence $ex = xe$, and e is in the center.

LEMMA 2. *If S is a subdirectly irreducible ring without nonzero nilpotent elements, then the only idempotents in S are the zero and the unit element in case S has a unit element.*

To prove this, let us assume that e is an idempotent in S, neither 0 nor 1, and show that S is not subdirectly irreducible. By Lemma 1, e is in the center of S and thus the set of all elements of the form $x - ex$, $x \in S$, is a two-sided ideal $\mathfrak{a}$ in S. Furthermore, $\mathfrak{a}$ is not (0) since otherwise e would be the unit element, contrary to hypothesis. Let $\mathfrak{b}$ be the two-sided ideal in S consisting of those elements of the form ex, $x \in S$. Then $\mathfrak{b} \neq (0)$ since it contains $e^2 = e \neq 0$. Now $\mathfrak{a} \cap \mathfrak{b} = (0)$, for if $x - ex = ey$, multiplication by e shows that $ey = 0$. Hence S cannot be subdirectly irreducible.

A ring R is a *regular ring* [6] if for each element a of R there exists an element x such that

$$axa = a. \tag{1}$$

We do not assume the existence of a unit element. From (1), it follows that ax is idempotent. If R has no nonzero nilpotent elements, Lemma 1 shows that ax is in the center of R, and hence (1) may be written in the form $a^2x = a$. Conversely, if $a \neq 0$, the existence of an x such that $a^2x = a$ certainly implies that a is not nilpotent. We have thus established the following lemma:

LEMMA 3. *A regular ring R has no nonzero nilpotent elements if and only if for each element a of R there is an element x such that $a^2x = a$.*

We may now easily complete the proof of the following theorem which is the principal result of this note:

THEOREM 3. *A regular ring R is isomorphic to a subdirect sum of division rings if and only if R has no nonzero nilpotent elements.*

The necessity of the condition is obvious. To prove the sufficiency, we consider a regular ring R without nonzero nilpotents which, by Theorem 2, is isomorphic to a subdirect sum of subdirectly irreducible rings R_i. Now each R_i is a homomorphic image of R and therefore a regular ring and, by use of Lemma 3, we see that no R_i has nonzero nilpotent elements. Thus, by Lemma 2, R_i can have no idempotents other than the zero and the unit element. Now if a_i is any nonzero element of R_i there is an x_i such that $a_ix_ia_i=a_i$. Thus $a_ix_i\neq 0$ and is idempotent. It follows that a_ix_i is the unit element of R_i and, by a similar argument, x_ia_i is also seen to be the unit element. Since every nonzero element of R_i has an inverse, R_i is a division ring.

Theorem 3 clearly contains, as a special case, the desired reduction of Theorem 1. For a ring R satisfying the condition of Theorem 1 is obviously regular and has no nonzero nilpotents. Hence such a ring is isomorphic to a subdirect sum of division rings each of which satisfies the condition of the theorem.

We are indebted to the referee for the following remark. If a is any nonzero element of an algebraic algebra without nonzero nilpotent elements, the subalgebra generated by a is of finite order and semi-simple. It is therefore a full direct sum of a finite number of fields and thus the equation $axa=a$ has a solution. Such an algebra is therefore regular and we have another proof of the following result due to Jacobson [2, Corollary to Theorem 6]:

COROLLARY. *An algebraic algebra without nonzero nilpotent elements is isomorphic to a subdirect sum of division algebras.*

3. **A special case.** We now let R be a ring of prime characteristic p with the further property that $a^p=a$ for every element a of R, and shall give an elementary proof that R is commutative.[2]

If a and b are any two elements of R, we must have

$$(a+b)^p = a+b = a^p + [a^{p-1}b] + [a^{p-2}b^2] + \cdots + [ab^{p-1}] + b^p, \tag{2}$$

where

$$[a^{p-1}b] = a^{p-1}b + a^{p-2}ba + \cdots + aba^{p-2} + ba^{p-1}, \tag{3}$$

and, in general, $[a^ib^j]$ is a sum of products of elements a and b, in each term of which a appears i times and b appears j times. Since $a^p=a$, $b^p=b$, we see from (2) that

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[2] Professor Jacobson has informed us that he has also obtained a simple proof in this case.

$$[a^{p-1}b]+[a^{p-2}b^2]+\cdots+[ab^{p-1}]=0. \tag{4}$$

Now (4) is true for all elements a and b, hence we may replace b in turn by $2b$, $3b$, $\cdots$, $(p-1)b$, thus getting the following system of equations, the first of which is merely equation (4):

$$\begin{aligned} [a^{p-1}b]+[a^{p-2}b^2]+\cdots+[ab^{p-1}] &= 0,\\ 2[a^{p-1}b]+2^2[a^{p-2}b^2]+\cdots+2^{p-1}[ab^{p-1}] &= 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdots\\ (p-1)[a^{p-1}b]+(p-1)^2[a^{p-2}b^2]+\cdots+(p-1)^{p-1}[ab^{p-1}] &= 0. \end{aligned} \tag{5}$$

Now let m denote the Vandermonde determinant $|i^j|$ $(i,j=1,2,\cdots,p-1)$, and $m_1, m_2, \cdots, m_{p-1}$ the cofactors of the elements of the first column of m. If we multiply the equations (5) by $m_1, m_2, \cdots, m_{p-1}$, respectively, and add, we get $m[a^{p-1}b]=0$. Now m is known to be a product of positive integers less than p, hence is prime to p; thus we must have $[a^{p-1}b]=0$. A simple calculation, using the explicit formula (3), and making use of $a^p=a$, shows that

$$0=a[a^{p-1}b]-[a^{p-1}b]a=ab-ba.$$

Since a and b are arbitrary elements of R, this proves that R is commutative.

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SOME REMARKS ABOUT ADDITIVE AND MULTIPLICATIVE FUNCTIONS

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The present paper contains some results about the classical multiplicative functions $\phi(n)$, $\sigma(n)$ and also about general additive and multiplicative functions.

(1) It is well known that $n/\phi(n)$ and $\sigma(n)/n$ have a distribution function.[1] Denote these functions by $f_1(x)$ and $f_2(x)$. ($f_1(x)$ denotes the density of integers for which $n/\phi(n) \leqq x$.) It is known that both $f_1(x)$ and $f_2(x)$ are strictly increasing and purely singular.[1] We propose to investigate $f_1(x)$ and $f_2(x)$; we shall give details only in case of $f_1(x)$. First we prove the following theorem.

THEOREM 1. *We have for every* ϵ *and sufficiently large* x

$$(1) \quad \exp(-\exp[(1+\epsilon)ax]) < 1 - f_1(x) < \exp(-\exp[(1-\epsilon)ax])$$

where $a = \exp(-\gamma)$, γ *Euler's constant.*

We shall prove a stronger result. Put $A_r = \prod_{i=1}^{r} p_i$, p_i consecutive primes. Define A_k by $A_k/\phi(A_k) \geqq x > A_{k-1}/\phi(A_{k-1})$. Then we have

$$(2) \qquad 1/A_k < 1 - f_1(x) < 1/A_k^{1-\epsilon}.$$

First of all it is easy to see that Theorem 1 follows from (2), since from the prime number theorem we easily obtain that $\log\log A_k = (1+o(1))ax$, which shows that (1) follows from (2).

(2) means that the density of integers with $\phi(n) \leqq (1/x)n$ is between $1/A_k$ and $1/A_k^{1-\epsilon}$.

We evidently have for every $n \equiv 0 \pmod{A_k}$, $n/\phi(n) \geqq x$, which proves

$$1/A_k \leqq 1 - f_1(x).$$

To get rid of the equality sign, it will be sufficient to observe that there exist integers u with $u/\phi(u) \geqq x$, $(u, A_k) = 1$, and that the density of the integers $n \equiv 0 \pmod{u}$, $n \not\equiv 0 \pmod{A_k}$ is positive. This proves the first part of (2). The proof of the second part will be much harder. We split the integers satisfying $n/\phi(n) \geqq x$ into two classes. In the first class are the integers which have more than $[(1-\epsilon_1)k] = r$ prime factors not greater than Bp_k, where $B = B(\epsilon_1)$ is a large number. In

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[1] These results are due to Schonberg and Davenport. For a more general result see P. Erdös, J. London Math. Soc. vol. 13 (1938) pp. 119–127.

the second class are the other integers satisfying $n/\phi(n) \geqq x$. It is easy to see that the number of integers of the first class does not exceed

$$2^{\pi(Bp_k)}/A_r = 2^{o(p_k)}/A_r < 1/A_k^{1-\epsilon} \tag{3}$$

since $\pi(Bp_k) = o(p_k)$ ($\pi(x)$ denotes the number of primes not greater than x), and from the prime number theorem $\log A_r > (1-\epsilon)p_k$ if ϵ_1 is small.

Let now n be any integer of the second class. A simple argument shows that

$$\prod_{p|n}{}'\left(1 - \frac{1}{p}\right) < \prod_{t=r+1}^{k-1}\left(1 - \frac{1}{p_t}\right) < 1 - \frac{c_1\epsilon_1}{\log p_k}\cdot$$

The prime indicates that the product is extended over the $p > Bp_k$. The first inequality follows from the definition of A_k, and from the fact that n is of the second class, the second inequality follows from the prime number theorem. Thus we have

$$\sum_{p|n}{}' \frac{1}{p} > \frac{c_1\epsilon_1}{\log p_k}\cdot \tag{4}$$

Denote now by J_t the interval $(B^t p_k, B^{t+1}p_k)$, $t = 1, 2, \cdots$. It follows from (4) that for every integer of the second class there exists some t such that

$$\sum_{p|n}{}_t \frac{1}{p} > c_1 \frac{\epsilon_1}{2^t \log p_k} \tag{5}$$

where in $\sum_t$ the summation is extended over the primes in J_t. Thus for some t, n must divide more than

$$c_1\epsilon_1(B^t/2^t)(p_k/\log p_k) = B_t \tag{6}$$

primes in J_t. The density of the integers satisfying (6), that is, the density of the integers of the second class, is less than

$$\sum_{t=1}^{\infty}\left(\sum_{p \text{ in } J_t} \frac{1}{p}\right)^{B_t} \Big/ [B_t]! < \frac{1}{[B_t]!} < e^{-2p_k} < \frac{1}{A_k}, \tag{7}$$

that is, $\sum_{p \text{ in } J_t} 1/p < 1$ for large enough k (B is independent of k), if $B = B(\epsilon_1)$ is large enough. Theorem 1 now follows from (3) and (7).

From Theorem 1 we easily obtain that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n=1}^{x} \exp\,(\phi(n))$$

exists. In fact we can also prove that for $\alpha < a$

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n=1}^{x}\exp\left(\exp\left(\phi(n)\right)\right)$$

exists. For $\alpha > a$ the limit is infinite.

THEOREM 2.

$$1/A_k^{1+\epsilon} < 1 - f_1(x) < 1/A_k^{1-\epsilon}.$$

We omit the proof since it is very similar to that of Theorem 1.

THEOREM 3. *Let* $\epsilon \to 0$, *then*

$$f_1(1+\epsilon) = (1+o(1))a/\log \epsilon^{-1}, \qquad f_2(1+\epsilon) = (1+o(1))a/\log \epsilon^{-1}.$$

We prove only the first statement since the proof of the second is essentially the same. Let n be an integer with $n/\phi(n) \leqq 1+\epsilon$. Clearly n does not divide any prime $p < (1-(1+\epsilon)^{-1})^{-1} = \epsilon^{-1} + O(1)$. Thus

$$f_1(1+\epsilon) < (1+o(1))a/\log \epsilon^{-1}. \tag{8}$$

Denote by J_t the interval

$$(4^{t-1}(1-(1+\epsilon)^{-1})^{-1},\ 4^t(1-(1+\epsilon)^{-1})^{-1}).$$

If an integer $n \not\equiv 0 \pmod{p_i}$, $p_i < (1-(1+\epsilon)^{-1})^{-1}$, does not satisfy $n/\phi(n) \leqq 1+\epsilon$, then a simple computation shows that for some t it must have at least t prime factors in J_t. Thus the number of these integers does not exceed

$$(1+o(1))\frac{a}{\log \epsilon^{-1}}\sum_{t=1}^{\infty}\left(\sum_{p \text{ in } J_t}\frac{1}{p}\right)^t \Big/ t! = o(a/\log \epsilon^{-1}),$$

which together with (8) proves Theorem 3.

It follows from Theorem 3 that $f_1'(1) = \infty$. It would be easy to show that $f_1'(n/\phi(n)) = \infty$ for every n.

Denote by f_1^α and f_2^α the distribution functions of

$$\prod_{p|n}\left(1-\frac{1}{p}\right)^{-\alpha} \quad \text{and} \quad \sum_{d|n}\frac{1}{d^\alpha}, \qquad \alpha > 0.$$

THEOREM 4.

$$f_1^{(\alpha)}(1+\epsilon) = (1+o(1))\frac{a\alpha}{\log \epsilon^{-1}}, \qquad f_2^{(\alpha)} = (1+o(1))\frac{a\alpha}{\log \epsilon^{-1}}.$$

We omit the proof since it is very similar to that of Theorem 3.

Let us denote by $F_\alpha(x)$, $\alpha > 0$, the distribution function of $\prod_{p|n}(1-1/\log p^\alpha)^{-1}$, $\alpha > 0$.

THEOREM 5.
$$F_1(1+\epsilon) = (1+o(1))b\epsilon,$$

that is, $F_1'(1)=b$. *Also* $F_\alpha'(1)=0$ *for* $\alpha<1$ *and* $F_1'(1)=\infty$ *for* $\alpha>1$.

We do not give the details of the proof since it would be long and similar to that of Theorem 3. We just make the following remarks: If n satisfies

$$\sum_{p|n}\frac{1}{\log p}\leqq 1+\epsilon$$

then n does not divide any prime $p\leqq\exp(1/\epsilon)$. Thus $F_1'(1+\epsilon)\leqq(1+o(1))a\epsilon$. But here (unlike in Theorem 3) we have $F_1(1+\epsilon)=(1+o(1))b$, $b<a$. We obtain analogous results if we consider the additive function $\sum_{p|n}1/\log$ p. It is possible that $F_1'(x)$ exists for every $1\leqq x$, but this we can not prove.

(2) The following results are well known:

$$\sum_{m=1}^{x}\frac{\phi(m)}{m}=(1+o(1))\frac{6}{\pi^2}x,\qquad \sum_{m=1}^{x}\frac{\sigma(m)}{m}=(1+o(1))\frac{\pi^2}{6}x.$$

The density of integers for which $\sigma(n+1)/(n+1)>\sigma(n)/n$ is 1/2, also the density of integers for which $\phi(n+1)/(n+1)>\phi(n)/n$ is 1/2.[2] Now we prove the following theorem.

THEOREM 6. *Let* $g(n)/\log\log\log n\to\infty$. *Then we have*

(i)
$$\sum_{m=n}^{n+g(n)}\frac{\phi(m)}{m}=(1+o(1))\frac{6}{\pi^2}g(n).$$

(ii) *The number of integers* m *in* $(n, n+g(n))$ *which satisfy* $\phi(m+1)/(m+1)>\phi(m)/m$ *equals* $(1+o(1))g(n)/2$.

(iii) *The number of integers* m *in* $(n, n+g(n))$ *which satisfy* $m/\phi(m)\leqq c$ *equals* $(1+o(1))g(n)f_1(c)$. *In other words the distribution function of* $\phi(m)/m$ *in* $(n, n+g(n))$ *is the same as the distribution function of* $\phi(m)/m$.

All these results are best possible; they become false if for infinitely many n, $g(n)<c\log\log\log n$.

We prove only (i); the proof of (ii) and (iii) are similar. Let $A=A(n)$ tend to infinity sufficiently slowly. Put

$$\frac{\phi(m)}{m}=D_1(m)D_2(m),$$

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[2] P. Erdös, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 530–540.

where

$$D_1(m) = \prod_{p\mid m}{}' \left(1 - \frac{1}{p}\right), \qquad D_2(m) = \prod_{p\mid m}{}'' \left(1 - \frac{1}{p}\right).$$

The prime indicates that $p \leqq A$, the two primes that $p > A$. We evidently have

$$\begin{aligned} \sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} < \sum_{m=n}^{n+g(n)} D_1(m) &= \sum_d{}''' \left(\frac{g(n)}{d}\right) \frac{\mu(d)}{d} \\ &= (1 + o(1)) g(n) \prod_{p \leqq A} \left(1 - \frac{1}{p^2}\right) = (1 + o(1)) \frac{\pi^2}{6} g(n) \end{aligned} \tag{9}$$

where the three primes indicate that the prime factors of d are not greater than A, and $(g(n)/d)$ denotes the number of multiples of d in $(n, n+g(n))$. Now we show that for sufficiently large A the number of integers in $(n, n+g(n))$ which satisfy

$$D_2(m) < 1 - \epsilon \tag{10}$$

is $o(g(n))$. It will be sufficient to show that

$$\prod_m D_2(m) > (1 - \eta)^{g(n)} \tag{11}$$

for every $\eta > 0$, the product over m runs in $(n, n+g(n))$. We evidently have

$$\prod_m D_2(m) > \prod{}_1 \left(1 - \frac{1}{p}\right)^{2g(n)/p-1} \prod{}_2 \left(1 - \frac{1}{p}\right)$$

where, in $\prod_1$, $A < p \leqq g(n)$, and in $\prod_2$, p runs through the prime factors greater than $g(n)$ of $n(n+1) \cdots (n+g(n))$. Clearly

$$\prod{}_1 > \prod_{p>A} \left(1 - \frac{c}{p^2}\right)^{g(n)} > (1 - \eta_1)^{g(n)}.$$

From the prime number theorem we have $\prod_{p \leqq x} p < e^{2x}$. Thus

$$\prod{}_2 > \prod_{p \leqq 2y} \left(1 - \frac{1}{p}\right) > \frac{c_1}{\log y}$$

where $y = \log [n(n+1) \cdots (n+g(n))]$. Hence using $g(n)/\log \log \log n \to \infty$, we obtain by a simple calculation that

$$\prod{}_2 > (1 - \eta_2)^{g(n)}$$

which proves (11) and therefore (10). From (9) and (10) we obtain by a simple argument that

$$(12)\qquad \sum_{m=n}^{n+g(n)} \frac{\phi(m)}{m} > (1 - o(1)) \sum_{m=n}^{n+g(n)} D_1(m) = (1 + o(1))g(n) \frac{\pi^2}{6}\cdot$$

(i) now follows from 9 and (12).[3]

Now we are going to prove that (i) is best possible. Put $g(N) = c \log\log\log N$, $n/2 < N < n$. Further let $A_1, A_2, \cdots, A_r$, $r = [2^{-1} \log\log\log n]$ be relatively prime integers all of whose prime factors are less than $2^{-1} \log n$ and for which

$$1/4 < \phi(A_i)/A_i < 1/2, \qquad i = 1, 2, \cdots, r.$$

This is obviously possible since

$$\prod_{p<(\log n)/2} \left(1 - \frac{1}{p}\right) < \frac{c}{\log\log n} < \left(\frac{1}{4}\right)^{(\log\log\log n)/2}.$$

Now choose $n/2 < N < n$ so that $N + j \equiv 0 \pmod{A_j}$, $j \leqq r$. This is possible since by the prime number theorem $A_1 \cdot A_2 \cdots A_r < n/2$. (In all cases where we refer to the prime number theorem a more elementary result would be sufficient.) Clearly

$$\sum_{m=N+1}^{N+(\log\log\log n)/2} \frac{\phi(m)}{m} < \frac{\log\log\log n}{4}.$$

From (9) we have

$$(13)\qquad \sum_{N+(\log\log\log n)/2}^{N+g(N)} \frac{\phi(m)}{m} < (1 + o(1)) \frac{6}{\pi^2}\left(g(N) - \frac{\log\log\log n}{2}\right).$$

Thus finally from (10) and (11) we obtain by a simple calculation

$$\sum_{m=N}^{N+g(N)} \frac{\phi(m)}{m} < (1 - c) \frac{6}{\pi^2} g(N),$$

which shows that (i) is best possible.[4]

THEOREM 7. *Let* $g_1(n)/\log\log n \to \infty$. *Then we have*

$$\text{(i)}\qquad \sum_{m=n}^{n+g_1(n)} \frac{\sigma(m)}{m} = (1 + o(1)) \frac{\pi^2}{6} g_1(n).$$

(ii) *Let* $g_2(n)/\log\log\log n \to \infty$. *The number of integers* m *in* $(n, n+g_2(n))$ *which satisfy* $\sigma(n+1)/(n+1) > \sigma(n)/n$ *equals* $(1+o(1)) \cdot g(n)/2$.

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[3] This proof is similar to a proof in P. Erdös, J. London Math. Soc. vol. 10 (1935) pp. 128–131.

[4] This proof is similar to a proof of Chowla and Pillai, J. London Math. Soc. vol. 5 (1930) pp. 95–101.

(iii) *The number of integers* m *in* $(n, n+g(n))$ *which satisfy* $\sigma(m)/m < c$ *equals* $(1+o(1))\ g(n)\ f_2(c)$.[5] *All these results are best possible.*

We omit the proof of Theorem 7, since it is similar to that of Theorem 6. We must allow $g_1(n)/\log\log n \to \infty$, since it is well known that for some $m \leqq n$, $\sigma(m) > c \log\log n$ (for example, $m = \prod_{p<(\log n)/2} p$).

Let $f(n) \leqq 1$ and $F(n) \geqq 1$ be multiplicative functions with

$$\sum_p \frac{1-f(p)}{p} < \infty \quad \text{and} \quad \sum_p \frac{F(p)-1}{p} < \infty.$$

Then we have:

THEOREM 8. *Let* $A = A(n)$ *tend to infinity arbitrarily slowly, then*

$$\frac{1}{A}\sum_{m=n}^{n+A} f(m) < (1 + (o(1))\frac{1}{n}\sum_{m=1}^{n} f(m)$$

and

$$\frac{1}{A}\sum_{m=n}^{n+A} F(m) > (1 + o(1))\frac{1}{n}\sum_{m=1}^{n} F(m).$$

The proof is quite trivial; it is similar to that of (9). It can be shown that $\lim\ (1/n)\sum_{m=1}^{n} f(m)$ and $\lim\ (1/n)\sum_{m=1}^{n} F(m)$ exist.

Denote by $V(n)$ the number of prime factors of n and by $d(n)$ the number of divisors of n. We can prove analogs to Theorem 6 for these functions. But the results are very unsatisfactory since for $v(n)$ we have to choose $g(n) = n^{\epsilon/\log\log n}$ and for $d(n)$, $g(n) = n^c$ for some suitable c. These results are probably very far from best possible.

(3) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_k^{\alpha_k}$. Put $(p_i^{\alpha_i})^{b_i} = p_{i+1}^{\alpha_{i+1}}$. We prove the following theorem.

THEOREM 9. *Let* $1 < x$, *then for almost all* n *the number of* b*'s greater than* x *equals*

$$x^{-1} \log\log n + o(\log\log n).$$

REMARK. We immediately obtain that every interval $(x, x+\epsilon)$ contains $(1+o(1))\ (\epsilon/x(x+\epsilon)) \log\log n$ b's.

We are going to give only an outline of the proof. First of all we can assume that all the α's are 1, since for large r the number of integers not greater than n for which r or more of the α's is greater than 1 is less than ϵn, since the number of these integers is clearly less than

$$\left(\sum_p \frac{1}{p^2}\right)^r \Big/ r! < \epsilon n.$$

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[5] This result has been stated previously, see footnote 4.

Denote by $F(n)$ the number of prime factors p of n such that no prime q in (p, p^x) divides n. $F(n)$ is thus the number of b's not less than x. We have

$$\sum_{m=1}^{n} F(m) = \frac{1}{x} \log\log n + o(\log\log n). \tag{14}$$

We now give a sketch of the proof. Clearly

$$\sum_{m=1}^{n} F(m) = \sum_{p} f_p(n)$$

where $f_p(n)$ denotes the number of integers $m \leqq n$, with $m \equiv 0 \pmod p$ and $m \not\equiv 0 \pmod q$, $p < q < p^x$. It is easy to see that for $p < n^\epsilon$

$$f_p(n) = (1 + o(1))n/px \qquad (p \text{ large}).$$

Also for all p

$$f_p(n) \leqq n/p.$$

Thus

$$\sum_{m=1}^{n} F(m) = \sum_{p \leqq n^\epsilon} \frac{n}{px} + O \sum_{n^\epsilon < p < n} \frac{n}{p} + o(\log\log n)$$
$$= (1 + o(1)) \frac{\log\log n}{x},$$

which proves (14). Now we have to show that

$$F(m) = (1 + o(1))(\log\log n)/x$$

for almost all $m \leqq n$. We use Turán's method.[6] We have

$$\sum_{m=1}^{n} \left(F(m) - \frac{1}{x}\log\log n\right)^2$$
$$= \sum_{m=1}^{n} F^2(m) - \frac{2}{x}\log\log n \sum_{m=1}^{n} F(m) + n\left(\frac{\log\log n}{x}\right)^2.$$

Now

$$\sum_{m=1}^{n} F^2(m) = (1 + o(1))n\left(\frac{\log\log n}{x}\right)^2. \tag{15}$$

We omit the proof of (15), it is similar to the proof of (14). Thus

$$\sum_{m=1}^{n} \left(F(m) - \frac{1}{x}\log\log n\right)^2 = o(n(\log\log n)^2)$$

which proves Theorem 9.

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[6] P. Turán, J. London Math. Soc. vol. 9 (1934) pp. 274–276.

THEOREM 10. *For almost all n we have*

$$\sum_{p_i \mid n} b_i = (1 + o(1)) \log \log n \log \log \log n.$$

THEOREM 11. *Let* $1<x$ *be any number. For almost all n there exist intervals* (m, m^x), $m^x \leqq n$, *such that for every* $m \leqq y \leqq m^x$, $n \not\equiv 0 \pmod{y}$.

We omit the proofs of Theorems 10 and 11. They are similar to that of Theorem 9.

For some time I have not been able to decide the following question: Is it true that almost all integers n have divisors d_1 and d_2, such that $d_1<d_2<2d_1$.

(4) Let $f(n)$ be an additive function which has a distribution function. Then it is well known that[7]

$$\sum_p \frac{f(p)'}{p} < \infty, \qquad \sum_p \frac{(f(p)')^2}{p} < \infty, \tag{16}$$

$f(p)'=f(p)$ if $|f(p)| \leqq 1$ and $f(p)'=1$ if $|f(p)|>1$. Assume now that $|f(p^\alpha)| \leqq C$ ($f(n)$ is assumed to be real valued). We prove the following theorem.

THEOREM 12. *Let* $|f(p^\alpha)| \leqq c$. *Denote by* $F(x)$ *the distribution function of* $f(x)$. *We have*

$$F(x) > 1 - \exp(-cx),$$

for every c and sufficiently large x. In other words the density of integers with $f(n) \geqq x$ *is less than* $\exp(-cx)$.

Put $g(n)=\exp(2cf(n))$, $g(n)$ is multiplicative and clearly has a distribution function. Define

$$f_k(n) = \sum_{p \mid n, p \leqq k} f(p), \qquad g_k(n) = \exp(2cf_k(n)).$$

For sake of simplicity we assume that $f(p^\alpha)=f(p)$. It is well known that the distribution function $F_k(x)$ of $f_k(n)$ converges to $F(x)$, thus the distribution function $G_k(x)$ of $g_k(x)$ converges to $G(x)$ ($G(x)$ is the distribution function of $g(x)$). Suppose now that Theorem 12 is false, then there exists a constant c and infinitely many x_r with $x_r \to \infty$ and

$$F(x_r) > 1 - \exp(-cx_r).$$

Therefore for any r there exists a k so large that

$$F_k(x_r) > 1 - \exp(-cx_r).$$

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[7] P. Erdös and A. Wintner, Amer. J. Math. vol. 61 (1939) pp. 713–721.

Thus the density of integers with $g_k(n) > \exp(2cx_r)$ is greater than $\exp(-cx_r)$ and hence

$$\sum_{m \leqq n} g_k(m) > (1-\epsilon)\exp(cx_r)\cdot n$$

for n sufficiently large. Thus for any A there exists k and n_0, such that for all $n > n_0$

$$\sum_{m \leqq n} g_k(m) > An. \tag{17}$$

On the other hand

$$\sum_{m \leqq n} g_k(m) = \sum_{m=1}^{n} \prod_{p \mid m} g_k(p) = \sum_{m=1}^{n} \prod_{p \mid m} (1 + (g_k(p) - 1)).$$

Put $g_k(p) - 1 = h_k(p)$. Clearly

$$\sum_{m=1}^{n} g_k(m) = \sum_{m=1}^{n} \prod_{p \mid m} (1 + h_k(p)) = \sum_{d} \left[\frac{n}{d}\right] h_k(d)$$

where $h_k(d) = \prod_{p \mid d} h_k(p)$. Thus

$$\sum_{m=1}^{n} g_k(m) \leqq n \sum_{d} \frac{h_k(d)}{d} = n \prod_{p} \left(1 + \frac{h_k(p)}{p}\right).$$

From the fact that $g(n)$ has a distribution function and that $f(p^\alpha)$ is bounded, it easily follows that (we shall give the details in the proof of Theorem 13)

$$\sum_{p} \frac{h(p)}{p} < \infty, \qquad \sum_{p} \frac{(h(p))^2}{p} < \infty, \qquad h(p) = g(p) - 1.$$

Thus finally

$$\sum_{m=1}^{n} g_k(m) < c_1 n \prod_{p} \left(1 + \frac{h(p)}{p}\right) < c_2 n,$$

which contradicts (17), and this contradiction establishes the theorem.

It is easy to see that Theorem 12 is best possible. Let $\phi(x)$ tend to infinity arbitrarily slowly; then there exists an additive function $f(n)$ such that its distribution function $F(x)$ satisfies $F(x_i) < 1 - \exp(-\phi(x_i)x_i)$ for an infinite sequence x_i with $x_i \to \infty$. We omit the proof.

THEOREM 13. *Let $g(n) \geqq 0$ be multiplicative. Then the necessary and sufficient condition for the existence of a distribution function is that*

$$\sum_p \frac{(g(p)-1)'}{p} < \infty, \qquad \sum_p \frac{((g(p)-1)')^2}{p} < \infty, \tag{18}$$

where $(g(p)-1)' = g(p)-1$ *if* $|g(p)-1| \leqq 1$ *and* 1 *otherwise.*

The proof follows very easily from (16). Put $\log(g(n)) = f(n)$. $g(n)$ has a distribution function if and only if $f(n)$ has a distribution function. Thus from (16)

$$\sum_p \frac{(\log g(p))'}{p} < \infty, \qquad \sum_p \frac{((\log g(p))')^2}{p} < \infty. \tag{19}$$

Now it follows from (19) that if we neglect a sequence of primes q with $\sum 1/q < \infty$ that $|g(p)-1| < 1/2$. Thus

$$\log g(p) = \log(1 + (g(p)-1)) = g(p) - 1 + (1/2)(g(p)-1)^2 + \cdots.$$

Also simple computation shows that $(\log g(p))^1 > (1/4)(g(p)-1)^2$. Thus from (19)

$$\sum_p \frac{(g(p)-1)^2}{p} < \infty$$

and

$$\sum_p ((1/2)(g(p)-1)^2 + (g(p)-1)^3 + \cdots) < \infty.$$

Thus $\sum_p (g(p)-1)/p < \infty$, which shows that (18) is necessary.

If the two series in (18) converge, then clearly

$$\sum_p \frac{\log g(p)}{p} = \sum_p \left(\frac{(g(p)-1)}{p} + \frac{(1/2)(g(p)-1)^2}{p} + \cdots \right) < \infty$$

and

$$\sum_p \frac{(\log g(p))^2}{p} < c \sum_p \frac{(g(p)-1)^2}{p} < \infty,$$

which shows that $f(n)$, and therefore $g(n)$, has a distribution function. Thus (18) is necessary, which completes the proof of Theorem 13.

These results suggest that if $g(n)$ is multiplicative, satisfies (18), $|g(p^\alpha)| < c$, then $g(n)$ has a mean value, that is, $\lim(1/x)\sum_{n=1}^{x} f(n)$ exists. I have not yet been able to prove this.

UNIVERSITY OF MICHIGAN

TWO NONEXISTENCE THEOREMS ON PARTITIONS

D. H. LEHMER

The theory of partitions contains a number of theorems which assert that the number of partitions of a given number into parts subjected to a certain restriction is the same as the number of partitions restricted in some other way. A common type of restricted partition is one in which all parts are distinct. We have for example the famous theorem of Euler[1] (1748):

EULER'S THEOREM. *The number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts.*

The notion of distinctness of parts may be altered in two directions. One may relax it to some extent and admit partitions in which no part is repeated more than a given number of times. In this case we have the beautiful theorem of Glaisher[2] (1883).

GLAISHER'S THEOREM. *The number of partitions of n in which no part is repeated more than $r-1$ times is the same as the number of partitions of n into parts not divisible by r.*

This theorem obviously becomes Euler's theorem when $r=2$.

On the other hand the notion of distinctness may be further restricted so as to include only those partitions in which the parts differ by d or more. For $d=0$, we have completely unrestricted partitions. For $d=2$ we have a celebrated and difficult theorem discovered independently by Rogers[3] (1894), Schur[4] (1917) and Ramanujan[5] (1919).

ROGERS' THEOREM. *The number of partitions of n into parts differing by 2 or more is the same as the number of partitions of n into parts taken from the set* $1, 4, 6, 9, 11, 14, 16, \cdots, (5k+1, 4), \cdots$.

Attempting to go further in this direction, Schur[6] later (1926) proved the following theorem:

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[1] L. Euler, *Introductio analysin infinitorum*, vol. 1, Lausanne, 1748, pp. 253–275.

[2] J. W. L. Glaisher, Messenger of Mathematics vol. 12 (1883) pp. 158–170.

[3] L. J. Rogers, Proc. London Math. Soc. (1) vol. 25 (1894) pp. 328–329.

[4] I. Schur, Akademie der Wissenschaften, Berlin, Sitzungsberichte (1917) pp. 302–321.

[5] S. Ramanujan, Proc. Cambridge Philos. Soc. vol. 19 (1919) pp. 211–216.

[6] I. Schur, Akademie der Wissenschaften, Berlin, Sitzungsberichte (1926) pp. 488–495. See also W. Gleissberg, Math. Zeit. vol. 28 (1928) pp. 372–382.

SCHUR'S THEOREM. *The number of partitions of n into parts which differ by 3 or more, and such that parts divisible by 3 differ by at least 6, s the same as the number of partitions of n into parts taken from the set* 1, 7, 13, 19, 25, 31, 37, $\cdots$, $(6k+1)$, $\cdots$.

The above theorems have corresponding analytical formulations. f we denote by $q_d(n)$ the number of partitions of n into parts differing y d or more, then Euler's and Rogers' theorems can be written:

$$\tag{1}\begin{aligned}\sum_{n=0}^{\infty} q_1(n)x^n &= \sum_{s=0}^{\infty} \frac{x^{s(s+1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)}\\ &= \prod_{\nu=1}^{\infty}(1-x^{2\nu-1})^{-1} = \prod_{\nu=1}^{\infty}(1+x^{\nu})\end{aligned}$$

nd

$$\tag{2}\begin{aligned}\sum_{n=0}^{\infty} q_2(n)x^n &= \sum_{s=0}^{\infty} \frac{x^{s^2}}{(1-x)(1-x^2)\cdots(1-x^s)}\\ &= \prod_{\nu=0}^{\infty}(1-x^{5\nu+1})^{-1}(1-x^{5\nu+4})^{-1}.\end{aligned}$$

As a matter of fact, (1) and (2) include rather more than is stated in the theorems, namely the facts that $q_1(n)$ and $q_2(n)$ are generated by the sums over s. They suggest also the well known[7] case due to Euler in which $d=0$, so that $q_0(n)=p(n)$, the number of unrestricted partitions of n:

$$\tag{3}\sum_{n=0}^{\infty} p(n)x^n = \sum_{s=0}^{\infty} \frac{x^s}{(1-x)(1-x^2)\cdots(1-x^s)} = \prod_{\nu=1}^{\infty}(1-x^{\nu})^{-1}.$$

The first parts of (1), (2) and (3) are special cases of a general generating identity

$$\tag{4}\sum_{n=0}^{\infty} q_d(n)x^n = \sum_{s=0}^{\infty} \frac{x^{s+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)}$$

which is proved in Theorem 1. As for the products in (1), (2), and (3), we show in Theorems 2 and 3 that no further examples exist; that is, that neither of the identities

$$\sum q_d(n)x^n = \prod_{\nu=1}^{\infty}(1-x^{a_\nu})^{-1} \qquad (d \neq 0, 1, 2),$$

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[7] See for instance Hardy and Wright, *Introduction to the theory of numbers*, Oxford, 1938, chap. 19, where also can be found proofs of Euler's and Rogers' theorems.

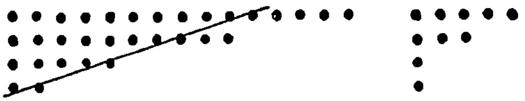
$$\sum q_d(n)x^n = \prod_{\nu=1}^{\infty} (1 + x^{a_\nu}) \qquad (d \neq 1)$$

can hold, no matter what set of distinct a's is chosen. In particula this means that no more theorems like those of Euler and Rogers ar possible. In commenting on (2), P. A. MacMahon[8] points out tha $q_2(n)$ is also the number of partitions of n in which each part is a least equal to the number of parts. For the general d there exists corresponding second interpretation of $q_d(n)$. If d is even, $q_d(n)$ the number of partitions of n into parts greater than $(s-1)d/2$, whe s is the number of parts. For d odd, $q_d(n)$ is the number of partitio of $2n$ into parts greater than $1+(s-1)d$ and having the same parit as $s-1$. The case $d=2$ is certainly the most elegant one. Howev for $d>0$ such sets of parts depend on s and so such partitions ar not of the kinds considered in Theorems 2 and 3.

THEOREM 1. *Let $q_d(n)$ denote the number of partitions of n into parts differing by d or more; then*

$$\sum_{n=0}^{\infty} q_d(n)x^n = \sum_{s=0}^{\infty} \frac{x^{s+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)}.$$

PROOF. The following graphical proof may be given. Any partition of the sort enumerated by $q_d(n)$ may be represented graphically by a series of rows of dots or "nodes," each row representing a part. Thus the following figure represents the case of the partition $15+10+5+2$ of 32 into parts differing by three or more.



A line passing just below the last node in the first column having a slope of $1/d$ divides the graph into a triangle or "head" and an irregular "tail." Graphs of this kind may be classified by the number of their rows or parts. It is clear that the number of nodes in the head of a graph of s rows is the polygonal number

$$s + d(s-1) + d(s-2) + \cdots + d = s + d(s-1)s/2$$

of order $d+2$. The $n-[s+ds(s-1)/2]$ nodes in the tail may be translated to the right so as to form a "regular" graph which, when read by columns, gives a partition of $n-[s+ds(s-1)/2]$ into parts not exceeding s. Conversely, one may start with a graph of such a parti- ion and, by translating the rth row $(r-1)d$ spaces to the left for $=1, 2, \cdots, s$ and affixing the head of s rows, produce a graph of a rtition of n into parts differing by d or more. Hence we may write

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[8] P. A. MacMahon, *Combinatorial analysis*, vol. 2, Cambridge, 1916.

$$q_d(n) = \sum_{s=0}^{\infty} p_s(n - s - ds(s-1)/2)$$

ere $p_s(m)$ denotes the number of partitions of m into parts not ater than s, $p_0(m)=0$ for $m>0$, $p_s(0)=1$. If we define $p_s(m)$ to zero for $m<0$ we have, as is well known,[7] the generating function

$$\sum_{m=-\infty}^{\infty} p_s(m)x^m = \frac{1}{(1-x)(1-x^2)\cdots(1-x^s)}.$$

ultiplying (5) by x^n and summing over n we have, in view of (6),

$$\begin{aligned}\sum_{n=0} q_d(n)x^n &= \sum_{s=0}^{\infty} x^{s+ds(s-1)/2} \sum_{n=0}^{\infty} p_s(n - s - ds(s-1)/2)x^{n-s-ds(s-1)/2} \\ &= \sum_{s=0}^{\infty} \frac{x^{s+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)},\end{aligned}$$

hich proves the theorem.

We now proceed to prove two nonexistence theorems.

THEOREM 2. *The number $q_d(n)$ of partitions of n into parts differing by d or more is not equal to the number of partitions of n into parts taken from any set S of integers whatsoever, except when $d=0$, 1, or 2.*

REMARKS. The exceptional case $d=0$ is trivial since S is in this case the set of all integers. For $d=1$, the set S may be taken to be the set of all odd integers, giving us Euler's theorem. For $d=2$ the set S consists of all numbers of the form $5x+1$, 4 $(x=0, 1, 2, \cdots)$, giving us Rogers' theorem. Hence we may suppose that $d \geqq 3$ in what follows.

PROOF. Suppose that the theorem is false and that there exists such a set S of integers $a_1 < a_2 < a_3 < \cdots$ for $d \geqq 3$; then by Theorem 1 we would have

$$\begin{aligned}\prod_{\nu=1}^{\infty} (1 - x^{a_\nu})^{-1} &= \sum_{s=0}^{\infty} \frac{x^{s+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)} \\ &= \frac{1}{1-x} + \frac{x^{d+2}}{(1-x)(1-x^2)} \qquad (7)\\ &\quad + \frac{x^{3d+3}}{(1-x)(1-x^2)(1-x^3)} + \cdots.\end{aligned}$$

Therefore $a_1=1$. Multiplying (7) by $1-x$ we have

$$\prod_{\nu=2}^{\infty}(1-x^{a_\nu})^{-1}=1+\frac{x^{d+2}}{1-x^2}+\frac{x^{3d+3}}{(1-x^2)(1-x^3)}+\cdots. \tag{8}$$

Hence $a_2=d+2$. Multiplying (8) by $(1-x^{d+2})$ we have

$$\begin{aligned}\prod_{\nu=3}^{\infty}(1-x^{a_\nu})^{-1}&=1-x^{d+2}+\frac{x^{d+2}}{1-x^2}-\frac{x^{2d+4}}{1-x^2}\\&\quad+\frac{x^{3d+3}}{(1-x^2)(1-x^3)}+\cdots.\end{aligned} \tag{9}$$

If d is odd the coefficient of x^{2d+4} on the right of (9) is seen to be -1. This contradicts the fact that the product on the left, when expanded in powers of x, has non-negative coefficients. Hence d must be even. In this case the right-hand member of (9) simplifies so that we have

$$\begin{aligned}\prod_{\nu=3}(1-x^{a_\nu})^{-1}&=1+x^{d+4}+x^{d+6}+\cdots+x^{2d+2}+x^{3d+3}\\&\quad+x^{3d+5}+x^{3d+6}+\cdots.\end{aligned} \tag{10}$$

Hence $a_3=d+4$, and since $a_\nu+a_\mu>2a_3=2d+8$ we can conclude that $a_4=d+6$, $a_5=d+8,\cdots, a_{d/2+2}=2d+2$. Multiplying both sides of (10) by the corresponding factors we have

$$\begin{aligned}\prod_{\nu=3+d/2}(1-x^{a_\nu})^{-1}&=\prod_{\lambda=4}^{d+2}(1-x^{d+\lambda})(1+x^{d+4}+x^{d+6}+\cdots\\&\quad+x^{2d+2}+x^{3d+3}+x^{3d+5}+x^{3d+6}+\cdots).\end{aligned} \tag{11}$$

We now consider the coefficient of x^{2d+8} on both sides of (11). Since d is even, $3d+3\neq 2d+8$ and $d\geqq 4$, so that $2d+8<3d+5$. Therefore the coefficient of x^{2d+8} on the right of (11) is -1. As before this contradicts the fact that the coefficients on the left are non-negative. Hence we have the theorem.

THEOREM 3. *The number $q_d(n)$ of partitions of n into parts differing by d or more is not equal to the number of partitions of n into distinct parts taken from any set S of integers whatsoever, except when $d=1$.*

REMARKS. The theorem for $d=0$ is obvious since the first mentioned partitions are, in this case, unrestricted and hence more numerous than any other kind of partition. For $d=1$, the exception is of course necessary but trivial since in this case S may be taken as the set of all integers. Hence we are interested in proving the theorem for $d\geqq 2$.

PROOF. Suppose the theorem to be false and let S exist consisting of the integers $a_1 < a_2 < a_3 < \cdots$. Then comparing the generating functions of the two kinds of partitions described in the theorem we have, by Theorem 1,

$$\begin{aligned}\prod_{\nu=1}(1 + x^{a_\nu}) = \frac{1}{1-x} + \frac{x^{d+2}}{(1-x)(1-x^2)} \\ + \frac{x^{3d+3}}{(1-x)(1-x^2)(1-x^3)} + \cdots .\end{aligned} \tag{12}$$

Therefore $a_1 = 1$. Multiplying both sides of (12) by $1-x$ we have

$$(1 - x^2)\prod_{\nu=2}(1 + x^{a_\nu}) = 1 + \frac{x^{d+2}}{1-x^2} + \frac{x^{3d+3}}{(1-x^2)(1-x^3)} + \cdots . \tag{13}$$

Let λ be defined by

$$2^{\lambda-1} < d + 2 \leqq 2^\lambda. \tag{14}$$

Since the coefficients of x^k on the right of (13) vanish for $0 < k < d+2$ we must have $a_2 = 2$, $a_3 = 2^2$, $a_4 = 2^3, \cdots, a_\lambda = 2^{\lambda-1}$. Hence (13) may be written

$$(1 - x^{2^\lambda})\prod_{\nu>\lambda}(1 + x^{a_\nu}) = 1 + \frac{x^{d+2}}{1-x^2} + \frac{x^{3d+3}}{(1-x^2)(1-x^3)} + \cdots . \tag{15}$$

We now separate two cases:

Case 1. d is even. Let us compare the coefficient of x^{2^λ} on both sides of (15). Since $3d+3 > 2^\lambda$ by (14), the coefficient of x^{2^λ} on the right of (15) is clearly $+1$. Since the sum of any two a's occurring in (15) s greater than $2a_\lambda = 2^\lambda$, the coefficient of x^{2^λ} on the left of (15) is either 0 or -1, according as one of the a_ν has the value 2^λ or not. n either case a contradiction exists disposing of Case 1.

Case 2. d is odd and greater than or equal to 3. In this case the powers of x occurring on the right of (15) are the odd powers from x^{d+2} to x^{3d+2}. Hence x^{2^λ} does not appear on the right. Since the sum of two a's on the left exceeds 2^λ, one of these a's must be 2^λ. Setting the corresponding factor $1+x^{2^\lambda}$ outside the product sign and developing the rest in powers of x we have

$$\begin{aligned}(1 - x^{2^{\lambda+1}})\Big(1 + \sum_{\nu>\lambda,\ a_\nu\neq 2^\lambda} x^{a_\nu} + \sum_{\mu>\nu>\lambda,\ a_\mu\neq 2^\lambda\neq a_\nu} x^{a_\nu+a_\mu} + \cdots\Big) \\ = 1 + \frac{x^{d+2}}{1-x^2} + \frac{x^{3d+3}}{(1-x^2)(1-x^3)} + \cdots .\end{aligned} \tag{16}$$

Comparing the early powers of x we see that $a_{\lambda+1}=d+2$, $a_{\lambda+2}=d+4, \cdots$, so that $a_\nu+a_\mu \geqq a_{\lambda+1}+a_{\lambda+2} \geqq 2d+6$. Hence the a's in (16) are the successive odd numbers beginning with $d+2$ and extending at least as far as $2d+5$ since this latter number is less than $2^{\lambda+1}$ on the one hand (in view of (14)) and $3d+3$ on the other, since $d \geqq 3$. We now consider the coefficient of x^{3d+6} on both sides of (16). Since $3d+6$ is odd, it is not the sum of two a's, since the a's are odd, nor is $3d+6$ the sum of three or more a's, since such a sum would exceed $3a_{\lambda+1}=3d+6$. Hence any contribution to the coefficient of x^{3d+6} on the left of (16) must arise from the multiplication of $(1-x^{2^{\lambda+1}})$ by $\sum x^{a_\nu}$ and so is either ±1 or 0. However it is clear that the coefficient of x^{3d+6} on the right of (16) is precisely 2. This contradiction proves the theorem.

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THE TRIANGULATION PROBLEM AND ITS ROLE IN ANALYSIS

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1. Introduction. The triangulation problem is fundamental in the topology of manifolds and is closely related to certain methods employed in analysis. The results to date are, perhaps, of greater interest from the viewpoint of connections between topology and differential geometry (or other branches of analysis) than from a purely topological viewpoint. Most of the published work on this problem has appeared during approximately the last fifteen years, save for well known results in two dimensions. A brief discussion of a 2-dimensional result and its role in proving a theorem of analysis (adapted from Osgood's *Funktionentheorie* [31][1]) may throw light on the problem and a certain class of applications.

Let B denote a simple closed curve in the (x, y)-plane, and let R denote its interior. It will be assumed that B is differentiable in the sense that some neighborhood of any point on B can be represented by giving y (or x) as a single-valued function of x (or y) with a continuous first derivative. Let the entire (x, y)-plane be subdivided into squares by the lines

$$x = m\delta, \quad y = n\delta, \qquad m, n = 0, \pm 1, \pm 2, \cdots, \tag{1.1}$$

where δ is a positive number so small that a circle of radius 3δ about any point of B cuts from B a single arc, any two tangents to which form an angle less than $\pi/6$. If, in or on the boundary of one of the squares determined by (1.1), B is (1) parallel at some point to an axial direction and (2) meets an edge, β, parallel to that same direction, then let the two squares incident with β be amalgamated into a single rectangular region. The amalgamated region cuts from B a single arc with end points on the sides perpendicular to β. This arc divides the rectangle into two parts, one in R and one outside. The part inside R will be a 2-cell of the subdivision of $(R+B)$. Each of the squares which meets $(R+B)$ and is not involved in such an amalgamation has in its interior just a 2-cell of R, to be reckoned as a 2-cell of the subdivision. The 1-cells of the subdivision consist of (1) the edges entirely in R of the squares determined by (1.1), (2) the

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

segments in R of those edges which are met by B, and (3) the arcs cut from B by amalgamated rectangles and by unmodified squares of the set (1.1). The 0-cells are, of course, the end points of the 1-cells just defined.

Osgood employed the above subdivision to prove rigorously, among other results, the 2-dimensional case of Green's Theorem, which can be expressed in the form

$$\int\int_R \left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) dS = -\int_B (X \cos \tau + Y \sin \tau) ds \tag{1.2}$$

where (1) (X, Y) is a vector field with continuous first partial derivatives, (2) B is oriented counterclockwise, and (3) τ is the inclination of the directed tangent line to B. The theorem is invariant under euclidean transformations of coordinates. After a suitable rotation of axes, it is easy to prove it for a typical 2-cell of the subdivision. This gives the result in the small. Its proof in the large is obtained by summing the identities for all the 2-cells of the subdivisions. Each inner 1-cell is common to the boundaries of two 2-cells and is oppositely oriented on these boundaries. Hence the contributions from all inner 1-cells add up to zero, and the desired result follows.

Kellogg [28] directly generalized the above procedure to three dimensions, and the writer [5, 18] carried through such a generalization in n dimensions.

The word *triangulation* suggests a subdivision of a curved surface, or a plane region, into (curvilinear) triangles. However, cellular subdivisions of the sort just described are also referred to as *triangulations*, as are the higher-dimensional generalizations defined in §4.

The above work reveals a cellular subdivision as a useful tool in proving a theorem of analysis. In general, such applications require not merely a knowledge of the triangulability of the region or locus in question, but require a subdivision thereof into cells possessing special properties.

2. Manifolds of various classes. Intrinsic definition.

The most fundamental and interesting triangulation problems relate to manifolds. A *topological m-manifold* is a connected topological space which can be covered by a denumerable set of neighborhoods, called *m-cells*, each of which is homeomorphic to the interior of an $(m-1)$-dimensional hypersphere in euclidean m-space.

The manifolds of differential geometry and analysis are subject to differentiability conditions, and the most general triangulation theorems to date have been proved only with the aid of such conditions.